Introduction

The topic of this book is 3-*manifold groups,* that is, fundamental groups of compact 3-manifolds. This class of groups sits between the class of fundamental groups of surfaces, which is very well understood, and the class of fundamental groups of higher-dimensional manifolds, which is very badly understood for the simple reason that, given any finitely presented group π and integer $n \geq 4$, there exists a closed n-manifold with fundamental group π. (See [CZi93, Theorem 5.1.1] or [SeT80, Section 52] for a proof.) This fact poses a serious obstacle to understanding high-dimensional manifolds; for example, the unsolvability of the Isomorphism Problem for finitely presented groups [Ady55, Rab58] leads to a proof that closed manifolds of dimensions ≥ 4 cannot be classified in an algorithmically feasible way; see [Mav58, Mav60, BHP68, Sht04] and [Sti93, Section 9.4].

The study of the fundamental groups of 3-manifolds goes hand in hand with that of 3-manifolds themselves, since the latter are essentially determined by the former. More precisely, a closed, orientable, irreducible 3-manifold that is not a lens space is uniquely determined by its fundamental group. (See Theorem 2.1.3 below.) Despite the great interest and progress in 3-manifold topology during the last decades, survey papers focussing on the group-theoretic properties of fundamental groups of 3-manifolds seem to be few and far between. See [Neh65, Sta71, Neh74, Hem76, Thu82a], [CZi93, Section 5], and [Kir97] for some earlier surveys and lists of open questions.

This book grew out of an appendix originally planned for the monograph [AF13]. Its goal is to fill what we perceive as a gap in the literature, and to give an extensive overview of properties of fundamental groups of compact 3-manifolds with a particular emphasis on the impact of the Geometrization Conjecture of Thurston [Thu82a] and its proof by Perelman [Per02, Per03a, Per03b], the Tameness Theorem of Agol [Ag04] and Calegari–Gabai [CaG06], and the Virtually Compact Special Theorem of Agol [Ag13], Kahn–Markovic [KM12a] and Wise [Wis12a].

Our approach is to summarize many of the results in several flowcharts and to provide detailed references for each implication appearing in them. We will mostly consider fundamental groups of 3-manifolds which are either closed or have toroidal boundary, and we are interested in those properties of such 3-manifold groups π that can be formulated purely group-theoretically, i.e., without reference to the 3-manifold whose fundamental group is π. Typical examples are: *torsion-freeness*, *residual* properties such as being *residually finite* or *residually p*, *linearity* (over a field of characteristic zero), or *orderability*. We do not make any claims to originality—all results are either already in the literature, are simple consequences of established facts, or are well known to the experts.

Organization of this book. As a guide for the reader, it may be useful to briefly go through some of the building blocks for our account of 3-manifold groups in the order they are presented in this book (which is roughly chronologically).

An important early result on 3-manifolds is the Sphere Theorem, proved by Papakyriakopoulos [Pap57a]. (See Section 1.3 below.) It implies that every orientable, irreducible 3-manifold with infinite fundamental group is an Eilenberg–Mac Lane space, and so its fundamental group is torsion-free. (See (A.2) and (C.3) in Sections 3.1 and 3.2, respectively.)

Haken [Hak61a, Hak61b] introduced the concept of a *sufficiently large* 3-manifold, later baptized *Haken manifold*. (See (A.10) in Section 3.1 for the definition.) He proved that Haken manifolds can be repeatedly cut along incompressible surfaces until the remaining pieces are 3-balls; this allows an analysis of Haken manifolds to proceed by induction. Soon thereafter, Waldhausen [Wan68a, Wan68b] produced many results on the fundamental groups of Haken 3-manifolds, e.g., the solution to the Word Problem.

A decade later, the Jaco–Shalen–Johannson (JSJ) decomposition [JS79, Jon79a] of an orientable, irreducible 3-manifold with incompressible boundary gave insight into the subgroup structure of the fundamental groups of Haken 3-manifolds. (See Section 1.6.) The JSJ-Decomposition Theorem also prefigured the Geometrization Conjecture. This conjecture was formulated and proved for Haken 3-manifolds by Thurston [Thu82a], and in the general case finally by Perelman [Per02, Per03a, Per03b]. (See Theorems 1.7.6 and 1.9.1.) After Perelman's epochal results, it became possible to prove that 3-manifold groups have many properties in common with linear groups: for example, they are residually finite [Hem87] (in fact, virtually residually p for all but finitely many prime numbers p [AF13]; see (C.28) in Section 3.2 below) and satisfy the Tits Alternative (see items (C.26) and (L.2) in Sections 3.2 and 6.2, respectively).

The developments outlined in the paragraphs above (up to and including the proof of the Geometrization Conjecture and its fallout) are discussed in Chapters 1–3 of the present book. Flowchart 1 on page 49 collects properties of 3-manifold groups that can be deduced using classical results of 3-manifold topology and Geometrization alone.

The Geometrization Conjecture also laid bare the special rôle played by hyperbolic 3-manifolds, which has become a major focus of study in the last 30 years. During this period, our understanding of their fundamental groups has reached a level of completeness which seemed almost inconceivable only a short while ago. This is the subject of Chapters 4–6.

An important stepping stone in this process was the Subgroup Tameness Theorem (Theorem 4.1.2 below), which describes the finitely generated, geometrically infinite subgroups of fundamental groups of finite-volume hyperbolic 3-manifolds. This theorem is a consequence of the proof to Marden's Tameness Conjecture by Agol [Ag04] and Calegari–Gabai [CaG06], in combination with Canary's Covering Theorem [Cay96]. As a consequence, in order to understand the finitely generated subgroups of fundamental groups of hyperbolic 3-manifolds of this kind, one can mainly restrict attention to *geometrically finite* subgroups.

The results announced by Wise in [Wis09], with proofs provided in [Wis12a] (see also [Wis12b]), revolutionized the field. First and foremost, together with Agol's Virtual Fibering Theorem [Ag08], they imply that every Haken hyperbolic 3-manifold is virtually fibered (i.e., has a finite cover which is fibered over S^1). Wise in fact proved something stronger, namely that if N is a hyperbolic 3-manifold with an embedded geometrically finite surface, then $\pi_1(N)$ is virtually compact special. See Section 4.3 for the definition of a (*compact*) *special group.* (These groups arise as particular types of subgroups of *right-angled Artin groups* and carry a very combinatorial flavor.) As well as virtual fibering, Wise's theorem also implies that the fundamental group of a hyperbolic 3-manifold N as before is *subgroup separable* (i.e., each of its finitely generated subgroups is closed in the profinite topology) and *large* (i.e., has a finite-index subgroup which surjects onto a non-cyclic free group), and has some further, quite unexpected corollaries: for instance, $\pi_1(N)$ is linear over $\mathbb{Z}$.

Building on the aforementioned work of Wise and the proof of the Surface Subgroup Conjecture by Kahn–Markovic [KM12a], Agol [Ag13] was able to give a proof of Thurston's Virtually Haken Conjecture: every closed hyperbolic 3-manifold has a finite cover which is Haken. Indeed, he proved that *the fundamental group of any closed hyperbolic* 3-*manifold is virtually compact special.* (See Theorem 4.2.2 below.) Flowchart 2 on page 74 contains the ingredients involved in the proof of this astounding fact, and the connections between various 'virtual' properties of 3-manifolds are summarized in Flowchart 3 on page 86.

Complementing Agol's work, Przytycki–Wise [PW12] showed that fundamental groups of compact, orientable, irreducible 3-manifolds with empty or toroidal boundary which are not graph manifolds are virtually special. In particular such manifolds are also virtually fibered and their fundamental groups are linear over $\mathbb{Z}$. These and many other consequences of being virtually compact special are summarized in Flowchart 4 on page 94, and we collect the consequences of these results for finitely generated infinite-index subgroups of 3-manifold groups in Flowchart 5 on page 117.

The combination of these results of Agol, Przytycki–Wise, and Wise, with a theorem of Liu [Liu13] also implies that *the fundamental group of a compact, orientable, aspherical* 3-*manifold N with empty or toroidal boundary is virtually special* *if and only if N is non-positively curved.* This very satisfying characterization of virtual speciality may be seen as a culmination of the work on 3-manifold groups in the last half-century.

We conclude the book with a discussion of some outstanding open problems in the theory of 3-manifold groups (in Chapter 7).

What this book is *not* about. As with any book, this one reflects the tastes and biases of the authors. We list some of the topics which we leave basically untouched:

(1) *Fundamental groups of non-compact* 3*-manifolds.* We note that Scott [Sco73b] showed that given a 3-manifold with finitely generated fundamental group, there exists a compact 3-manifold with the same fundamental group.
(2) *'Geometric' and 'large-scale' properties of* 3*-manifold groups.* For some results in this direction see [Ger94, KaL97, KaL98, BN08, Bn12, Sis11a].
(3) *Automaticity, formal languages, Dehn functions, and combings.* We refer to, for instance, [Brd93, BrGi96, Sho92, ECHLPT92, Pin03].
(4) *Recognition problems.* These problems are treated in [Hen79, JO84, JLR02, JT95, Sel95, Mng02, JR03, Mae03, KoM12, SSh14, GMW12]. We survey some of these results in a separate paper[AFW13].
(5) 3*-dimensional Poincaré duality groups.* We refer to, e.g., [Tho95, Davb00, Hil11] for further information. (But see also Section 7.1.1.)
(6) We rarely discuss specific properties of fundamental groups of *knot complements* (known as 'knot groups'), although they were some of the earliest and most popular examples of 3-manifold groups to be studied. We note that in general, irreducible 3-manifolds with non-empty boundary are not determined by their fundamental groups, but interestingly, prime knots in S^3 are in fact determined by their groups [CGLS85, CGLS87, GLu89, Whn87].
(7) *Fundamental groups of distinguished classes of* 3*-manifolds.* For example, arithmetic hyperbolic 3-manifolds exhibit a lot of special features [MaR03, Lac11, Red07]. But they also tend to be quite rare [Red91], [Bor81, Theorem 8.2], [Chi83], [BoP89, Section 7], [GrL12, Appendix], [Mai14].
(8) The *representation theory of* 3*-manifolds* is a substantial field in its own right, which fortunately is served well by Shalen's survey paper [Shn02].
(9) The *history* of the study of 3-manifolds and their fundamental groups. We refer to [Epp99, Gon99, McM11, Mil03, Mil04, Sti12, Vo96, Vo02, Vo13c, Vo14] for some articles, dealing mostly with the early history of 3-manifold topology and the Poincaré Conjecture.

This book is not intended as a leisurely introduction to 3-manifolds. Even though most terms will be defined, we will assume that the reader is already somewhat acquainted with 3-manifold topology. We refer to [Hem76, Hat, JS79, Ja80, Scs14] for background material. Another gap we perceive is the lack of a post-Geometrization textbook on 3-manifolds. We hope that someone else will step forward to fill this gaping hole.

What is a 3-manifold? Throughout this book we have tried to state the results in maximal generality. It is one of the curses of 3-manifold topology that at times authors make implicit assumptions on the 3-manifolds they are working with, for example that they are orientable, or compact, or closed, or that the boundary is toroidal. When we give a reference for a result, then to the best of our knowledge our assumptions match the ones given in the reference. For results concerning non-compact or non-orientable 3-manifolds, it is recommended to go back to the original reference.

A few sections in our book state in the beginning some assumptions on the 3-manifolds considered in that section, and that are in force throughout that section. The reader should be aware of those assumptions when studying a particular section, since we do not repeat them when stating definitions and theorems.

Conventions and notation. All topological spaces are assumed to be connected unless it says explicitly otherwise, but we do not put any other a priori restrictions on our spaces. All rings have an identity, and m, n range over the set $\mathbb{N} = \{0,1,2,3,\dots\}$ of natural numbers.

Notation	Definition
$\mathbb{Z}_n$	the cyclic group $\mathbb{Z}/n\mathbb{Z}$ with n elements ($n \geq 1$)
D^n	the closed n-ball $\{v \in \mathbb{R}^n : \lvert v\rvert \leq 1\}$
S^n	the n-dimensional sphere $\{v \in \mathbb{R}^{n+1} : \lvert v\rvert = 1\}$
I	the closed interval $[0,1]$
T^2	the 2-dimensional torus $S^1 \times S^1$
K^2	the Klein bottle
$K^2 \widetilde{\times} I$	the unique oriented total space of (a necessarily twisted) I-bundle over K^2

Various notions of 'n-dimensional manifold' agree for $n = 3$; see Section 1.1. Unfortunately, there are different conventions for what a *topological n-dimensional submanifold* of a topological 3-manifold N should be. For us it is a subset S of N such that for any point in $p \in N$ there exists a homeomorphism $\varphi\colon U \to V$ from an open neighborhood of p in N to an open subset of

$$\mathbb{R}^3_{\geq 0} := \left\{(x,y,z) \in \mathbb{R}^3 : x,y \in \mathbb{R}, z \in \mathbb{R}_{\geq 0}\right\}$$

such that $\varphi(U \cap S) \subseteq (\{0\}^{3-n} \times \mathbb{R}^n) \cap \mathbb{R}^3_{\geq 0}$. For example, according to our definition, Alexander's horned sphere [Ale24b], [Rol90, Section I] is not a topological 2-dimensional submanifold of S^3.

Let S be a submanifold of N. We denote by νS a tubular neighborhood of S in N. Given a surface Σ in N we refer to $N \setminus \nu\Sigma$ as N *cut along* Σ. Moreover, if N is orientable and Σ is an orientable surface in N, then at times we pick a product structure $\Sigma \times [-1,1]$ for a neighborhood of Σ and we identify $\nu\Sigma$ with $\Sigma \times (-1,1)$.

When we write 'a manifold with boundary' then we also include the case that the boundary is empty. If we want to ensure that the boundary is in fact non-empty, then we will write 'a manifold with non-empty boundary.' Finally, beginning with Convention 1.7, a hyperbolic 3-manifold is understood to be orientable and to have finite volume, unless we say explicitly otherwise.

A note about the bibliography. The bibliography to this book contains well over 1300 entries. We decided to refer to each paper or book by a combination of letters. We followed the usual approach: for a single-author paper we use the first two or three letters of the author's last name, for a multiple-author paper we use the first letters of each of the authors' last names. Unfortunately, with so many authors this approach breaks down at some point. For example, there are three single-author papers by three different Hamiltons. We tried to deal with each problem on an ad hoc basis. We are aware that this produced some unusual choices for abbreviations. Nonetheless, we believe that using letter-based names for papers (rather than referring to each entry by a number, say) will make it easier to use this book. For example, for many readers it will be clear that [Lac06] refers to a paper by Marc Lackenby and that [Wan68b] refers to a paper by Waldhausen.

Acknowledgments. Aschenbrenner acknowledges support from the NSF through grant DMS-0969642. Friedl's work on this book was supported by the SFB 1085 'Higher Invariants' at the Universität Regensburg, funded by the Deutsche Forschungsgemeinschaft (DFG). Wilton was partially supported by an EPSRC Career Acceleration Fellowship.

The authors thank Ian Agol, Jessica Banks, Igor Belegradek, Mladen Bestvina, Michel Boileau, Steve Boyer, Martin Bridson, Jack Button, Danny Calegari, Daryl Cooper, Jim Davis, Dave Futer, Cameron Gordon, Bernhard Hanke, Pierre de la Harpe, Matt Hedden, John Hempel, Jonathan Hillman, Neil Hoffman, Jim Howie, Takahiro Kitayama, Thomas Koberda, Viktor Kulikov, Marc Lackenby, Mayer A. Landau, Tao Li, Peter Linnell, Wolfgang Lück, Curtis McMullen, Matthias Nagel, Mark Powell, Piotr Przytycki, Alan Reid, Igor Rivin, Saul Schleimer, Kevin Schreve, Dan Silver, András Stipsicz, Stephan Tillmann, Stefano Vidussi, Liam Watson, Susan Williams, and Raphael Zentner for helpful comments, discussions, and suggestions. We are also grateful for the extensive feedback we got from many other people on earlier versions of this book. We especially thank the referee for a very long list of useful comments. Finally we thank Anton Geraschenko for bringing the authors together.