The years 1880–1882 are crucial to our theme. It was then that Klein and Poincaré announced and then “proved” that *all algebraic curves of genus at least 2 can be uniformized by the disc.* This came as a great surprise to the mathematicians of the time. Examples were known — we saw some of them earlier — but that the result held in such generality seemed incredible. Even today it has the status of a major and highly nontrivial fact about the geometry of algebraic curves — to such an extent, indeed, that many mathematicians profess to know it “so well” that they forget that it is so highly nontrivial and all too often confuse it with one or another of two theorems which, although certainly important, are much older (and much simpler): the *Riemann Mapping Theorem* (the first convincing proof of which, as we shall see, was given by Osgood) according to which a nontrivial simply connected open subset of the plane is conformally equivalent to the disc, and *Gauss's theorem* (often wrongly attributed to Riemann) stating that a (real analytic) surface is locally conformal to an open subset of the plane.

Even though the present work is not a history book, a brief introduction to the protagonists may nonetheless be useful.

In 1880 Poincaré was a 26-year-old assistant professor[17]. He had defended his thesis two years earlier on the research topic of differential equations. It is undeniable that differential equations were at the root of almost all of his subsequent discoveries. The Paris Academy had proposed in 1878, as the theme of its competition for the Grand Prize in the mathematical sciences to be awarded in 1880, the following problem: “To bring to perfection in some significant aspect the theory of linear differential equations in a single independent variable”. Since he had founded the qualitative theory of dynamical systems a few months earlier[18], Poincaré now began to investigate differential equations in a single variable. In March 1880 he submitted a first memoir on the real theory, and then withdrew it in June of the same year. In the meantime he had become aware — in May 1880 — of an article by Fuchs on second order linear differential equations with algebraic coefficients. The memoir that he finally submitted to the Academy — in June 1880 — contains reflections inspired by Fuchs's article, reproduced in [Poin1951, Tome I, pp. 336–373]. In the work that so stimulated Poincaré, Fuchs sought to generalize Jacobi's inversion. He considered in particular the inverse function of the quotient of two independent solutions of a second-order differential equation and gave a necessary and sufficient condition for this function to be meromorphic. Since Fuchs's theory is essentially only local, Poincaré was struck by the result but found it unconvincing. He understood that Fuchs's

[17]Actually a *maître de conférences*, equivalent to senior lecturer in a British university or assistant professor in North America. *Trans*

[18]At this time Poincaré was also feeling the need to develop an autonomous topological theory (which, as we know, he subsequently realized).

result was an (excessively strong) version of uniformization[19]. Be that as it may, at that time Poincaré was in the midst of an attempt to understand second-order linear differential equations with algebraic coefficients *via* Fuchs's theory — and it was in connection with this aim that he created the theory of Fuchsian groups. Details of this first stage in this engaging story[20] are omitted from the present book. Happily, the existence of [Poin1997] excuses us somewhat. We might summarize these first months by saying simply that Poincaré immersed himself, with all his genius but also with a certain "naïveté", in the new theory. His correspondence with Klein shows, for instance, that at that time he had not read Riemann!

Klein was six years older than Poincaré. He had by that time been a professor already for ten years, and, possessed of an immense mathematical culture, was probably the most prominent mathematician of the era. He was certainly one of the finest connoisseurs of Riemann's works and knew the theory of elliptic functions thoroughly. He was one of the most influential propagators of the group concept in mathematics: his "*Erlanger Programm*" of 1872, announced on the occasion of his nomination to a professorship (at the age of 23), shows astonishing perspicacity. He had at that time already published major articles on the uniformization of certain particular algebraic curves arising in number theory. He had also established the projective character of (real) non-Euclidean geometry. When Professor Klein learns of Poincaré's first notes on Fuchsian groups (dating from February 1881) he is astounded both at the generality of the latter's constructions and his ignorance of the literature — in particular German — on the topic. On June 12, 1881 he begins a correspondence with his young colleague on the other side of the Rhine, destined to continue till September 22, 1882.

We reproduce this celebrated correspondence in an appendix, and strongly recommend it to the reader. One sees there a (scientific!) confrontation between a beginner and an established professor, tinged with oblique political references. Also in evidence is the increase in mutual respect over the course of the correspondence. But best of all one sees there the genesis of the uniformization theorem, gaining in precision of formulation almost day by day. It should also be mentioned how Poincaré's genius compels Klein's respect — respect he gladly acknowledges subsequently.

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[19]The day after the submission of his memoir to the Academy, Poincaré also sent to Fuchs the first of a series of letters in which the young assistant professor tried — without success — to explain to Professor Fuchs that a local diffeomorphism need not necessarily be a covering. Note in this connection that, throughout his work on uniformization, the ease with which Poincaré deals with what is not yet explicitly covering-space theory is certainly one of his essential assets — to such an extent that some have wished to see in the construction of the universal covering space Poincaré's main contribution to the problem. However, as we will see, the latter contention is largely an exaggeration.

[20]That is, the part consisting not only of the memoir submitted to the Academy but also the three supplements brought to light by Gray in 1979 and published in [Poin1997].

The first Fuchsian functions Poincaré constructs (in a note of May 23, 1881, [Poin1951, T. II, pp. 12–15]) uniformize surfaces obtained by removing a finite number of *real* points from a sphere (Poincaré also allows what are now called "orbifold singularities"[21]). He arrives independently at functions introduced earlier, as Klein points out to him, by Schwarz (see Chapter IV). Poincaré's method is quite different, however. He considers (Fuchsian) groups generated by reflections in the sides of ideal n-sided hyperbolic polygons. These groups depend on $n-3$ real parameters $1 < x_1 < \ldots < x_{n-3}$ and he identifies the space of these groups with the space of moduli of spheres with n real points removed. This represents the first appearance of the *method of continuity*[22]. It is clear that "from the beginning Poincaré has a lead that Klein can no longer make up" [Freu1955]. On August 8, 1881 Poincaré makes the following announcement [Poin1951, Tome II, pp. 29–31]:

> We conclude from this that:
>
> 1. Every linear differential differential equation with algebraic coefficients is integrable by means of zeta Fuchsian functions;
> 2. The coordinates of the points of every algebraic curve can be expressed by means of functions of an auxiliary variable.

This represents the very first enunciation of the uniformization theorem. It is, however, always necessary to moderate the enthusiasm of the young Poincaré a little. What he had actually proved (but completely rigorously) was appreciably weaker: *every algebraic curve can be "uniformized" by means of a function from the disc to the curve except for at most a finite number of points*. For Poincaré, motivated as he was by the integration of differential equations by means of functions given explicitly by series, excepting a finite number of points was not a problem. Moreover the proof of his result was in fact especially simple and elegant: given an algebraic curve branched over the sphere, up to removing the branch points one obtains a covering of the sphere with a finite number of points removed. It then only remains to show that up to the removal of finitely many more points, one has a covering of the sphere with finitely many *real* points removed. (This last step is an elementary exercise which we recommend to the reader.)

It is in fact Klein to whom the honor belongs of enunciating the uniformization theorem for algebraic curves as we now understand it. Klein, less interested in differential equations, in effect prefers finite polygons. Moreover his intimate

[21] An "orbifold" is a certain generalization of a manifold with singularities. *Trans*

[22] The method of continuity, as conceived by Poincaré, is explicitly described in Chapter IX in the case of spheres with 4 points removed. We leave to the reader as an exercise the verification that the method becomes considerably simpler when the 4 points are real.

knowledge of Riemann's work allows him to identify the number of moduli of curves of a fixed genus with the number of parameters on which Poincaré's polygons of the same genus depend. He is thus more naturally inclined to produce the "correct formulation" (see Freudenthal [Freu1955] and Scholz [Schol1980]); "this is the only essential point in which Klein, in his research on automorphic functions, surpassed Poincaré" [Freu1955]. The great principle is still the method of continuity, however implementing it in the needed generality is difficult. The correspondence between Klein and Poincaré shows very clearly just how each interprets it according to his own point of view.

Thus Klein observes that Poincaré's construction of Fuchsian groups produces uniformizable algebraic curves, and that these depend on parameters equal in number to those of the moduli space of curves of fixed genus. He notes also that if a Riemann surface can be uniformized, then this is possible in one way only. Thus the problem reduces to showing that the space of uniformizable curves is both open and closed. The question of the connectedness of the moduli space is mentioned by Klein as established in his book [Kle1882c], which we have already described[23].

Poincaré, on the other hand, was interested in second-order linear differential equations on an algebraic curve and showed that their description depended on a "monodromy" representation of the fundamental group (which he had then not as yet "invented") in $SL(2,\mathbb{C})$. When a differential equation on a fixed algebraic curve is allowed to vary, so also does this representation vary. In his examples of uniformizable curves (given by Fuchsian groups) one of these differential equations is privileged and has real monodromy group: Poincaré calls this equation *Fuchsian*. He asserts that every algebraic curve possesses a Fuchsian equation and that this allows him to show that his construction of Fuchsian groups is flexible enough to yield a description of all algebraic curves. The "proof" that he proposes also contains a component devoted to openness and another to closure. His attachment to ideal polygons allows him to more easily identify the difficulties associated with closure; see [Schol1980].

Both Klein and Poincaré later published descriptions of this period in their lives. Poincaré's text on "mathematical invention", dating from 1908, is famous [Poin1908]. There he describes his discovery of the link between differential equations and hyperbolic geometry as pre-dating his first epistolary contact with Klein.

> At that time I left Caen, where I was then living, in order to take part in a geology course undertaken by the School of Mines. The hazards of the trip caused me to forget my mathematical labors; when we arrived in Coutances

[23]His "proof" is hardly convincing.

> we climbed into an omnibus to go I knew not whither. At the instant I placed my foot on the step, the idea came to me, seemingly without anything in my mind having prepared me for it earlier, that the transformations I had used to define Fuchsian functions were identical to those of non-Euclidean geometry. I carried out no verification of this, I wouldn't have had the time since scarcely had I entered the omnibus when I resumed an earlier conversation; nonetheless I immediately felt complete certitude. Once back in Caen, I checked the result at leisure to satisfy my conscience.

It is indisputable that Poincaré had grasped the essentials of the theory before beginning his correspondence with Klein. In his third supplement to the memoir for the prize of the Academy, submitted on December 20, 1880, he "conjectures" that Fuchsian functions allow one to solve *all* linear differential equations with algebraic coefficients [Poin1997]:

> I have no doubt, moreover, that the many equations envisaged by M. Fuchs in his memoir in Volume 71 of Crelle's journal... will furnish an infinity of transcendentals... and that these new functions will allow the integration of all linear differential equations with algebraic coefficients.

One observes here, however, the absence of any formulation of the situation in terms of the uniformization of algebraic curves.

As for Klein, in his book on the development of 19th century mathematics [Kle1928] he explains:

> During the last night of my journey, that from March 22 to March 23 [1882], which I spent sitting on a couch on account of an attack of asthma, suddenly, towards 3:30, the central theorem dawned on me as if it had been sketched in the figure of the 14-sided polygon. Next morning, in the coach which at that time travelled between Norden and Emden, I thought about what I had found, going over all the details once more. I knew then that I had found an important theorem. Once arrived in Düsseldorf, I wrote up the memoir, dated March 27, sent it off to Teubner, and had copies sent to Poincaré and Schwarz, and also to Hurwitz.

In [Kle1921a, Vol. 3, pp. 577–586], there is an addendum to the effect that he considered that neither he nor Poincaré had a complete proof and that the proof using the method of continuity had been firmly established only by Koebe in 1912 [Koe1912]. He also describes that episode in his life as marking "the end of his productive period". He fell ill in the autumn of 1882[24].

[24]"Leipzig seemed to be a superb outpost for building the kind of school he now had in mind:

Unfortunately the second part of his book makes only a superficial contribution to the description of this mathematical adventure. Freudenthal's fine article [Freu1955] served us as a point of departure. Klein's book [Kle1928] is an essential reference for the history of 19th century mathematics, written by one of the heroes of the present work. By way of complementing these, the reader may also consult the relevant chapter of the historical book by J. Gray [Gra1986], the remarkable analysis [Die1982] by J. Dieudonné, the introduction [Poin1997] to Poincaré's three supplements to his memoir on the discovery of Fuchsian functions, J. Stillwell's commentary to his translation into English of Poincaré's articles on Fuchsian functions [Poin1985], the relevant chapter of the impressive thesis by Chorlay [Cho2007], the commentaries attached to the French version of the Klein–Poincaré correspondence [Poin1989], or Fricke's article [Fric1901] in the *Encyklopädie der mathematischen Wissenschaften*. Finally, there is the article by Abikoff [Abi1981], from which, while interesting also mathematically speaking, we quote, for our present historical purposes, only his version of the reception by Hurwitz, Schwarz, and Poincaré of the latter's proof of the uniformization theorem:

– Hurwitz: I accept it without reservation.

– Schwarz: It's false.

– Poincaré: It's true. I knew it and I have a better way of looking at the problem.

Chapter VI is an introduction to Fuchsian groups. The reader will find there, for instance, the construction of the Fuchsian group associated with a fundamental polygon, and also the construction of automorphic forms and Fuchsian functions invariant under the action of a given Fuchsian group. As current references for Fuchsian groups we might mention the books [Kat1992] and [Bea1983], the latter dealing also with their generalization to higher dimensions: discrete groups of isometries of hyperbolic space, notably the Kleinian groups in dimension 3. For Kleinian groups one may also consult [Dal2007] and [Mas1988]. The paper [Mas1971] gives the first complete and correct proof of Poincaré's polygon theorem (Theorem VI.1.10 below).

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one that would draw heavily on the abundant riches offered by Riemann's geometric approach to function theory. But unforeseen events and his always delicate health conspired against this plan. [In him were] two souls [. . .] one longing for the tranquil scholar's life, the other for the active life of an editor, teacher, and scientific organiser. [. . .] It was during the autumn of 1882 that the first of these two worlds came crashing down upon him [. . .] his health collapsed completely, and throughout the years 1883–1884 he was plagued by depression" [Row1989].

Chapter VII is a variation on Klein's approach, and there we make no attempt to pronounce on the validity of the proofs proposed by him[25]. We propose a "reconstitution" of what purported to be a proof of the theorem on the uniformization of algebraic curves along the lines of the method of continuity as viewed by Klein. This proof uses tools developed later, but in a weak form. In sum, Chapter VII is in some sense the article Klein might have written if he had more tools at his disposal. In the space of twenty years the literature on the representations of surface groups has grown enormously. The article [GolW1988] is an important reference for the questions evoked in this chapter. For a presentation adhering more closely to the ideas of Klein, the reader may consult the classic book [FrKl1897].

Chapter VIII is an introduction to the approach of Poincaré. We first explain there how uniformization theory can be expressed in terms of second-order linear differential equations, and then give a proof of the openness of the space of uniformizable curves. In this connection it is necessary to complete some of Poincaré's arguments, but in a relatively light-handed manner. However, as far as his approach to closure is concerned, we do not expound it because it fails to convince us, and also because we cannot see our way to "repairing" it without in fact using the arguments of Chapter VII.

Finally, in Chapter IX we put Poincaré's approach to work in the analysis of special cases, and also describe the subsequent life of these methods. In particular, we expound there the explicit examples of uniformization obtained by Schwarz in his investigation of the hypergeometric equation.

As we have already mentioned, the uniformization theorem is not confined to algebraic curves. Emboldened by this "special case" (yet an already enormously general one), Poincaré went on to attempt to generalize it to all simply-connected Riemann surfaces not necessarily universal covering spaces of compact surfaces. Here he can no longer resort to finite-dimensional moduli spaces or monodromy groups. Koebe and Poincaré succeeded in 1907, and we shall explain how in Part C.

[25]Except to say that his approach to closure does not appear convincing to us.