General introduction: The uniformization theorem

The study of plane curves is one of the chief preoccupations of mathematicians. The ancient Greeks investigated in detail straight lines, circles, as well as the conic sections and certain more exotic curves such as Archimedean spirals. A systematic study of general curves became possible only with the introduction of Cartesian coordinates by Fermat and Descartes during the first half of the 17th century [Fer1636, Desc1637], marking the beginning of algebraic geometry. For the prehistory of algebraic geometry the reader may consult [BrKn1981, Cha1837, Die1974, Weil1981].

Two ways of representing a curve

A plane curve can be modelled mathematically in two — in some sense dual — ways:

— by an *implicit equation* $F(x, y) = 0$, where $F : \mathbb{R}^2 \to \mathbb{R}$ is a real function of two real variables;

— as a parametrized curve, that is, as the image of a map $\gamma : \mathbb{R} \to \mathbb{R}^2$.

We shall see that the *uniformization theorem* allows one to pass from the first of these representations to the second. If F is a polynomial, the curve is said to be *algebraic* (formerly such curves were called "geometric"), otherwise *transcendental* (formerly "mechanical"). A significant part of this book is concerned with algebraic curves but, as we shall see, the uniformization theorem in its final version provides an *entrée* into the investigation of (almost) all curves.

Among transcendental curves we find various kinds of spirals and catenaries, the brachistochrone and other tautochrones, which played a fundamental role in the development of mathematics in the 17th century.

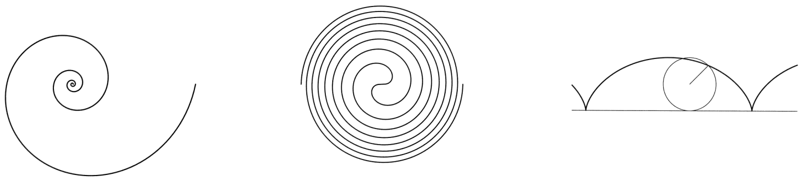

Figure 1: Some transcendental curves

Formerly the study of algebraic curves consisted in a case-by-case examination of a large number of examples of curves with complicated names (lemniscates, cardioids, folia, strophoids, cissoids, etc.) which used to be found among the exercises in undergraduate textbooks and which continue to give pleasure to amateur mathematicians[1].

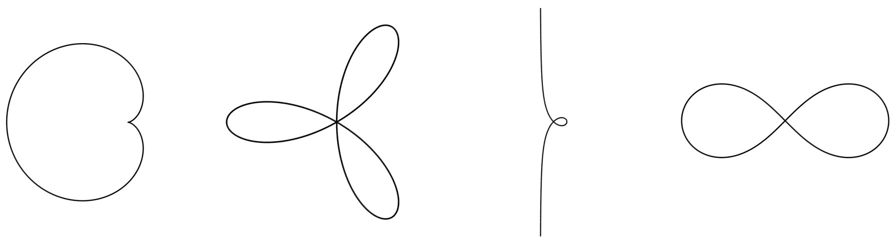

Figure 2: Some algebraic curves

The first invariant that suggests itself for an algebraic curve is the *degree* of the polynomial F, readily seen to be independent of the system of plane (Cartesian) coordinates to which the curve is referred. It is clear that straight lines are precisely the curves of degree 1, and it is not difficult to show that the venerable conic sections of the ancient Greeks are just the curves of degree 2. In a celebrated work [New1704] Newton took up the task of producing a "qualitative"

[1]See e.g. http://www.mathcurve.com/ or http://www.2dcurves.com/.

classification of the curves of degree 3, concluding that there are 72 different types[2]. Evidently it would be difficult if not impossible to continue in this fashion since the number of possible "types" increases very rapidly with the degree and the situation soon becomes impenetrable.

Three innovations

Three major mathematical innovations led to significant clarification of the situation. First came the understanding that the projection from a point in 3-dimensional space of one plane onto another, both situated in that space and neither containing the point of projection, transforms an algebraic curve in one plane into an algebraic curve on the other, moreover of the same degree, said to be *projectively equivalent* to the first. For example, every non-degenerate conic section is the image of a circle under a suitable such projection; hence from the projective point of view the distinction between ellipses, parabolas, and hyperbolas disappears: there now exists just a single equivalence class of non-degenerate conic sections. Similarly, after having defined a *diverging parabola* to be a curve given by an equation of the form $y^2 = ax^3 + bx^2 + cx + d$, Newton states that:

> Just as the circle lit by a point-source of light yields by its shadow all curves of the second degree, so also do the shadows of diverging parabolas give all curves of the third degree.

Here we are at the beginning of *projective geometry*, initiated by Girard Desargues [Desa1639]. Rather than considering a curve $F(x, y) = 0$ in the plane coordinatized by pairs (x, y), one considers it in the *projective plane*, coordinatized by means of homogeneous coordinates $[X : Y : Z]$, where now the curve is given by a homogeneous polynomial $\bar{F}(X, Y, Z) = 0$. Each set of points of the projective plane with $Z \neq 0$ and fixed values for the ratios X/Z and Y/Z corresponds to the point of the affine plane with coordinates $x = X/Z$ and $y = Y/Z$, so that the projective plane is in effect the ordinary affine plane with a *line at infinity* adjoined, each of whose points corresponds to a line through $[0 : 0 : 0]$ in XYZ-space with $Z \equiv 0$ and with $[0 : 0 : 0]$ omitted. It follows that the two branches of a hyperbola in the affine plane join up at two points on the line at infinity, namely the points of that line determined by its two asymptotes, while a parabola is actually tangential to the line at infinity. Thus utilization of projective

[2]Note however that he "missed" 6, his definition of "type" in this context was criticized by Euler, and Plücker, using a different criterion, distinguished 219 types.

geometry simplifies the geometrical picture in a significant way, reducing apparently distinct cases down to an examination of the relative positions of a *projective algebraic curve* and a (projective) line. The adjunction of the line at infinity has other advantages: for instance, every pair of distinct projective lines intersects in a point, which will be at infinity precisely when the corresponding affine lines are parallel.

The second major innovation, dating back to the turn of the 19th century, was *the systematic use of complex numbers in geometry*, leading to the need to consider the complex points of the algebraic curve under investigation, that is, the complex solutions of the equation $F(x,y) = 0$, where furthermore the polynomial $F(x,y)$ is, naturally, now allowed to have complex coefficients. The fact that the field of complex numbers is algebraically closed — awareness of which grew gradually until it was finally established in the 19th century — entails a substantial consolidation of geometrical statements. Clearly projective geometry and complex geometry represent natural enlargements of the original context of the study of plane curves, and indeed until relatively recently were together taken as providing the most natural framework for algebraic geometry.

To take a simple example, the straight line $y = 0$ now meets every "parabola" $y = ax^2 + bx + c$ (with not all of a, b, c zero) in two points. The sign of the discriminant is no longer of any importance — indeed it no longer really has a sign! — but if it vanishes then the two roots merge into one. If $a = 0, b \neq 0$ one of the points is at infinity and if $a = b = 0, c \neq 0$ there is a "double root at infinity"[3]. (The case $a = b = c = 0$ is exceptional.) Thus does one see the unifying power of complex projective algebraic geometry. An even more compelling example concerns the *cyclic points*, which are both imaginary and on the line at infinity. These are just the points $[1 : i : 0]$ and $[1 : -i : 0]$. It is not difficult to see that a conic section in the Euclidean plane is a circle if and only if, considered as a conic in the complex projective plane, it passes through the cyclic points. From this fact many of the properties of circles can be inferred, since they in fact reduce to the question of the position of a conic relative to two points.

Even if we study complex algebraic curves only up to projective coordinate changes, a systematic classification still eludes us except in small degrees. To see this it suffices to note, as Cramer did in 1750, that the vector space of algebraic curves of degree d has dimension $d(d+3)/2$, while the group of projective transformations has "only" dimension 8 [Cra1750].

The third major innovation, due to Poncelet, Plücker, and Steiner [Ponc1822, Plü1831, Ste1832] among others, rested on the discovery that one can investigate curves by means of *non-linear coordinate changes*. Among such coordinate trans-

[3] To see this rewrite the equation in terms of homogeneous coordinates.

formations *inversion* plays an important role. (Up until the 1960s many chapters of high school geometry textbooks used to be devoted to inversion.) One very simple algebraic version is the transformation (to which the name De Jonquières is attached) sending each point with affine coordinates (x, y) to the point $(1/x, 1/y)$, or, in its "homogenized" variant, mapping the point with projective coordinates $[X : Y : Z]$ to $[YZ : XZ : XY]$. This prompts two remarks. First, this "transformation" σ is not everywhere defined. When two of the homogeneous coordinates are zero — the three such points forming the vertices of a triangle with one side on the line at infinity — the image is not defined (since $[0 : 0 : 0]$ does not correspond to a point of the projective plane). Secondly, the transformation is not injective: the line at infinity $Z = 0$ is sent entirely to the point $x = y = 0$. However, apart from such "details", which hardly bothered our predecessors, this transformation may be regarded as a legitimate change of variables. It is "almost" bijective in view of the fact that it is involutory: if σ is defined both at a point p and its image $\sigma(p)$, then $(\sigma \circ \sigma)(p) = p$. On transforming an algebraic curve via σ we obtain another algebraic curve *but of different degree*. For example, the image of the straight line $x + y = 1$ is the conic $1/x + 1/y = 1$, or, to be precise, a conic with certain points removed.

The non-linear transformations we have in mind form a group (named after Cremona) which is much larger that the projective group, so that one can hope for a precise and at the same time tractable classification of algebraic curves up to such a non-linear transformation. Here we have the beginnings of *birational geometry*, one of Riemann's great ideas. We say that two projective algebraic curves $\bar{F}(X,Y,Z) = 0$ and $\bar{G}(X,Y,Z) = 0$ are *birationally equivalent* if there is a (possibly non-linear) transformation of the form $(X,Y,Z) \mapsto (p(X,Y,Z), q(X,Y,Z), r(X,Y,Z))$ where the coordinates p, q, r are homogeneous polynomials of the same degree, which maps the first curve "bijectively" to the second. Here the quotation marks are meant to indicate that, as in the above example, the transformation may not be defined everywhere. One insists only that each of the two curves has a finite set of points such that the transformation sends the complement of the finite subset of the first curve bijectively to the complement of the subset of the second.

A signal virtue of birational transformations is that they allow us to avoid the problem of *singular points*. Early geometers were soon confronted with the need to study double points, cusp points, etc. In the real domain the theory of such points is relatively simple, at least in its topological aspects. Every point of a real algebraic curve has a neighbourhood in which the curve is made up of an *even* number of arcs. Such a curve cannot have an end-point, for instance.

On the other hand for complex algebraic curves, local analysis of their singular points has established that they can have an extraordinarily intricate structure: investigations begun by Newton and continued by Puiseux [New1671, Pui1850, Pui1851] show that their topological structure is connected with the theory of knots, which theory does not, however, come within the compass of the present book. For us it suffices to know that every algebraic curve is birationally equivalent to a curve possessing only especially simple singular points, namely ordinary double points (Noether, Bertini [Noe1873, Bert1882]) — in other words, points in a neighbourhood of which the curve consists of two smooth arcs with distinct tangents.

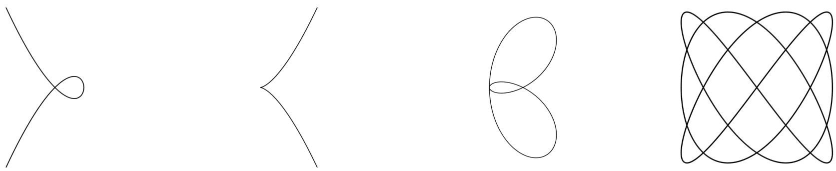

Figure 3: Some types of singular points

To summarize, geometers have progressively reduced the study of algebraic plane curves to that of algebraic curves which, to within a birational transformation, have only ordinary double points.

Rational curves

The introduction of complex numbers had consequences far beyond projective geometry: the beginning of the 19th century also witnessed the advent of the geometric theory of holomorphic functions, which are at one and the same time functions of a single complex variable and of two real variables. Gauss not only knew that it is useful to coordinatize the plane with the complex numbers, but understood equally well that any surface in space can be coordinatized by the complex numbers *conformally* (see Chapter I). Thus a surface is locally determined by a single number. The step had been taken: *a real surface can be considered a complex curve.* Some thirty years later Riemann understood that there is, reciprocally, some advantage in regarding a complex curve as a real surface (see Chapter II).

We are now in a position to address the question of *parametrized curves*. A curve is called *rational* if it is birationally equivalent to a straight line. (Formerly such curves were called *unicursal*, meaning that they could be "traced out with a single stroke of the pen".) More concretely, a curve $F(x, y)$ is rational if it can be parametrized by means of rational functions

$$x = \frac{p(t)}{r(t)}, \quad y = \frac{q(t)}{r(t)},$$

where p, q and r are polynomials in a single (complex) variable t, and the parametrization is a bijection outside a finite subset of values of t. Here are some simple examples.

Non-degenerate conics are rational. To see this it suffices to take a point m on the conic C and a projective line D not passing through m (see Figure 4). Then for each point t on D, the line determined by m and t meets the conic in two points, of which one is of course m. Denoting the other point by $\gamma(t)$, one readily checks that the map $\gamma : D \to C$ is a birational equivalence.

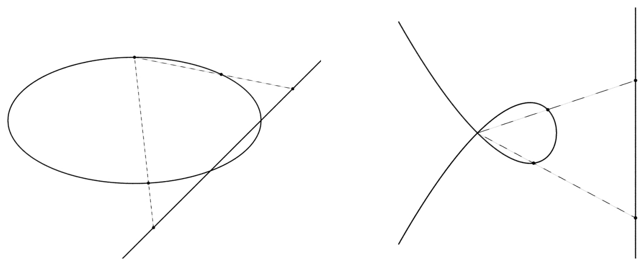

Figure 4: Parametrization of a conic and a singular cubic

A cubic curve with a double point is also a rational curve. For this it suffices to choose a straight line not passing through the singular point, and consider for each point p of that line the line through p and the singular point (see Figure 4). Each such line meets the conic in three points, two of which coincide with the double point of the cubic. The third point of intersection then determines a birational equivalence between the initially chosen line and the given cubic. For example, the origin is a double point of the curve $y^2 = x^2(1 - x)$. We choose $x = 2$ as our parametrizing line. The line passing through the origin and the point $(2, t)$ has equation $y = tx/2$, so intersects the given cubic where $t^2x^2/4 = x^2(1 - x)$, which has the expected double root $x = 0$ and the third solution $x = 1 - t^2/4$, yielding the desired rational parametrization $y = t(1 - t^2/4)/2$ of the curve.

Although rational curves are of considerable interest, they represent just a small proportion of all algebraic curves. We do not know exactly when mathematicians became fully aware of this, that is, of the fact that most algebraic curves are not rational. There are several elementary means of convincing oneself of it, and later on we shall give a topological argument rendering it "obvious". Or one can argue as follows. Note first that a curve given in the form $x = p(t)/r(t), y = q(t)/r(t)$ is of degree d where d is the largest of the degrees of the polynomials p, q, r: one can see this by counting the number of points of intersection with a generic straight line, which points will be given as the solutions of an equation of degree d. The vector space of triples of polynomials of degree d has dimension $3(d+1)$. However multiples of p, q, r by any non-zero scalar yield the same curve, and replacement of t by a suitable rational function of t (depending on at least three parameters) will also leave the curve unchanged. Thus the space of rational curves of degree d depends on at most $3d-1$ parameters. As noted earlier, a count of the number of coefficients of a polynomial of degree d in two variables yields $d(d+3)/2$ for the number of parameters. Since $d(d+3)/2 > 3d-1$ for $d \geq 3$, we conclude that *in general algebraic curves of degree at least three are not rational curves.*

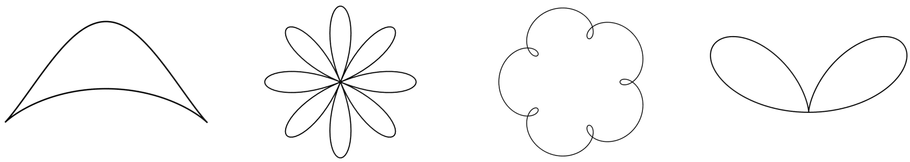

Figure 5: Some rational curves

Elliptic curves

It is completely natural that effort should first be concentrated on the cubics. As we have seen, Newton himself produced an initial classification which was neither projective nor complex, even though he found hints of certain features of projectivity and the complex numbers. His aim was to understand in some fashion the possible topological dispositions of cubic curves in the plane: the positions of asymptotes, singular points, etc. We saw above that singular cubics are rational. However non-singular cubics are never rational; we recommend that the reader attempt to prove this by elementary means.

We limit ourselves here to a brief overview of the principal results. First, every smooth cubic curve is projectively equivalent to a curve with equation in the following normal form (named for Weierstrass although it should properly be attributed to Newton):

$$y^2 = x^3 + ax + b,$$

where a, b are complex numbers. If $4a^3 + 27b^2 \neq 0$, this cubic is smooth. From the inception of the theory mathematicians struggled to evaluate integrals of the form

$$f(x) = \int \frac{dx}{y} = \int \frac{dx}{\sqrt{x^3 + ax + b}}.$$

They called such integrals "elliptic" since evaluation of the length of an arc of an ellipse leads to such a formula. Difficulties arise when one tries to make sense of such integrals with x and y allowed to be complex. The first problem is that the value of the integral depends on which square root one chooses for the denominator of the integrand. The second, linked to the first, consists in the dependence of the integral on the path of integration. Faced with these difficulties, one is forced to the conclusion that one must resign oneself to regarding f as a "many-valued[4] function", that is, that each point x may have several images, all denoted by $f(x)$ however — a situation somewhat distasteful to present-day mathematicians, brought up as they are on the modern set-theoretic definition of a function.

Gauss, Abel, and Jacobi conceived the ingenious idea (to be expounded in Chapter I) that it is not so much the function f that is of interest but its *inverse*. They were perhaps led to this by the analogy with the circle

$$x^2 + y^2 = 1$$

(which is certainly a rational curve) and the integral

$$\int \frac{dx}{y} = \int \frac{dx}{\sqrt{1 - x^2}} = \arcsin x.$$

The "function" arcsin as so defined is multivalued, but it is the inverse function of sin, a function in the strict sense of the word — each point x has a uniquely defined image $\sin x$. The many-valuedness of arcsin arises from the *periodicity* of the sine function, and in like manner the inverse $\wp$ of f is a "genuine" meromorphic

[4]The French term is "multiforme".

function (to emphasize that the modern interpretation of the word "function" is the one intended, the adjective *single-valued*[5] is sometimes used), and the fact that f is many-valued is explained by the periodicity of the single-valued function $\wp$.

It is important to stress this periodicity. While the sine function is periodic of period 2π, the periodicity of the meromorphic function $\wp$ is much richer: it has *two* linearly independent periods. In more precise terms, there is a subgroup Λ of $\mathbb{C}$ of rank 2 (depending on a and b) such that

$$\forall \omega \in \Lambda, \quad \wp(z) = \wp(z + \omega).$$

(In fact the elements of Λ are just the integrals of dx/y around closed curves in the x-plane.) It follows that we may regard $\wp$ as defined on the quotient of $\mathbb{C}$ by the lattice Λ. Topologically, the quotient space $\mathbb{C}/\Lambda$ is a 2-dimensional torus. Locally, each point of the torus is associated with a complex number in such a way that it inherits the structure of a holomorphic manifold of complex dimension 1, an example of a *Riemann surface* (see Chapter II).

Since $\wp$ is periodic, its derivative $\wp' = d\wp/dz$ is also, and we then obtain a map $(\wp, \wp')$ from the Riemann surface $\mathbb{C}/\Lambda$ with the poles of $\wp$ and $\wp'$ removed, to $\mathbb{C}^2$. It is not difficult to prove[6] that this map extends from $\mathbb{C}/\Lambda$ to the original cubic curve in the complex projective plane (with the three excluded points restored, now sent to three points at infinity). In this way one obtains an identification of the projective cubic curve and the torus $\mathbb{C}/\Lambda$.

A few remarks are apropos. First, it now becomes *topologically* clear why such cubics are not rational: a complex projective line is homeomorphic to a sphere (the Riemann sphere) and the removal of finitely many points will not make it homeomorphic to a torus.

We also see from the above that every smooth cubic, considered as a real surface (in the complex projective plane), is homeomorphic to the torus. On the other hand, considered as Riemann surfaces, these tori are not holomorphically equivalent to one another: given two distinct lattices Λ_1, Λ_2 in $\mathbb{C}$, there is in general no holomorphic bijection between $\mathbb{C}/\Lambda_1$ and $\mathbb{C}/\Lambda_2$. (There is such a bijection if and only if $\Lambda_2 = k\Lambda_1$ for some non-zero k.) Hence in contrast with rational curves, which are all parametrized by the complex projective line (the Riemann sphere), smooth cubics are not all parametrized by the same complex torus $\mathbb{C}/\Lambda$: each of them is parametrized by a complex parameter (determined by a lattice in $\mathbb{C}$ defined to within a homothety[7]), called a *modulus*.

[5]The French term is "uniforme". "Uniformization" is thus the process of representing many-valued functions by single-valued ones. *Translator*

[6]Using the fact that $(\wp')^2 = \wp^3 + a\wp + b$. *Trans*

[7]That is, defined to within a similarity.

The reader may now take the measure of the progress achieved since Newton's attempt at classification: the birational equivalence classes of smooth cubic curves also depend on a single complex parameter.

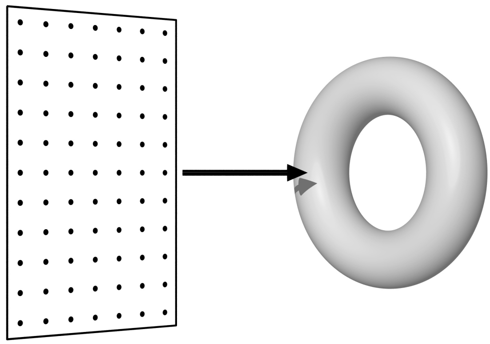

Figure 6: Uniformization of an elliptic curve

Even though the domain $\mathbb{C}/\Lambda$ of the parametrization of a smooth cubic depends on the cubic, it should be noted that the universal cover of $\mathbb{C}/\Lambda$, the complex line $\mathbb{C}$ considered as a Riemann surface, is in fact independent of the cubic. We shall now elaborate on this point — at the expense of perpetrating an anachronism since the concept of the universal cover evolved only gradually in the course of the 19th century, and reached final form only in the 20th. (In this connection one should also mention that some of the motivation for the development of topology came from the study of curves.) A topological space X is said to be *simply connected* if every loop $c : \mathbb{R}/\mathbb{Z} \to X$ can be contracted to a point, that is, if there is a continuous family of loops $c_t, t \in [0,1]$, with $c_0 = c$ and c_1 a constant loop. It can be shown that provided X is a "reasonably well-behaved space" — which is certainly the case for manifolds — there exists a simply-connected space $\tilde{X}$ and a projection map $\pi : \tilde{X} \to X$ whose fibres are the orbits of a discrete group Γ acting on $\tilde{X}$ (fixed point) freely and properly[8]. The space $\tilde{X}$ is then called *the universal covering space* of X, and Γ *the fundamental group* of X. In the case where X is the torus $\mathbb{C}/\Lambda$, it is obvious from its very construction that its universal cover is $\mathbb{C}$ and its fundamental group is Λ, which is isomorphic to the group $\mathbb{Z}^2$. When X is endowed with the additional structure of a Riemann surface, such a structure

[8]That is, with the map $\Gamma \times \tilde{X} \to \tilde{X} \times \tilde{X}$ given by $(g,x) \mapsto (gx,x)$ proper, meaning that complete inverse images of compact sets are compact.

is naturally induced on its universal cover, most often non-compact, so from the above it follows that the universal cover of every non-singular cubic curve is isomorphic to the complex line $\mathbb{C}$. Thus even though the isomorphism classes of smooth cubic curves depend on a modulus, their universal covers are all isomorphic. We summarize this, bringing in for the first time the term "uniformization":

Every smooth cubic curve C in the complex projective plane has a holomorphic uniformization $\pi : \mathbb{C} \to C$ which parametrizes the curve in the sense that two points of $\mathbb{C}$ have the same image under π if and only if their difference belongs to a certain lattice Λ of $\mathbb{C}$.

And the converse rounds off the theory into a harmonious whole: corresponding to each lattice Λ of $\mathbb{C}$, there exists a smooth cubic curve that is holomorphically isomorphic to $\mathbb{C}/\Lambda$.

Beyond elliptic curves

Our Chapter II constitutes an invitation to read the papers of Riemann devoted to algebraic functions and their integrals. These texts, so important for the history of mathematics, are difficult of access, and it took a considerable time for them to be finally assimilated. Although there are historical articles commenting on these, our approach is quite different, in particular in not at all attempting to be exhaustive. Riemann's great contribution was to turn Gauss' idea on its head: although it is useful to think of real surfaces as complex curves, it turns out to be more fruitful to think of a complex curve — with equation $P(x, y) = 0$, say — as a real surface. It is on this that Riemann bases his theory of surfaces, in which one-dimensional and two-dimensional notions become associated with one another. For example, he makes no bones about cutting a surface along a real curve, thereby introducing topological methods into algebraic geometry. Regarding an algebraic curve — that is, an object of one complex dimension situated in the complex projective plane — as a real two-dimensional surface presents no difficulties if the given curve is smooth, since then the real surface is also smooth. However, as we have already seen, this is far from representing the general situation since singular points arise frequently. In this case, however, one can to within a birational equivalence assume that the singularities are ordinary double points, and then it is not difficult to make the surface smooth: for this it suffices to regard the double point as actually two distinct points, on separate branches, and one constructs in this way a smooth surface associated with the original algebraic curve. This is how Riemann associates with each given algebraic curve a Riemann surface, that is, a holomorphic manifold of dimension 1, or, to put it another way, a real manifold of dimension 2 endowed with a complex structure. (We shall revisit this theme throughout the book.) Riemann went on to (almost) prove the following two statements:

— *Two algebraic curves are birationally equivalent if and only if their associated Riemann surfaces are holomorphically isomorphic.*

— *Every "abstract" compact Riemann surface is holomorphically isomorphic to the Riemann surface of some algebraic curve.*

Thus the algebraic problem of describing algebraic curves is transformed into the transcendental one of describing Riemann surfaces. The first invariant derived by Riemann was a purely topological one (and had a major impact on the development of topology since, among other things, it was in attempting to generalize it that Poincaré was led to the modern form of that discipline). It is well known that every compact orientable surface is homeomorphic to a sphere with a certain number of handles attached, which number is nowadays termed the *genus* of the surface. It follows that every algebraic curve has a specific associated genus which is invariant with respect to birational equivalence and so of much greater significance than the degree.

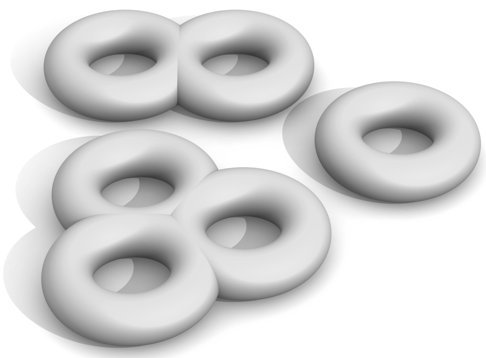

Figure 7: Topological surfaces of genus 1, 2, and 3

Here are some of the results concerning the genus that we shall encounter later on.

Having genus zero means that the curve's associated Riemann surface is homeomorphic to the 2-sphere. It does not then follow immediately that it is holomorphically isomorphic to the Riemann sphere. This fact was established in two different ways by Alfred Clebsch (see Chapter II) and Hermann Schwarz (see Chapter IV): every Riemannian metric on the sphere is globally conformally equivalent to that of the standard sphere. In other words (closer to those of Schwarz) every Riemann surface homeomorphic to the sphere is holomorphically equivalent to the Riemann sphere. In yet other words:

The algebraic curves of genus zero are precisely the rational curves.

This represents a further stage on the way to general uniformization: a single topological datum about a curve determines whether or not it has a rational parametrization.

Having genus 1 signifies that the Riemann surface is homeomorphic to a torus of two real dimensions. It follows, although not obviously — Clebsch proved it in 1865 — that it is holomorphically isomorphic to a quotient of the form $\mathbb{C}/\Lambda$ (see Chapter II). Thus:

The algebraic curves of genus 1 are precisely those birationally equivalent to smooth cubics (the so-called "elliptic" curves).

The case of genus greater than or equal to 2 is more complicated, and it is to this case that the present book is devoted. Before summarizing the situation, we clarify the connection between genus and degree: it can be shown that if C is a curve of degree d with k singular points, all ordinary double points, the genus is given by the formula

$$g = \frac{(d-1)(d-2)}{2} - k.$$

It is then immediate that straight lines and conics have genus zero, smooth cubics genus 1, singular cubics genus zero, and smooth quartics genus 3.

Riemann demonstrates great mastery by the manner in which he generalizes from the case of elliptic curves. For instance, for each fixed value $g \geq 2$ of the genus, he seeks to describe the space of moduli of the curves of that genus — that is, the space of algebraic curves of genus g considered to within a birational transformation — showing it has complex dimension $3g - 3$. Among other results of Riemann, we should also mention the celebrated one asserting that every non-empty simply connected open subset of $\mathbb{C}$ is biholomorphically equivalent to the open unit disc — a result of fundamental importance, although Riemann's proposed proof leaves a little to be desired (see Chapter II). It sometimes happens that this result, albeit an important special case, is confused with the "great" uniformization theorem forming the theme of the present book, which has to do not just with open sets of $\mathbb{C}$ but, much more impressively, with *all* Riemann surfaces.

Riemann's work in this field exerted a considerable influence on his immediate successors. In Chapter IV we describe Schwarz's attempts to establish explicitly certain particular cases of the conformal representation theorem while skirting the technical difficulties on which Riemann's proof founders.

Among the best expositions of Riemann's ideas, that of Felix Klein, another hero of the present work, stands out. In 1881 he wrote up what he believed to be the idea behind Riemann's intuition, even though Riemann's actual articles make no mention of it. We will never know if Klein was right in this, but the resulting new approach, via Riemannian metrics, seems to us especially illuminating. It relies on an electrostatic or perhaps hydrodynamic interpretation, making it particularly accessible to the intuition. We describe this way of looking at the subject and its modern developments in Chapter III.

Uniformizing algebraic curves of genus greater than 2

The question of parametrizing curves of general genus g remained open, or, more precisely, no one suspected that every algebraic curve might be parametrizable by *single-valued* holomorphic functions. However, following Riemann's work, evidence for this began to accumulate from the examination of certain remarkable examples.

In a marvellous article Klein studied the curve C given by the homogeneous equation $x^3y + y^3x + z^3x = 0$ as a Riemann surface, showing that it is isomorphic to the quotient of the upper half-plane by an explicit group of holomorphic transformations. In other words, he constructed a (single-valued) holomorphic function π with domain the upper half-plane $\mathbb{H}$ and with fibres the orbits of a group Γ of holomorphic transformations acting freely and properly. The analogy with the situation of elliptic curves was striking: the half-plane replaces the complex line and the group Γ of Möbius transformations replaces the group Λ acting via translations. *Thus is Klein's quartic uniformized by* π.

Even though this remarkable specimen was actually the first example of uniformization in higher genus, it was nonetheless taken at the time for an unparalleled gem, as it were, incapable of generalization like the regular polyhedra. As such it marked an interlude prior to attempts at establishing general uniformization. We shall expound Klein's example in Chapter V.

Motivated by quite different considerations arising in the theory of linear differential equations, Poincaré was led to the systematic investigation of the discrete subgroups Γ of the group $\mathrm{PSL}(2,\mathbb{R})$, which he called *Fuchsian*, and the quotients $\mathbb{H}/\Gamma$ obtained from them. He saw that among such quotients there are compact Riemann surfaces of genus at least 2. He showed that there is some latitude in the choice of the group, depending on certain parameters (see Chapter VI).

In light of Poincaré's results, Klein realized that the algebraic curves uniformized by $\mathbb{H}$ are in fact not isolated examples as he had thought, but form continuous families depending on parameters to be determined. Almost simultaneously Klein and Poincaré saw that the latter's constructions might be of sufficient flexibility for *all* compact Riemann surfaces to be uniformizable by $\mathbb{H}$. A dimensional count quickly showed that the space of Poincaré's Fuchsian groups, considered up to conjugation, yielding a surface of genus g depends on $6g-6$ real parameters — highly suggestive given Riemann's result that Riemann surfaces of genus g depend on $3g-3$ complex moduli. The race was on between Klein and Poincaré to prove the theorem. We encourage the reader to read the impassioned correspondence on this topic between our two heros reproduced at the end of the book. Here Klein and Poincaré introduce a new method of proof, namely *by continuity*.

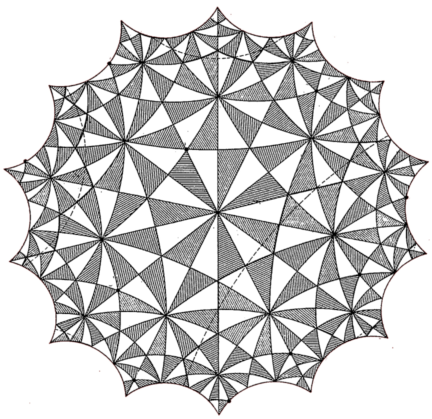

Figure 8: Klein's Fuchsian group (shown here as a group of automorphisms of the unit disc rather than the upper half-plane).

To us neither Klein's proof nor Poincaré's is totally convincing. In Chapter VII we try to resurrect Klein's proof[9]; to obtain a rigorous proof we had to use modern tools derived from quasiconformal techniques, which Klein and Poincaré certainly did not have at their disposal. Then in Chapter VIII we make an attempt to resuscitate — at least in part — Poincaré's approach, which was not motivated

[9]The matter is actually more complex; in fact some parts of the proof given in Chapter VII are closer to certain of Poincaré's arguments than to those of Klein.

by uniformization but rather by the desire to solve linear differential equations. The reader will observe there the emergence for the first time of a great number of concepts familiar to modern mathematicians. Chapter IX is devoted to the explicit investigation of some examples of uniformization of surfaces of higher genus.

By 1882 Klein and Poincaré had become fully convinced of the truth of the following uniformization theorem:

Theorem. *Let X be any compact Riemann surface of genus ≥ 2. There exists a discrete subgroup Γ of $PSL(2,\mathbb{R})$ acting freely and properly on $\mathbb{H}$ such that X is isomorphic to the quotient $\mathbb{H}/\Gamma$. In other words, the universal cover of X is holomorphically isomorphic to $\mathbb{H}$.*

To summarize, Klein and Poincaré had now effectively solved one of the main problems handed down by the founders of algebraic geometry: to parametrise an algebraic curve $F(x, y) = 0$ (of genus at least 2) by single-valued meromorphic functions $x, y : \mathbb{H} \to \mathbb{C}$. This magnificent result rounded out the theory dealing with the particular cases of rational and elliptic curves. Thus *Fuchsian functions* were now seen to be the appropriate generalizations of elliptic functions. Of course, as in the case of the elliptic functions, it was now necessary to admit new transcendental functions into the menagerie of basic mathematical objects, find their (convergent) series representations, etc. In fact Poincaré subsequently devoted a number of papers to such questions.

Beyond algebraic curves

But why should we confine ourselves to algebraic curves? What is the situation with "transcendental" curves? Spurred by his success with algebraic curves, Poincaré went on to address the problem of non-compact Riemann surfaces, which *a priori* have no relation to algebraic geometry. Although the method of continuity could no longer be applied, nonetheless already by 1883 Poincaré had managed to show that every Riemann surface admitting a non-constant meromorphic function can be uniformized in a certain weakened sense of the word "uniformize": one has now to allow parametrisations that may not be locally injective, that is, with ramification points. This result is the subject of Chapter XI. The question of the uniformization of non-algebraic surfaces seems to have stagnated for a while thereafter, until, in 1900, Hilbert stressed the incomplete nature of Poincaré's result, and encouraged mathematicians to re-apply themselves to it; this was Hilbert's twenty-second problem. At last, in 1907, Poincaré and Koebe arrived independently at the general uniformization theorem:

Theorem. *Every simply connected Riemann surface is holomorphically isomorphic to the Riemann sphere $\bar{\mathbb{C}}$, the complex plane $\mathbb{C}$, or the upper half-plane $\mathbb{H}$.*

Koebe's and Poincaré's approaches to this theorem are described in Chapters XII and XIII.

Of course, this classification of simply connected Riemann surfaces yields immediately a characterisation of all Riemann surfaces, since every Riemann surface is a quotient of its universal cover by a group acting holomorphically, freely, and properly. Thus by the theorem of Koebe and Poincaré every Riemann surface is identical with either the Riemann sphere or a quotient of $\mathbb{C}$ by a discrete group of translations, or a quotient of the half-plane $\mathbb{H}$ by a Fuchsian group. The work of Poincaré and Koebe, occupying Part C, allowed a new page to be turned in potential theory, and represents the end of an important epoch in the history of mathematics.

Meanwhile, over the decade 1890–1900, Picard and Poincaré worked out a new proof of the uniformization theorem based on a suggestion by Schwarz, valid in the compact case at least, and depending of the solution of the equation $\Delta u = e^u$. We present this in Chapter X.

The uniformization theorem was at the centre of the evolution of mathematics in the 19th century. In the diversity of its algebraic, geometric, analytic, topological, and even number-theoretic aspects it is in some sense symbolic of the mathematics of that century.

Our book ends in 1907, even though the story of the uniformization theorem continues. Among later developments, one might mention Teichmüller's work on moduli spaces, or those of Ahlfors and Bers in the 1960s relating to the concept of quasiconformal mappings (see for example [Hub2006]). There is also the progress in higher dimensions, in particular Kodaira's classification of complex surfaces, that is, of 2 complex dimensions. But that's another story!