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# Four Faces of Number Theory



European Mathematical Society

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# Foreword

Number theory is a fascinating branch of mathematics. There are many examples of number theoretical problems that are easy to understand but difficult to solve, often enough only by use of advanced methods from other mathematical disciplines. This might be one of the reasons why number theory is considered to be such an attractive field with many connections to other areas of mathematical research.

In August 2012 the number theory group of the Department of Mathematics at Würzburg University, under the aegis of Jörn Steuding, organized an international summer school entitled Four Faces of Number Theory. In the frame of this event about fifty participants, mostly PhD students from all over the world, but also a few local participants, even undergraduate students, learned in four courses about different aspects of modern number theory. These courses highlight a strong interplay between number theory and other fields like combinatorics, functional analysis and graph theory. They will be of interest to (under)graduate students aiming to discover various aspects of number theory and their relationship with other areas of mathematics.

Kathrin Bringmann from Cologne gave an introduction to the theory of modular forms and, in particular, so-called Mock theta-functions, a topic which had been untouched for decades but has obtained much attention during the last five years.

Yann Bugeaud from Strasbourg lectured about expansions of algebraic numbers. Despite some recent progress, presented in his essay, questions like ‘does the digit 7 occur infinitely often in the decimal expansion of square root of two?’ remain very far from being answered. Here combinatorics on words and transcendence theory are combined to derive new information on the sequence of decimals of algebraic numbers and on their continued fraction expansions.

Titus Hilberdink from Reading lectured about a recent and rather unexpected approach to extreme values of the Riemann zeta-function by use of (multiplicative) Toeplitz matrices and functional analysis.

Finally, Jürgen Sander from Hildesheim gave an introduction to algebraic graph theory and the impact of number theoretical methods on fundamental questions about the spectra of graphs and the analogue of the Riemann hypothesis.

In this volume the reader can find the course notes from this summer school and in some places further additional material. Each of these courses is essentially self-contained (although a background in number theory and analysis might be useful). In all four courses recent research results are included indicating how easily one can approach frontiers of current research in number theory by elementary and basic analytic methods.

The picture on the front page shows the poster created by Nicola Oswald from Würzburg University for the summer school. The editing of the course notes had been

done by Rasa Steuding from Würzburg University. The authors are most grateful to both of them for their help. Last but not least, we sincerely thank Jörn Steuding from Würzburg University for initiating the summer school and the publication of these notes as well as for his editorial work.

The authors, March 2014