The last part of this book is devoted to describing the path that, from 1882 to 1907, led from the uniformization of algebraic Riemann surfaces by the method of continuity to the general uniformization theorem as we know it today. Gray has written a very detailed study [Gra1994] devoted to the Riemann Mapping Theorem [Gra1994] to which we may refer the reader. We recommend also earlier entries in the *Encyklopädie der mathematischen Wissenschaften*: [OsgW1901, Bie1921].

In 1882, Klein and Poincaré became convinced that every algebraic Riemann surface could be uniformized by the sphere, the plane, or the unit disc. Although some of the details of the proof of this marvellous result remained to be filled in, Poincaré, never lacking in mathematical audacity, was already launched on the conquest of much wider territory, attempting to uniformize Riemann surfaces associated with arbitrary, so not necessarily algebraic, *germs of analytic functions*.

The memoir [Poin1883b] Poincaré published in 1883 begins with a statement of *the theorem of uniformization of functions* that he proposes to prove:

> Let y be any analytic function of x, not single-valued. One can always find a variable z such that x and y are single-valued functions of z.

What is the missing link between this statement and what we call today the uniformization theorem for Riemann surfaces? In his memoir, Poincaré recalls how to construct from a "non-single-valued analytic function y of the variable x" an abstract Riemann surface extended over the plane of the variable x, on which y is naturally defined as a single-valued analytic function. In modern terminology, given a germ of an analytic function y of a variable x, one constructs the maximal Riemann surface on which one can extend the germ y to a (single-valued) analytic function (see Box II.1): this is the Riemann surface associated with the germ y. Finding a variable z such that x and y are single-valued functions of z comes down to uniformizing the Riemann surface associated with the germ y, that is, to parametrizing this surface with a single complex variable z. In 1883, Poincaré did not succeed in obtaining a parametrization that is a local biholomorphism at every point, and was forced to allow for branch points. His precise result was as follows:

Theorem. — *Let S a Riemann surface admitting a non-constant meromorphic function. Then there exists a branched covering map $\pi : U \to S$, where U is a bounded open subset of $\mathbb{C}$.*

The uniformization theorem for functions announced by Poincaré follows immediately from this result: if S is the Riemann surface associated with a germ of an analytic function y of a complex variable x, and if U is the open set in $\mathbb{C}$ given by the above theorem, then x and y may be viewed as single-valued functions on the surface S, and therefore as single-valued functions of the coordinate z of the complex plane containing U.

The concept of the universal cover of a Riemann surface plays an important role in Poincaré's memoir. As far as we know, it is in this memoir that there appears for the very first time a definition of the universal cover of the Riemann surface associated with a germ of a function (or with a finite family of germs of functions; see Box XI.2 below). In 1898, Osgood reckoned this definition a crucial feature (and perhaps the most important contribution) of Poincaré's memoir ([OsgW1898]). To establish the above theorem, Poincaré shows the existence of a Riemann surface Σ that is a branched covering space of S and is such that its universal covering space $\widetilde{\Sigma}$ is biholomorphic to a bounded open subset of $\mathbb{C}$. To achieve this, it suffices — as Riemann had observed — to find a Riemann surface Σ that is a branched covering of S such that $\widetilde{\Sigma}$ admits a positive harmonic function with a logarithmic pole.

The basic tool in Poincaré's proof is the following result, which he attributes to Schwarz, and which does indeed follow immediately from techniques invented by the latter in [Schw1870a] (even if it would seem that Schwarz himself was unaware in 1870 of having effectively established such a general result):

Theorem. — *Let Ω be a region of compact closure, with analytic or polygonal boundary, of a Riemann surface. Then Ω admits a Green's function*[13]*. It follows that if Ω is simply connected, then it is biholomorphic to the unit disc in $\mathbb{C}$.*

Poincaré considers an exhaustion of a simply connected Riemann surface $\widetilde{\Sigma}$ by means of simply connected regions with compact closure (however without justifying its existence); he applies Schwarz's theorem to each of these regions, obtaining thereby a sequence of Green's functions; if this sequence converges, then the limit will automatically be a positive harmonic function defined on $\widetilde{\Sigma}$ with a logarithmic pole, and $\widetilde{\Sigma}$ will therefore be biholomorphic to an open subset of the unit disc. However, in general one does not obtain a convergent sequence of Green's functions, and this is why, instead of considering the universal cover $\widetilde{S}$ of the Riemann surface S of interest, Poincaré has to resort to the universal cover $\widetilde{\Sigma}$ of a branched covering space Σ of S.

The result Poincaré obtained in 1883 represents an exceptional advance from the point of view of analytic functions, but is much less satisfactory if one is interested in Riemann surfaces for their own sake, and not merely as a simple tool to be used to investigate analytic functions.

Recall that Klein and Poincaré had shown (or at least believed they had shown) that the universal cover $\widetilde{S}$ of an algebraic Riemann surface S is always biholo-

[13] Recall that a Green's function on Ω is a positive, harmonic function with a logarithmic pole, that tends to zero in the neighborhood of the boundary of Ω.

morphic to the sphere, complex plane, or unit disc, and that therefore S can be identified with the quotient of one of these surfaces by the action of a group of automorphisms. In the case where S is not algebraic, Poincaré managed "only" to prove in 1883 that S has a branched covering space $\widetilde{\Sigma}$ biholomorphic to a bounded simply connected region U of $\mathbb{C}$. The primary drawback in this result consists in the fact that one has no control over the region U, which *a priori* depends on the surface S. (Note that at that time the Riemann Mapping Theorem had been proved rigorously only in special cases.) And even if one knew how to identify the region U, the presence of branch points makes for a considerably weaker result: indeed, for a fixed Riemann surface S and region U of $\mathbb{C}$, there exist in general infinitely many branched coverings $\pi : U \to S$ not obtained from one another by composing with biholomorphisms of U. And lastly, one is hard put to content oneself with Poincaré's result when one reflects that it yields a "uniformization" of the complex plane by means of an open subset of the unit disc![14]

In his address to the International Congress of Mathematicians in 1900 [Hil1900b], Hilbert praises Poincaré's work on algebraic Riemann surfaces and also his uniformization theorem for analytic functions, but also emphasizes the imperfections of the latter result. In view of the importance of the question, he reckons it essential to try to obtain a result for general Riemann surfaces as satisfying as that obtained by Klein and Poincaré for algebraic surfaces. This constitutes his 22nd problem.

An initial advance was made on the problem in 1900 by W. Osgood, in proving the following result:

Theorem. — *Every simply connected region of the complex plane that admits a positive harmonic function with a logarithmic pole (for example, every bounded simply connected region) is biholomorphic to the unit disc.*

Thus at this stage it was known that every Riemann surface has a branched covering biholomorphic to the unit disc in $\mathbb{C}$. It took another seven years before the uniformization theorem as we know it today was proved ...

Over the first several years of the 20th century there were various unsuccessful attempts to solve Hilbert's 22nd problem. We mention, in particular, Johansson ([Joh1906a, Joh1906b]). Then at the meeting of May 11, 1907 of the Göttingen Scientific Society, Klein presented a note by P. Koebe [Koe1907b] announcing that he had proved the general uniformization theorem:

Theorem. — *Every simply connected Riemann surface (supporting a non-constant meromorphic function[15]) is biholomorphic to the Riemann sphere, the complex plane, or the unit disc.*

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[14] It is interesting to read Osgood's presentation of Poincaré's result and its inadequacies in a series of talks given in Cambridge in 1898 [OsgW1898].

[15] At that time Riemann surfaces were always conceived as extended over the plane. However,

The case of compact simply connected Riemann surfaces (homeomorphic to the sphere $\mathbb{S}^2$) had been dealt with already in papers by Schwarz and Neumann: they are all biholomorphic to the Riemann sphere. Thus there remained only the case of non-compact simply connected Riemann surfaces. Given such a Riemann surface S, Koebe considers an exhaustion of it by means of an increasing sequence $(D_n)_{n\geq 0}$ of simply connected regions with compact closure and with polygonal boundaries, and chooses a fixed point $p_0 \in D_0$. Schwarz had shown the existence, for each n, of a biholomorphism φ_n from D_n onto the unit disc of $\mathbb{C}$, sending the prescribed point p_0 to the origin. If the sequence of moduli of the derivatives of the φ_n at p_0 could be shown to be bounded, then from work of Harnack and Osgood it would follow that the surface S is uniformized by the unit disc. Thus the whole of Koebe's paper is devoted to showing that, if the sequence of derivatives of the φ_n at p_0 should diverge, one can nonetheless construct from the sequence $(\varphi_n)_{n\geq 0}$ a different sequence $(\psi_n)_{n\geq 0}$ of biholomorphisms that converges to a biholomorphism between S and the complex plane. The key argument involved in constructing the sequence $(\psi_n)_{n\geq 0}$ is very subtle, and contains in embryo a version of the so-called *Koebe's Quarter Lemma*. But even if it is difficult to grasp[16], Koebe's proof is nevertheless perfectly rigorous.

Six months later, an article by Poincaré [Poin1907] appeared in *Acta Mathematica* in which he also proposed a proof of the general uniformization theorem, one very different from Koebe's[17]. For a given non-compact, simply connected Riemann surface S, Poincaré considers the region A obtained by removing a small disc. He notes that the surface S will be biholomorphic to the complex plane or the unit disc provided A admits a *Green's majorant*, that is, a positive harmonic function with at least one logarithmic pole. It then remains to construct such a function. To this end, Poincaré again generalizes the *alternating procedure* invented by Schwarz, and gives a physical interpretation of the procedure he defines, which he calls the "*sweeping method*".[18] Suppose one wishes to construct on a surface A a function u with a logarithmic pole at a point p_0, harmonic on $A \setminus \{p_0\}$, and tending to zero at infinity. Such a function may be thought of as given by the electric potential associated with a negative point charge situated at

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Koebe's proof works for abstract Riemann surfaces.

[16]It is appropriate to mention that the article [Koe1907a] was in the form of a communication to the Göttingen Scientific Society, and that the details suppressed in such communications were often intended for publication in a "real" mathematics journal. In fact Koebe continued for the rest of his life to reprise different proofs of the theorem in order to make it more accessible and more general, and improve its presentation. See, for instance, [Koe1907a, Koe1907b, Koe1908a, Koe1909a, Koe1909b, Koe1909c, Koe1909d, Koe1910b, Koe1911].

[17]Poincaré did not know of Koebe's proof when he was preparing his article, submitted in March 1907.

[18]Usually translated into English as the "scanning method". *Trans*

the point p_0. To construct it, Poincaré imagines the following:

— starting with an arbitrary function $u_0 : A \to \mathbb{R}$ with a logarithmic pole at p_0 and tending to zero at infinity, visualized as the potential associated with a distribution of charge $\rho_0 := \Delta u_0$;

— letting each small region of the surface gradually become more and more "conducting" in order to be able to "sweep" the charges (except for that at p_0, which is to be maintained artificially in place) towards the boundary of each of these regions. One hopes that at the end of this process, all charges (save that at p_0) will have been "swept to infinity"; the associated potential will then give the desired function. In mathematical terms, one covers A by holomorphic discs, and constructs a sequence $(u_n)_{n\geq 0}$ of continuous functions, with the property that u_{n+1} is the same as u_n everywhere except on one of the discs, on which it is harmonic.

Of course, the bulk of the work is involved in showing that the sequence $(u_n)_{n\geq 0}$ converges. As so often, Poincaré's proof, although not a model of rigor, contains luminous intuitions. In particular, he uses a physical argument (the conservation of the total electric charge when a disc in the Riemann surface is "made conducting") difficult to justify mathematically without using the theory of distributions.

Poincaré's memoir appeared at the beginning of November 1907. At the end of that same month, Koebe, who had read Poincaré's memoir avidly, submitted a new note to the Göttingen Scientific Society [Koe1907b], containing a proof of the general uniformization theorem largely inspired by Poincaré's proof. In fact Koebe reprises the global strategy of Poincaré's proof, but with the "sweeping method" replaced by a much more direct construction based on an exhaustion of A by regions of compact closure, thus gaining in simplicity (and rigor) what had been lost in physical intuition.

In the introduction to Part B, we explained how in 1881 Klein was an established professor who soon found himself outmatched by the young Poincaré. In 1907 it was Poincaré who was the established one and who must have felt a little hustled by the young Koebe, only 25 years old. The following anecdote shows clearly the difference in status between the two rivals: at the International Congress in Rome in 1908, both Koebe and Poincaré gave addresses. Koebe's was entitled "On the uniformization problem…", while Poincaré's was "On the future of mathematics"!

In sum, in 1883 Poincaré shows that every Riemann surface (on which a meromorphic function can be defined) admits a branched covering space biholomorphic to a bounded simply connected open subset of the plane. His proof depends on ideas of Schwarz allowing the uniformization of every relatively compact, simply connected region with polygonal boundary of a Riemann surface, and uses an exhaustion of his arbitrary non-compact simply connected Riemann surface by means of such regions, that is, by a sequence of relatively compact, simply connected regions with polygonal boundaries. The existence of such an exhaustion — which Poincaré does not prove — is not difficult to establish in the case of a Riemann surface extended over the plane[19] (see §XI.2). Then in 1900, Osgood shows that every bounded simply connected open subset of the plane is biholomorphic to the disc. Thus then it becomes known that every Riemann surface has a branched covering space biholomorphic to the disc. In his address to the International Congress of Mathematicians of that year, Hilbert emphasizes the inadequacy of this result, and urges mathematicians to try to prove a "true" uniformization theorem for non-algebraic Riemann surfaces. In May 1907, Koebe publishes the first proof of the general uniformization theorem, also based on work of Schwarz and on the existence of exhaustions by means of relatively compact, simply connected regions with polygonal boundary. (This proof seems to us to be perfectly correct and rigorous.) Just prior to the publication of Koebe's memoir, Poincaré also completes and prepares for publication a proof of the general uniformization theorem, which appears in early November 1907. (This proof, based on physical intuition, seems a very natural one to us; however, it cannot be made rigorous without recourse to the theory of distributions.) At the end of November 1907, Koebe publishes a "simplified" version of Poincaré's proof, with the "sweeping method" replaced by an appeal to work of Schwarz and the use of an exhaustion of a simply connected Riemann surface with a small disc removed, by means of relatively compact annuli. This "cleaned up" version of Poincaré's proof is, although certainly less intuitive than its original, especially brief, and seems to us rigorous as it stands.

Thus by the close of the year 1907, the uniformization theorem was firmly established. Of course, the process of assimilation of the result was far from complete, and it would take another fifteen years before the proofs began to appear that one finds in today's books (in this connection, see our annotated bibliography). Early on, mathematicians switched predominantly to the search for results beyond the uniformization theorem: Koebe was already beginning to think about uniformizing non-simply-connected Riemann surfaces [Koe1910b], and Hilbert was already inviting mathematicians to investigate the uniformizability of complex manifolds of higher dimensions... but our book stops in 1907.

Chapter XI is devoted to Schwarz's theorem on the uniformization of simply connected regions with compact closure, Poincaré's results of 1883 on the uniformization of functions, and Osgood's theorem. Then, in Chapter XII, we expound the first of Koebe's proofs of the general uniformization theorem from

[19]Prior to the work of Weyl in the 1910s, a Riemann surface was *by definition* extended over the plane.

his note [Koe1907a]. Finally, Chapter XIII is devoted to Poincaré's proof of the same theorem in [Poin1907], and also to the simplification of that proof proposed by Koebe in [Koe1907b].

Box: The classification of surfaces

In a completely natural way the theory of Riemann surfaces — the veritable topic of this book — evolved in parallel with the *topological* theory of surfaces, that is, of 2-dimensional manifolds not endowed *a priori* with a complex structure. While the history of these developments would furnish enough material for another book, we thought it nonetheless apropos to indicate here some of the most important milestones. In their progress towards the general uniformization theorem, Poincaré and Koebe used, proved, or quite simply anticipated the main results of the topology of surfaces. Often the borrowings from topology are completely implicit. Yet again do we find the situation somewhat confused.

The topological classification of compact surfaces took place gradually, progressively gaining in rigour and generality. The very concept of surface was some time in maturing, from the idea of a surface as embedded in 3-dimensional space to the conception of an abstract surface. Moreover two surfaces embedded in 3-space could be homeomorphic without there existing any homeomorphism of the ambient space sending one to the other: thus, for instance, a torus might be knotted in 3-space. And then it became necessary to distinguish degrees of regularity of surfaces under investigation, which might be smooth or merely topological. Fractal sets, arising at the same time as Kleinian groups, furnish many examples of curves that are not differentiable and whose local properties are such as to make one despair of any topological classification.

The main theorem, become classical, may be stated as follows:

Theorem. — *Every compact connected orientable surface is homeomorphic to a sphere or to a connected sum of tori.*

This theorem was "known" — and used — by B. Riemann, with no attempt made to justify it. For its history one may consult [Pont1974]: the most important names in this connection are F. A. Möbius [Möb1863], C. Jordan [Jor1866] and W. von Dyck [Dyc1888]. The first proofs meeting (almost) today's standards of rigour date from the 1860s and use two different kinds of ideas. They assume implicitly that the surfaces are smooth.

First Möbius produced a remarkable proof in the case of compact surfaces embedded in space (which is *a posteriori* equivalent to orientability).

His proof involves choosing a real-valued function on the surface and investigating the nature of its level curves. By means of successive modifications of the function he simplifies it so as to step by step eliminate critical points and reduce it to "standard" form. (One sees here the germ of what will much later be called Morse theory.) He also proves that the surface may be cut into two planar surfaces, that these are characterized to within a homeomorphism in terms of the number n of components of their boundary, and that this number is the only invariant of the initial closed surface. He observes also that $n-1$ is the greatest number of disjoint closed curves on the surface that do not disconnect it, thus recovering Riemann's definition of genus. These ideas were then elaborated on and consolidated by, among others, J.C. Maxwell [Max1870] and C. Jordan [Jor1872].

Jordan takes a different approach, in some sense reprising Riemann's method, which consists in cutting the surface along disjoint simple closed curves. His surfaces are compact, smooth, and without boundary, but not necessarily embedded in 3-space: curves of self-intersection are allowed, so that in fact he allows his surfaces to be immersed in 3-space.

The classification of nonorientable surfaces was also carried out progressively. In 1861, J.B. Listing (to whom, incidentally, we owe the word "topology" [Lis1847]) appears to have been the first to describe the nonorientable surface with boundary that today we call the Möbius strip [Lis1861], and in 1882 Klein described the "bottle" bearing his name in an article discussed earlier [Kle1882c]. In 1886, Möbius clearly defines the concept of orientability [Möb1886], and then Dyck obtains the classification of arbitrary compact, smooth surfaces, possibly with boundary, possibly nonorientable [Dyc1888]. Volume 6 of Poincaré's collected works includes a glossary allowing one to pass from the topological terminology of 1950 back to that of Poincaré. For example, opposite "Möbius strip" one finds Poincaré's term "the one-sided surface that everyone knows".

This was all made precise in an article by Dehn and Heegaard in 1907 [DeHe1907]. Here the surfaces are triangulated, and are allowed to be nonorientable and have non-empty boundary. The classification is combinatorial in nature, and the arguments are convincing. Klein comments on this article that it is "written in a rather abstract style.... It begins by formulating the concepts and facts fundamental to topology. Then the rest is deduced in a purely logical manner. This contrasts completely with the inductive presentation that I have always recommended. To be understood plainly, [let me say that] this article presupposes of the reader that he has already pondered the topic deeply in the inductive manner" [Kle1925].

The characterization of *topological* surfaces to within a homeomorphism will take more time, as we shall now see.

The Jordan Curve Theorem and the Osgood–Schoenflies Theorem

Theorem. — *The complement of a simple closed curve in the plane has exactly two connected components.*

This theorem was stated by Jordan in 1887 [Jor1887]. The "proof" he proposed did not seem convincing to those who commented on it [Veb1905, Ale1920, Schoe1906, DoTi1978], and it should be noted that it assumed the statement to be obvious in the case of a polygonal (or smooth) curve . . .
A proof for a polygonal curve was in fact first given by Schoenflies in 1896 [Schoe1896]. The first complete proof of the full theorem seems to be that given by Veblen in 1905 [Veb1905].

Consider these dates in relation to the period of relevant activity of the protagonists of this part of our book — Poincaré and Koebe — from 1883 to 1907. Since their interest lay with Riemann surfaces, which are necessarily smooth, all the theorems on the classification of surfaces were at their disposal, and indeed they exploited them to the full, though sometimes without mention.

The following theorem, especially delicate in the case of non-smooth curves, progressively makes its appearance during the same period.

Theorem. — *Every simple closed curve in the plane can be mapped onto a circle by means of a global homeomorphism of the plane.*

Here are some comments on the history of this result, traditionally called "Schoenflies' Theorem", drawn largely from a recent publication of Siebenmann [Sieb2005].

Even though the arguments Jordan used in his attempt to prove the Jordan Curve Theorem [Jor1887] were not convincing, they still showed essentially that the bounded component of the complement of a curve is homeomorphic to an open disc. This fact was explicitly established using conformal methods in 1900 by Osgood in an article we will be discussing later on [OsgW1900].

It was in 1902 that Osgood stated Schoenflies' Theorem [OsgW1902]; however it would take another ten years or so before the first complete proofs appeared, again using conformal methods [Car1913a, Car1913b, Car1913c, Koe1913a, Koe1913b, Koe1915, OsTa1913, Stu1913]. Schoenflies stated "his" theorem clearly enough in 1906 [Schoe1906]. His proof, fully correct in the case of a polygonal or smooth curve, was, however, lacking in the general case.

The first correct proof, using only topological arguments (and not conformal ones) seems to be due to Tietze in 1914 [Tie1913, Tie1914] or to Antoine in 1921 [Ant1921]. The name "Schoenflies' Theorem" was given to the result by Wilder in 1949 [Wil1949].

The *topological* classification of surfaces

We emphasize once again that since the interest of Poincaré and Koebe was concentrated on Riemann surfaces, and these are automatically smooth, the question of the structure of topological surfaces was of no direct interest to them at that time. It seemed to us nonetheless useful to give a quick description of the later developments concerning topological surfaces.

Schoenflies' theorem would be the key allowing Radó to prove in 1925 that every topological surface countable at infinity is triangulable, and thence to obtain a classification in the compact case [Rad1925].

The outstanding case of noncompact surfaces was dealt with thanks to the introduction of the idea of end compactification by Freudenthal, Kerékjártó and Schoenflies. The complete classification in the noncompact case was obtained by Kerékjártó in 1923 [Ker1923], and fully rigorized by Richards in 1963 and Goldman in 1971 [Ric1963, GolM1971].

We mention in conclusion a particular case that will be needed in the proof of Lemma XI.2.1: a noncompact, simply connected surface is homeomorphic to the plane.