Introduction

In this book, we study Gromov's metric geometric theory [22, §3$\frac{1}{2}$] on the space of metric measure spaces, which is based on the idea of the concentration of measure phenomenon due to Lévy and Milman. Although most of the details are omitted in the original article [22, §3$\frac{1}{2}$], we present complete and detailed proofs for some main parts, in which we prove several claims that are not mentioned in any literature. We also discuss concentration with a lower bound of curvature, which is originally studied in [20].

The concentration of measure phenomenon was first discovered by P. Lévy [30] and further put forward by V. Milman [37, 38]. It has many applications in various areas of mathematics, such as geometry, analysis, probability theory, and discrete mathematics (see [28, 41] and the references therein). The phenomenon states that any 1-Lipschitz continuous function is close to a constant on a domain with almost full measure, which is often observed for high-dimensional spaces. As a most fundamental example, we observe it in the high-dimensional unit spheres $S^n(1) \subset \mathbb{R}^{n+1}$, i.e., any 1-Lipschitz continuous function on $S^n(1)$ is close to a constant on a domain with almost full measure if n is large enough. In general, it is described for a sequence of metric measure spaces. In this book, we assume a metric measure space, an *mm-space* for short, to be a triple (X, d_X, μ_X), where (X, d_X) is a complete separable metric space and μ_X is a Borel probability[1] measure on X. A sequence of mm-spaces X_n, $n = 1, 2, \dots$ is called a *Lévy family* if

$$\lim_{n\to\infty} \inf_{c\in\mathbb{R}} \mu_{X_n}(|f_n - c| > \varepsilon) = 0$$

for any sequence of 1-Lipschitz continuous functions $f_n : X_n \to \mathbb{R}$, $n = 1, 2, \dots$, and for any $\varepsilon > 0$. The sequence of the unit spheres $S^n(1)$, $n = 1, 2, \dots$ is a Lévy family, where the measure on $S^n(1)$ is taken to be the Riemannian volume measure normalized as the total measure to be 1.

One of central themes in this book is the study of the observable distance. The *observable distance $d_{\mathrm{conc}}(X,Y)$ between two mm-spaces X and Y* is, roughly speaking, the difference between 1-Lipschitz functions on X and those on Y (see Definition 5.3 for the precise definition). A sequence of mm-spaces is a Lévy family if and only if it d_{conc}-converges to a one-point mm-space, where we note that any 1-Lipschitz function on a one-point mm-space is constant. Thus, d_{conc}-convergence of mm-spaces can be considered as a generalization of the Lévy property. We call d_{conc}-convergence of mm-spaces *concentration of mm-spaces*. A typical example of

[1] In [22, §3$\frac{1}{2}$], the measures of mm-spaces are not necessarily probability measures. However, all our proofs easily extend to the case of nonprobability mm-spaces with finite total measure.

a concentration $X_n \to Y$ is obtained by a fibration

$$F_n \to X_n \to Y$$

such that $\{F_n\}_{n=1}^{\infty}$ is a Lévy family, which example makes us notice that concentration of mm-spaces is an analogue of collapsing of Riemannian manifolds. Concentration is strictly weaker than measured-Gromov–Hausdorff convergence and is more suitable for the study of a sequence of manifolds whose dimensions are unbounded.

Although d_{conc} is not easy to investigate, we have a more elementary distance, called the *box distance*, between mm-spaces. The box distance function is closely related to the well-known measured-Gromov–Hausdorff convergence of mm-spaces (see Remark 4.34). Concentration of mm-spaces is equivalent to convergence of associated pyramids using the box distance function, where a *pyramid* is a family of mm-spaces that forms a directed set with respect to some natural order relation between mm-spaces, called the *Lipschitz order* (see Definitions 2.10 and 6.3). We have a metric ρ on the set of pyramids, say Π, induced from the box distance function (see Definition 6.21 and [54]). Each mm-space X is associated with the pyramid, say $\mathcal{P}_X$, consisting of all descendants of the mm-space (i.e., smaller mm-spaces with respect to the Lipschitz order). Denote the set of mm-spaces by $\mathcal{X}$. We prove that the map

$$\iota : \mathcal{X} \ni X \longmapsto \mathcal{P}_X \in \Pi$$

is a 1-Lipschitz continuous topological embedding map with respect to d_{conc} and ρ. This means that concentration of mm-spaces is expressed only by the box distance function, since ρ is induced from the box distance function. We also prove that Π is a compactification of $\mathcal{X}$ with d_{conc}. Such a concrete compactification is far more valuable than just an abstract one.

It is also interesting to study a sequence of mm-spaces that d_{conc}-diverges but have proper asymptotic behavior. A sequence of mm-spaces X_n, $n = 1, 2, \dots$ is said to be *asymptotic* if the associated pyramid $\mathcal{P}_{X_n}$ converges in Π. We say that a sequence of mm-spaces *asymptotically concentrates* if it is a d_{conc}-Cauchy sequence. Any asymptotically concentrating sequence of mm-spaces is asymptotic. For example, the sequence of the Riemannian product spaces

$$S^1(1) \times S^2(1) \times \cdots \times S^n(1), \quad n = 1, 2, \dots$$

d_{conc}-diverges and asymptotically concentrates (see Example 7.36). The sequence of the spheres $S^n(\sqrt{n})$ of radius $\sqrt{n}$, $n = 1, 2, \dots$ does not even asymptotically concentrate but is asymptotic (see Theorem 7.40, Corollary 7.42, and [54]). One of main theorems in this book states that the map $\iota : \mathcal{X} \to \Pi$ extends to the d_{conc}-completion of $\mathcal{X}$, so that the space Π of pyramids is also a compactification of the d_{conc}-completion of the space $\mathcal{X}$ of mm-spaces (see Theorem 7.27). Let γ^n denote the standard Gaussian measure on $\mathbb{R}^n$. Then, the associated pyramids $\mathcal{P}_{S^n(\sqrt{n})}$ and $\mathcal{P}_{(\mathbb{R}^n,\gamma^n)}$ both converge to a common pyramid as $n \to \infty$ (see Theorem 7.40 and [54]), which can be thought as a generalization of the Maxwell–Boltzmann distribution law (or the Poincaré limit theorem).

The spectral property is deeply related with the asymptotic behavior of a sequence of mm-spaces. The *spectral compactness* of a family of mm-spaces is defined by the Gromov–Hausdorff compactness of the energy sublevel sets of L_2 functions (see Definition 7.44) and is closely related with the notion of asymptotic compactness of Dirichlet energy forms (see [26, 27]). For a family of compact Riemannian manifolds, it is equivalent to the discreteness of the limit set of the spectrums of the Laplacians of the manifolds (see Proposition 7.50). We prove that any spectrally compact and asymptotic sequence of mm-spaces asymptotically concentrates if the observable diameter is bounded from above (see Theorem 7.52). We say that a sequence of mm-spaces *spectrally concentrates* if it is spectrally compact and asymptotically concentrates. For example, let

$$X_n := F_1 \times F_2 \times \cdots \times F_n$$

be the Riemannian product of compact Riemannian manifolds F_n, $n = 1, 2, \ldots$. If $\lambda_1(F_n)$ diverges to infinity as $n \to \infty$, then $\{X_n\}$ spectrally concentrates (see Corollary 7.54).

There is a notion of dissipation for a sequence of mm-spaces, which is opposite to concentration and means that the mm-spaces disperse into many small pieces far apart each other. A sequence of mm-spaces *δ-dissipates*, $\delta > 0$, if and only if any limit of the associated pyramids contains all mm-spaces with diameter $\leq \delta$. The sequence *infinitely dissipates* if and only if the associated pyramid converges to the space of mm-spaces (see Proposition 8.5). On the one hand, for a disconnected mm-space F, the sequence of the nth power product spaces F^n, $n = 1, 2, \ldots$, with l_∞ metric, δ-dissipates for some $\delta > 0$ (see Proposition 8.6). On the other hand, the nondissipation theorem (Theorem 8.8) states that the sequence $\{F^n\}$ does not δ-dissipate for any $\delta > 0$ if F is connected and locally connected. The proof of the nondissipation theorem relies on the study of the obstruction condition for dissipation. For example, a sequence of compact Riemannian manifolds X_n does not dissipate if $\lambda_1(X_n)$ is bounded away from 0 (see Corollary 8.14), which is one of the essential statements in the proof of the nondissipation theorem.

It is interesting to study the relation between curvature and concentration. The concept of Ricci curvature bounded below is generalized to the *curvature-dimension condition* for an mm-space by Lott–Villani–Sturm [34, 56, 57] via the optimal mass-transport theory. We prove that if a sequence of mm-spaces satisfying the curvature-dimension condition concentrates to an mm-space, then the limit also satisfies the curvature-dimension condition (see [20]). This stability result of the curvature-dimension condition has an application to the eigenvalues of the Laplacian on Riemannian manifolds. In fact, under the nonnegativity of Ricci curvature, the kth eigenvalue of the Laplacian of a closed Riemannian manifold is dominated by a constant multiple of the first eigenvalue, where the constant depends only on k and is independent of the dimension of the manifold. This dimension-free estimate cannot be obtained by the ordinary technique. Combining this estimate with Gromov–V. Milman's and E. Milman's results [21, 35, 36], we have the following equivalence:

$$\{X_n\} \text{ is a Lévy family} \iff \lambda_1(X_n) \to +\infty \iff \lambda_k(X_n) \to +\infty \text{ for some } k$$

for a sequence of closed Riemannian manifolds X_n, $n = 1, 2, \dots$, with nonnegative Ricci curvature.

The organization of this book is as follows.

In Chapter 1, we define weak and vague convergence of measures, the Prohorov distance, transport plan, the Ky Fan metric, convergence in measure of maps, and present those basic facts.

Chapter 2 is devoted to a minimal introduction to the Lévy–Milman concentration phenomenon. We define the observable diameter, the separation distance, and the Lipschitz order. We prove the normal law á la Lévy for $S^n(\sqrt{n})$ stating that any limit of the push-forward of the normalized volume measure on $S^n(\sqrt{n})$ by a 1-Lipschitz continuous function on $S^n(\sqrt{n})$ as $n \to \infty$ is the push-forward of the one-dimensional standard Gaussian measure by some 1-Lipschitz continuous function on $\mathbb{R}$. From this we derive the asymptotic estimate of the observable diameter of $S^n(1)$ and $\mathbb{C}P^n$. We also prove the relation between the kth eigenvalue of the Laplacian and the separation distance for a compact Riemannian manifold, which yields some examples of Lévy families.

Chapter 3 presents some basic facts on metric geometry, such as, the Hausdorff distance and the Gromov–Hausdorff distance. We also prove the equivalence between Gromov–Hausdorff convergence and convergence of the distance matrices of compact metric spaces.

Chapter 4 deals with the box distance between mm-spaces, which is one of fundamental tools in this book. We prove that the Lipschitz order is stable under box convergence, and that any mm-space can be approximated by a monotone nondecreasing sequence of finite-dimensional mm-spaces. We investigate the convergence of finite product spaces to the infinite product.

Chapter 5 discusses the observable distance and the measurements, where the *N-measurement of an mm-space* is defined to be the set of push-forwards of the measure of the mm-space by 1-Lipschitz maps to $\mathbb{R}^N$ with l_∞ norm. The measurements have the complete information of the mm-space and can be treated more easily than the mm-space itself. We prove that the concentration of mm-spaces is equivalent to the convergence of the corresponding measurements, which is one of the essential points for the investigation of convergence of pyramids.

Chapter 6 is devoted to the space of pyramids. We define a metric on the space of pyramids and prove its compactness. The metric is first introduced in this book and [54] simultaneously.

In Chapter 7, we finally complete the proof of the theorem that the d_{conc}-completion of the space of mm-spaces is embedded into the space of pyramids, which is one of main theorems in this book. We study the asymptotic concentration of finite product spaces and the asymptotic property of the pyramids $\mathcal{P}_{S^n(\sqrt{n})}$ and $\mathcal{P}_{(\mathbb{R}^n,\gamma^n)}$ (see [54]). We also study spectral compactness and prove that any spectrally compact and asymptotic sequence of mm-spaces asymptotically concentrates if the observable diameter is bounded from above.

Chapter 8 discusses dissipation. After the basics of dissipation, we present some examples of dissipation. One of the interesting examples is the sequence of the spheres $S^n(r_n)$ of radius r_n. It infinitely dissipates if and only if $r_n/\sqrt{n} \to +\infty$ as $n \to \infty$ (see [54]). We also study some conditions for nondissipation and prove the nondissipation property of product spaces.

The final Chapter 9 is an exposition of [20]. We prove the stability theorem of the curvature-dimension condition for concentration, and apply it to the study of the eigenvalues of the Laplacian on closed Riemannian manifolds. We also prove the stability of a lower bound of Alexandrov curvature.

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