Introduction

A quasi-Banach space $A(\mathbb{R}^n)$, with $A(\mathbb{R}^n) \hookrightarrow S'(\mathbb{R}^n)$ (continuous embedding), is called *homogeneous* if for some $\sigma \in \mathbb{R}$,

$$\|f(\lambda\cdot)\,|A(\mathbb{R}^n)\| = \lambda^\sigma \|f\,|A(\mathbb{R}^n)\|, \qquad \lambda > 0, \quad f \in A(\mathbb{R}^n). \tag{0.1}$$

The defining quasi-norm $\|\cdot\,|A(\mathbb{R}^n)\|$ is said to be *admissible* if it makes sense to test any $f \in S'(\mathbb{R}^n)$ for whether it belongs to this space or not. A typical example is

$$\|f\,|\overset{*}{B}{}^s_{p,q}(\mathbb{R}^n)\| = \Big(\int_0^\infty t^{-sq/2}\big\|W_t f\,|L_p(\mathbb{R}^n)\big\|^q \frac{\mathrm{d}t}{t}\Big)^{1/q} \tag{0.2}$$

with $0 < p,q \le \infty$, $s < 0$, based on the Gauss–Weierstrass semi-group

$$W_t f(x) = \frac{1}{(4\pi t)^{n/2}}\Big(f, e^{-\frac{|x-\cdot|^2}{4t}}\Big), \qquad f \in S'(\mathbb{R}^n), \tag{0.3}$$

$x \in \mathbb{R}^n$, $t > 0$. If it only makes sense to test some $f \in S'(\mathbb{R}^n)$ for whether they belong to $A(\mathbb{R}^n)$, then we speak about *regional* quasi-norms. A typical example is the Lebesgue space $L_p(\mathbb{R}^n)$, $1 \le p \le \infty$, normed by

$$\|f\,|L_p(\mathbb{R}^n)\| = \Big(\int_{\mathbb{R}^n} |f(x)|^p\,\mathrm{d}x\Big)^{1/p}, \qquad f \in S'(\mathbb{R}^n)^{\mathrm{reg}} = S'(\mathbb{R}^n) \cap L_1^{\mathrm{loc}}(\mathbb{R}^n) \tag{0.4}$$

(based on the usual identification of locally integrable functions with the equivalence classes generated by them). A less obvious example is given by

$$\|f\,|\overset{*}{F}{}^s_{p,q}(\mathbb{R}^n)\|_m = \Big\|\Big(\int_0^\infty t^{(m-\frac{s}{2})q}\big|\partial_t^m W_t f(\cdot)\big|^q \frac{\mathrm{d}t}{t}\Big)^{1/q}|L_p(\mathbb{R}^n)\Big\| + \|f\,|L_r(\mathbb{R}^n)\|, \tag{0.5}$$

$f \in S'(\mathbb{R}^n)^{\mathrm{reg}}$, where

$$0 < p < \infty, \quad 0 < q \le \infty, \quad n\Big(\max\big(\frac{1}{p},1\big) - 1\Big) < s < \frac{n}{p}, \quad -\frac{n}{r} = s - \frac{n}{p} \tag{0.6}$$

and $s/2 < m \in \mathbb{N}$. Equivalent quasi-norms in a fixed space $A(\mathbb{R}^n)$ are called *domestic*. A typical example is the famous Littlewood–Paley assertion for $L_p(\mathbb{R}^n)$, $1 < p < \infty$,

$$\|f\,|L_p(\mathbb{R}^n)\| \sim \Big\|\Big(\sum_{\substack{j\in\mathbb{Z}, G\in G^*,\\ m\in\mathbb{Z}^n}} 2^{jn}\big|\big(f, h^j_{G,m}\big)\big|^2 \chi_{j,m}(\cdot)\Big)^{1/2}|L_p(\mathbb{R}^n)\Big\|, \tag{0.7}$$

where $\chi_{j,m}$ is the characteristic function of the cube $Q_{j,m} = 2^{-j}m + 2^{-j}(0,1)^n$, $j \in \mathbb{Z}$, $m \in \mathbb{Z}^n$, whereas $h^j_{G,m}$ stands for the related homogeneous Haar functions (orthonormal basis in $L_2(\mathbb{R}^n)$). Quite obviously, the norm on the right-hand side of (0.7) is neither admissible in $S'(\mathbb{R}^n)$ nor regional with $S'(\mathbb{R}^n)^{\mathrm{reg}}$ as the underlying region. On the other hand, domestic norms based on homogeneous Haar bases $\{h^j_{G,m}\}$ can be extended to further homogeneous spaces, for example,

$$\overset{*}{B}{}^s_{p,q}(\mathbb{R}^n), \qquad 0 < p,q < \infty, \quad \max\Big(n\big(\frac{1}{p} - 1\big), \frac{1}{p} - 1\Big) < s < \min\big(\frac{1}{p}, 1\big). \tag{0.8}$$

Next we speak about *community* quasi-norms. In other words, within a fixed community (or family) of spaces $\overset{*}{A}{}^s_{p,q}(\mathbb{R}^n)$ one has equivalent quasi-norms based on the same building blocks (for example, the homogeneous Haar functions $h^j_{G,m}$). Further examples of domestic and (with some care) community quasi-norms are given by

$$\|f\,|\overset{*}{F}{}^s_{p,q}(\mathbb{R}^n)\| \sim \Big\|\Big(\sum_{j=-\infty}^{\infty} 2^{jsq}\big|(\varphi^j\widehat{f})^\vee(\cdot)\big|^q\Big)^{1/q}|L_p(\mathbb{R}^n)\Big\|, \quad f \in \overset{*}{F}{}^s_{p,q}(\mathbb{R}^n), \tag{0.9}$$

where $\{\varphi^j\}_{j\in\mathbb{Z}}$ is the usual homogeneous dyadic resolution of unity in $\mathbb{R}^n \setminus \{0\}$.

We deal mainly with *tempered homogeneous quasi-Banach spaces* $A(\mathbb{R}^n)$ in the framework of the dual pairing $\big(S(\mathbb{R}^n), S'(\mathbb{R}^n)\big)$. These spaces are requested to satisfy

$$S(\mathbb{R}^n) \hookrightarrow A(\mathbb{R}^n) \hookrightarrow S'(\mathbb{R}^n), \tag{0.10}$$

and for some $\sigma \in \mathbb{R}$,

$$\|f(\lambda\cdot)\,|A(\mathbb{R}^n)\| = \lambda^\sigma \|f\,|A(\mathbb{R}^n)\|, \qquad \lambda > 0, \quad f \in A(\mathbb{R}^n) \tag{0.11}$$

(also admitting equivalences). It is the main aim of these notes to develop a related theory for the spaces

$$\overset{*}{A}{}^s_{p,q}(\mathbb{R}^n), \qquad 0 < p,q \le \infty, \quad n\Big(\frac{1}{p} - 1\Big) < s < \frac{n}{p}, \tag{0.12}$$

$A \in \{B,F\}$. This will be done in Chapter 3, step by step, first for spaces with $s < 0$, then for some spaces with $s > 0$ and finally for all spaces according to (0.12). The respective Theorems 3.3, 3.5 for $s < 0$, 3.11, 3.20 for $s > 0$, and, in particular, 3.24 may be considered our main results. In particular, all these spaces can be introduced in terms of *admissible* quasi-norms, for example,

$$\begin{aligned}\|f\,|\overset{*}{B}{}^s_{p,q}(\mathbb{R}^n)\|^*_m = \Big(\int_0^\infty t^{(m-\frac{s}{2})q}\big\|\partial^m_t W_t f\,|L_p(\mathbb{R}^n)\big\|^q \frac{\mathrm{d}t}{t}\Big)^{1/q} \\ + \sup_{t>0,x\in\mathbb{R}^n} t^{n/2r}|W_t f(x)|,\end{aligned} \tag{0.13}$$

with p,q,s as in (0.12), $s/2 < m \in \mathbb{N}_0$ and $-\frac{n}{r} = s - \frac{n}{p}$. Then one can switch to domestic and community quasi-norms with (0.9) as a prototype. Within a fixed community one can argue quite often as in the case of the inhomogeneous spaces $A^s_{p,q}(\mathbb{R}^n)$. Afterwards one returns to the tempered homogeneous spaces $\overset{*}{A}{}^s_{p,q}(\mathbb{R}^n)$ in terms of admissible quasi-norms. This gives the possibility of transferring assertions for the inhomogeneous spaces $A^s_{p,q}(\mathbb{R}^n)$ to spaces $\overset{*}{A}{}^s_{p,q}(\mathbb{R}^n)$ with (0.12). However, we do not explore this ground comprehensively, leaving room for future research. But we touch briefly upon some relevant topics which may be worth studying in greater detail. There is even feedback where inhomogeneous spaces $A^s_{p,q}(\mathbb{R}^n)$ benefit from homogeneous spaces $\overset{*}{A}{}^s_{p,q}(\mathbb{R}^n)$ with $0 < p < \infty$, $0 < q \le \infty$, $n\big(\max(1/p,1) - 1\big) < s < n/p$. Then one has

$$\|f\,|A^s_{p,q}(\mathbb{R}^n)\| \sim \|f\,|\overset{*}{A}{}^s_{p,q}(\mathbb{R}^n)\|, \qquad \operatorname{supp} f \subset \{x:\ |x| < 1\}, \tag{0.14}$$

which means that these inhomogeneous spaces $A^s_{p,q}(\mathbb{R}^n)$ and their homogeneous counterparts $\overset{*}{A}{}^s_{p,q}(\mathbb{R}^n)$ coincide locally. This can be used to improve already known local homogeneity assertions for $F^s_{p,q}(\mathbb{R}^n)$ essentially (Corollary 3.55).

The preceding Chapters 1 and 2 are of an auxiliary nature. The restrictions for s in (0.12) may be disturbing from the point of view of a comprehensive theory of function spaces. But (0.12) covers in particular the tempered homogeneous spaces

$$\overset{*}{A}{}^s_{p,q}(\mathbb{R}^n), \qquad 0 < p,q \le \infty, \quad \frac{n}{p} - 1 \le s < \frac{n}{p}, \tag{0.15}$$

$n \ge 2$, which play a significant role in the recent theory of the Navier–Stokes equations ($\frac{n}{p} - 1 = s$ critical spaces, $\frac{n}{p} - 1 < s < \frac{n}{p}$ supercritical spaces). This may serve as motivation to deal with tempered homogeneous spaces as indicated above. We very briefly insert some comments about distinguished spaces and the role of homogeneity for Navier–Stokes equations in Chapter 1. Nothing will be used later on and it can simply be skipped (in the belief that there are good reasons for dealing with the spaces in (0.12)). Usually one studies homogeneous spaces

$$\dot{A}^s_{p,q}(\mathbb{R}^n), \qquad 0 < p,q \le \infty, \quad s \in \mathbb{R}, \tag{0.16}$$

$A \in \{B,F\}$, in the framework of $\big(\dot{S}(\mathbb{R}^n), \dot{S}'(\mathbb{R}^n)\big)$. In Chapter 2 we give a very brief introduction to some relevant aspects. But again, nothing will play a role later on. In addition we collect in Chapters 1 and 2 some basic notation and clarify what is meant by heat kernels, Gauss–Weierstrass semi-groups, and how they are related to function spaces, supported mainly by relevant references. In other words, the reader may concentrate on Chapter 3, occasionally consulting, as necessary, the preceding chapters for notation and references.

We fix our use of $\sim$ (equivalence) as follows. Let I be an arbitrary index set. Then

$$a_i \sim b_i \qquad \text{for } i \in I \text{ (equivalence)}, \tag{0.17}$$

for two sets of positive numbers $\{a_i : \; i \in I\}$ and $\{b_i : \; i \in I\}$, means that there are two positive numbers c_1 and c_2 such that

$$c_1 \, a_i \leq b_i \leq c_2 a_i \qquad \text{for all } i \in I. \tag{0.18}$$