

Contents

Introduction	1
Part I Basic spinorial material	9
1 Algebraic aspects	11
1.1 Clifford algebras	11
1.1.1 Definitions	11
1.1.2 Classification of Clifford algebras	18
1.2 Spin groups and their representations	20
1.2.1 Spin groups	20
1.2.2 Representations of spin groups	28
1.2.3 Real and quaternionic structures	36
2 Geometrical aspects	39
2.1 Spinorial structures	39
2.1.1 Spin structures and spinorial metrics	39
2.1.2 Spinorial connections and curvatures	42
2.2 Spin ^c and conformal structures	45
2.2.1 Spin ^c structures	45
2.2.2 Weyl structures	49
2.2.3 Spin and Spin ^c conformal manifolds	52
2.3 Natural operators on spinors	56
2.3.1 General algebraic setting	57
2.3.2 First-order differential operators	59
2.3.3 Basic differential operators on spinor fields	61
2.3.4 The Dirac operator: basic properties and examples	63
2.3.5 Conformal covariance of the Dirac and Penrose operators . .	69
2.3.6 Conformally covariant powers of the Dirac operator	72

2.4	Spinors in classical geometrical contexts	73
2.4.1	Restrictions of spinors to hypersurfaces	73
2.4.2	Spinors on warped products	75
2.4.3	Spinors on Riemannian submersions	76
2.5	The Schrödinger–Lichnerowicz formula	81
3	Topological aspects	85
3.1	Topological aspects of spin structures	85
3.1.1	Čech cohomology and principal bundles	86
3.1.2	Lifting principal bundles via central extensions	88
3.1.3	Stiefel–Whitney classes	90
3.2	Topological classification of Spin^c structures	91
3.3	Spin structures in low dimensions	93
3.3.1	Dimension 1	93
3.3.2	Dimension 2	94
3.3.3	Dimensions 3 and 4	95
3.4	Examples of obstructed manifolds	96
4	Analytical aspects	99
4.1	Fourier transform	99
4.2	Pseudo-differential calculus	101
4.2.1	Symbols	101
4.2.2	Asymptotic summation	102
4.3	Pseudo-differential operators	104
4.4	Composition of pseudo-differential operators	107
4.5	Action of diffeomorphisms on pseudo-differential operators	108
4.6	Pseudo-differential operators on vector bundles	109
4.7	Elliptic operators	112
4.8	Adjoints	113
4.9	Sobolev spaces	113
4.10	Compact operators	118
4.11	Eigenvalues of self-adjoint elliptic operators	121
Part II	Lowest eigenvalues of the Dirac operator on closed spin manifolds	125
5	Lower eigenvalue bounds on Riemannian closed spin manifolds	127
5.1	The Lichnerowicz theorem	127
5.2	The Friedrich inequality	129
5.3	Special spinor fields	133
5.4	The Hijazi inequality	137
5.5	The action of harmonic forms on Killing spinors	140

5.6	Other estimates of the Dirac spectrum	142
5.6.1	Moroianu–Ornea’s estimate	142
5.7	Further developments	146
5.7.1	The energy–momentum tensor	147
5.7.2	Witten’s proof of the positive mass theorem and applications	148
5.7.3	Further applications	158
6	Lower eigenvalue bounds on Kähler manifolds	161
6.1	Kählerian spinor bundle decomposition	161
6.2	The canonical line bundle	164
6.3	Kählerian twistor operators	166
6.4	Proof of Kirchberg’s inequalities	170
6.5	The limiting case	172
7	Lower eigenvalue bounds on quaternion-Kähler manifolds	175
7.1	The geometry of quaternion-Kähler manifolds	176
7.2	Quaternion-Kähler spinor bundle decomposition	179
7.3	The main estimate	182
7.4	The limiting case	190
7.5	A systematic approach	202
7.5.1	General Weitzenböck formulas	202
7.5.2	Application to quaternion-Kähler manifolds	206
7.5.3	Proof of the estimate	210
Part III Special spinor field and geometries		213
8	Special spinors on Riemannian manifolds	215
8.1	Parallel spinors on spin and Spin^c manifolds	215
8.1.1	Parallel spinors on spin manifolds	215
8.1.2	Parallel spinors on Spin^c manifolds	219
8.2	Special holonomies and relations to warped products	223
8.2.1	Sasakian structures	223
8.2.2	3-Sasakian structures	225
8.2.3	The exceptional group G_2	227
8.2.4	Nearly Kähler manifolds	229
8.2.5	The group Spin_7	233
8.3	Classification of manifolds admitting real Killing spinors	236
8.4	Detecting model spaces by Killing spinors	239
8.5	Generalized Killing spinors	241
8.6	The Cauchy problem for Einstein metrics	243

9 Special spinors on conformal manifolds	251
9.1 The conformal Schrödinger–Lichnerowicz formula	251
9.2 Parallel spinors with respect to Weyl structures	254
9.2.1 Parallel conformal spinors on Riemann surfaces	254
9.2.2 The non-compact case	256
9.2.3 The compact case	262
9.3 A conformal proof of the Hijazi inequality	263
10 Special spinors on Kähler manifolds	265
10.1 An introduction to the twistor correspondence	265
10.1.1 Quaternion-Kähler manifolds	265
10.1.2 3-Sasakian structures	267
10.1.3 The twistor space of a quaternion-Kähler manifold	268
10.2 Kählerian Killing spinors	269
10.3 Complex contact structures on positive Kähler–Einstein manifolds	272
10.4 The limiting case of Kirchberg’s inequalities	275
11 Special spinors on quaternion-Kähler manifolds	279
11.1 The canonical 3-Sasakian SO_3 -principal bundle	280
11.2 The Dirac operator acting on projectable spinors	283
11.3 Characterization of the limiting case	288
11.4 Conclusion	299
Part IV Dirac spectra of model spaces	301
12 A brief survey on representation theory of compact groups	305
12.1 Reduction of the problem to the study of irreducible representations	305
12.2 Reduction of the problem to the study of irreducible representations of a maximal torus	315
12.3 Characterization of irreducible representations by means of dominant weights	321
12.3.1 Restriction to semi-simple simply connected groups	321
12.3.2 Roots and their properties	322
12.3.3 Irreducible representations of the group SU_2	325
12.3.4 Proof of the fundamental properties of roots	327
12.3.5 Dominant weights	334
12.3.6 The Weyl formulas	339

12.4 Application: irreducible representations of the classical groups SU_n , Spin_n , and Sp_n	343
12.4.1 Irreducible representations of the groups SU_n and $U_n, n \geq 3$	343
12.4.2 Irreducible representations of the groups Spin_n and $\text{SO}_n, n \geq 3$	348
12.4.3 Irreducible representations of the group Sp_n	361
13 Symmetric space structure of model spaces	369
13.1 Symmetric space structure of spheres	372
13.2 Symmetric space structure of the complex projective space	374
13.3 Symmetric space structure of the quaternionic projective space	375
14 Riemannian geometry of model spaces	379
14.1 The Levi-Civita connection	381
14.2 Spin structures	388
14.3 Spinor bundles on symmetric spaces	390
14.4 The Dirac operator on symmetric spaces	392
15 Explicit computations of the Dirac spectrum	395
15.1 The general procedure	395
15.2 Spectrum of the Dirac operator on spheres	407
15.3 Spectrum of the Dirac operator on the complex projective space	409
15.4 Spectrum of the Dirac operator on the quaternionic projective space	412
15.5 Other examples of spectra	414
Bibliography	421
Index	451