# Introduction

According to Hartshorne ([96], Section I.8), one of the guiding problems in Algebraic Geometry is the classification of algebraic varieties up to isomorphy. Let us briefly mention two (related) variants of the problem. The first one is the classification of complex projective varieties up to isomorphy, usually assumed to be smooth or mildly singular. The strategy is first to accomplish the birational classification of so-called minimal varieties and then relate any projective variety to a minimal model by some specific operations such as blow-down. The reader may consult [14] for the classical case of surfaces and [121], [47], and [145] for results in higher dimension. In the second one, we place ourselves in one particular projective space $\mathbb{P}_n$ and wish to classify all closed subvarieties of this projective space up to projective equivalence. We remind the reader that two subvarieties of $\mathbb{P}_n$ are called *projectively equivalent*, if there is an automorphism of the ambient space $\mathbb{P}_n$, i.e., an element of $\mathrm{PGL}_{n+1}(\mathbb{C})$, which carries the first variety onto the second one. This problem is just the abstract formulation of putting a system of homogeneous equations in the variables $x_0, \ldots, x_n$ into a suitable normal form. In modern language, one states it as the slightly more general problem of classifying polarized varieties, i.e., pairs $(X, L)$ which consist of a projective variety $X$ and an ample line bundle $L$ on it. Treatises of this theory are in [68], [17], and [215]. The two problems do overlap: A Fano manifold $X$, for example, yields the polarized variety $(X, -K_X)$. Conversely, two polarized varieties $(X, L)$ and $(X', L')$ with $\dim(X) = \dim(X')$, $L^{\dim(X)} = L'^{\dim(X')}$, and $\mathrm{Pic}(X) \cong \mathrm{Pic}(X') \cong \mathbb{Z}$ are projectively equivalent, if and only if they are isomorphic. As Hartshorne also describes, these classification problems usually fall into two parts. First, one has some discrete numerical invariants such as the Hilbert polynomial of a polarized variety $(X, L)$ which yields a coarse subdivision of the class of all objects. Second, the objects with fixed numerical invariants usually come in positive dimensional families and one has to construct a **moduli space** for them, which is an algebraic variety whose points are in "natural" bijection to the set of isomorphy classes of the objects with fixed numerical data. Mumford has conceived his Geometric Invariant Theory (GIT) as a major tool for constructing such moduli spaces.

To get a more concrete idea, let us look at a special case of the second classification problem, namely the classification of hypersurfaces in $\mathbb{P}_n$ up to projective equivalence. The numerical invariant which we have to take into account is just the degree of a hypersurface, a positive integer. For fixed $d \in \mathbb{Z}_{>0}$, the hypersurfaces of degree $d$ form the linear system $|\mathscr{O}_{\mathbb{P}_n}(d)|$. If $\mathbb{C}[x_0, \ldots, x_n]_d$ denotes the vector space of homogeneous polynomials of degree $d$ in the variables $x_0, \dots, x_n$, then we have the identification

$$|\mathscr{O}_{\mathbb{P}_n}(d)| = P\Big(\mathbb{C}[x_0, \dots, x_n]_d\Big) := \Big(\mathbb{C}[x_0, \dots, x_n]_d \setminus \{0\}\Big)/\mathbb{C}^\star.$$

The group $\mathrm{PGL}_{n+1}(\mathbb{C}) = \mathrm{SL}_{n+1}(\mathbb{C})/(\mu_{n+1} \cdot \mathbb{E}_{n+1})$ acts on $\mathbb{P}_n$ as its automorphism group. It clearly induces an action of $\mathrm{PGL}_{n+1}(\mathbb{C})$ on $|\mathscr{O}_{\mathbb{P}_n}(d)|$ and $P(\mathbb{C}[x_0, \dots, x_n]_d)$. This action may also be obtained in a different way: The group $\mathrm{SL}_{n+1}(\mathbb{C})$ acts on the vector space $\mathbb{C}[x_0, \dots, x_n]_d$ as a group of linear automorphisms by change of variables. This action descends to an action of $\mathrm{SL}_{n+1}(\mathbb{C})$ on $P(\mathbb{C}[x_0, \dots, x_n]_d)$. Since the center $\mu_{n+1} \cdot \mathbb{E}_{n+1}$ acts trivially, it induces an action of $\mathrm{PGL}_{n+1}(\mathbb{C})$ on $P(\mathbb{C}[x_0, \dots, x_n]_d)$ which is—using the right conventions—the one that we have introduced before. Intuitively, the moduli space for hypersurfaces of degree $d$ in $\mathbb{P}_n$ will be the quotient $P(\mathbb{C}[x_0, \dots, x_n]_d)/\mathrm{PGL}_n(\mathbb{C})$.[1] So far, we have no clue whether or in which sense we can construct the quotient $P(\mathbb{C}[x_0, \dots, x_n]_d)/\mathrm{PGL}_{n+1}(\mathbb{C})$ as an algebraic variety. The same circle of ideas works in the more general context: Subvarieties of $\mathbb{P}_n$ with fixed Hilbert polynomial are parameterized by a projective scheme, a so-called **Hilbert scheme**, which replaces the linear system in the above example, and, as before, the action of $\mathrm{PGL}_{n+1}(\mathbb{C})$ on $\mathbb{P}_n$ yields an action of $\mathrm{PGL}_{n+1}(\mathbb{C})$ on this Hilbert scheme. Again, we are lead to the problem of forming quotients (see [215] for this general context).

The guiding problem has thus evoked our interest in the following problem: Let $G$ be an algebraic group, $X$ a variety or, more generally, a scheme, and $\alpha: G \times X \longrightarrow X$ an action of $G$ on $X$. In which sense can we form the quotient of $X$ by the action of $G$? One easily checks that, in general, the set of orbits does not carry a natural scheme structure. Thus, one first has to develop the appropriate notion of a quotient. The most general one is that of a **categorical quotient** which is denoted by $X/\!\!/G$. Still one finds examples where even the categorical quotient does not exist as a variety, separated scheme, or just scheme. Thus, let us formulate the following more concrete problem: In the above situation, suppose that $X$ is a variety or a scheme of finite type over $\mathbb{C}$. Then, the task is to find a $G$-invariant open subset $U \subseteq X$, as large as possible, such that $U/\!\!/G$ exists as a variety or a scheme of finite type over $\mathbb{C}$. Now, we restrict to the case where $G$ is a **reductive linear algebraic group** (e.g., $\mathrm{SL}_n(\mathbb{C})$ or $\mathrm{GL}_n(\mathbb{C})$). Then, Mumford's GIT as developed in [155] is a formalism for finding such open subsets. (More recently, various generalizations have been discovered, e.g., [99].)

To give the reader an impression, let us indicate the most basic case. For this, let $\varrho: G \longrightarrow \mathrm{GL}(V)$ be a representation of $G$, i.e., $\varrho$ is a homomorphism of linear algebraic groups. Then,

$$\begin{aligned} \varkappa: G \times V &\longrightarrow V \\ (g, v) &\longmapsto \varrho(g)(v) \end{aligned}$$

is an action of $G$ on $V$ by linear automorphisms. The $\mathbb{C}$-algebra of regular functions on the affine variety $V$ is $\mathbb{C}[V] = \mathrm{Sym}^\star(V^\vee) = \bigoplus_{d\geq 0} \mathrm{Sym}^d(V^\vee)$. We form the algebra

$$\mathbb{C}[V]^G = \bigoplus_{d\geq 0} \mathrm{Sym}^d(V^\vee)^G \tag{1}$$

[1] We make the following abuse of notation: Although the actions will be usually left actions, we always divide from the right, for typographical reasons.

of the functions that are constant on all $G$-orbits in $V$. A fundamental theorem of Hilbert asserts that $\mathbb{C}[V]^G$ is again a finitely generated $\mathbb{C}$-algebra, so that it belongs to an affine algebraic variety which we shall denote by $V/\!\!/_{\varrho}G$. Moreover, the inclusion $\mathbb{C}[V]^G \subseteq \mathbb{C}[V]$ gives a $G$-invariant morphism $\pi\colon V \longrightarrow V/\!\!/_{\varrho}G$. This maps exhibits $V/\!\!/_{\varrho}G$ as the categorical quotient of $V$ by the action of $G$. It has several nice properties. One of them is that $\pi$ maps the set of **closed** $G$-orbits in $V$ bijectively onto $V/\!\!/_{\varrho}G$. Thus, in this case, we may just take $U = V$ as the whole variety.

The representation $\varrho$ also supplies the action

$$\begin{array}{rcl} \overline{\varkappa}\colon G \times P(V) & \longrightarrow & P(V) := \big(V \setminus \{0\}\big)/\mathbb{C}^\star \\ \big(g, [v]\big) & \longmapsto & \big[\varrho(g)(v)\big]. \end{array}$$

As we have implicitly observed in (1), the algebra $\mathbb{C}[V]^G$ is graded. It therefore defines a projective variety $P(V)/\!\!/_{\varrho}G$. This time, the inclusion $\mathbb{C}[V]^G \subseteq \mathbb{C}[V]$ of graded algebras gives rise to a $G$-invariant **rational map** $\overline{\pi}\colon P(V) \dashrightarrow P(V)/\!\!/_{\varrho}G$. The map $\pi$ is defined in the point $x = [v]$, if there is a non-zero **homogeneous** function $f \in \mathbb{C}[V]^G$ of positive degree with $f(v) \neq 0$. Such a point is said to be *$\varrho$-semistable*. The set $P(V)^{\varrho\text{-ss}}$ of $\varrho$-semistable points in $P(V)$ is open and $\varrho$-invariant, and the morphism $\pi\colon P(V)^{\varrho\text{-ss}} \longrightarrow P(V)/\!\!/_{\varrho}G$ exhibits $P(V)/\!\!/_{\varrho}G$ as the categorical quotient of $P(V)^{\varrho\text{-ss}}$ by the induced $G$-action. Hence, we take $U = P(V)^{\varrho\text{-ss}}$ in this case. In general, $U$ will be a proper subset. However, its categorical quotient is a **projective** variety. It is clearly an important task to characterize the semistable points with a handy criterion. This is the so-called **Hilbert–Mumford criterion**. (This criterion is the main reason for the success of GIT in applications. It is still lacking in its strong form in the recent generalizations of GIT such as [99].)

There are two things noteworthy here: The question of determining the invariant ring $\mathbb{C}[V]^G$ with respect to the action of $G := \mathrm{SL}_{n+1}(\mathbb{C})$ on $V := \mathbb{C}[x_0, \dots, x_n]_d$ which we have introduced above was the topic of classical invariant theory. Hilbert managed to prove the finite generation of the invariant ring and the Hilbert–Mumford criterion in precisely that set-up. Along the way, he discovered his most famous results in commutative algebra, such as the Nullstellensatz. (The reader may have a look at the lecture notes [107].) The second point is that almost everything (especially in applications) is reduced in one way or another to the above results.

As the main abstract (i.e., isolated from applications) results of GIT in which we are interested we note the following: Given a representation $\varrho\colon G \longrightarrow \mathrm{GL}(V)$, leading to the actions $\varkappa\colon G \times V \longrightarrow V$ and $\overline{\varkappa}\colon G \times P(V) \longrightarrow P(V)$, GIT provides us with:

- The categorical quotient $\pi\colon V \longrightarrow V/\!\!/_{\varrho}G$.
- The $G$-invariant open subset $U := P(V)^{\varrho\text{-ss}}$ and the categorical quotient $\overline{\pi}\colon U \longrightarrow P(V)/\!\!/_{\varrho}G$.
- A characterization of $U$ by means of the Hilbert–Mumford criterion.

It is the aim of this book to provide a generalization of the above abstract results to a relative setting. To formulate it, let $X$ be a connected smooth projective curve over the complex numbers. The input datum for our theory will be again a representation

$\varrho\colon G \longrightarrow \mathrm{GL}(V)$ of the reductive group $G$ on the finite dimensional $\mathbb{C}$-vector space $V$. If $\mathscr{P} \longrightarrow X$ is any principal fiber bundle with structure group $G$, then we may associate to it a vector bundle $\mathscr{P}_\varrho$ with fiber $V$, using the representation $\varrho$. Thinking in terms of cocycles, $\mathscr{P}_\varrho$ is glued together with a cocycle in $\mathrm{GL}(V)$ which is the image under $\varrho$ of a cocycle in $G$ which gives $\mathscr{P}$. According to the actions $\varkappa\colon G \times V \longrightarrow V$ and $\overline{\varkappa}\colon G \times P(V) \longrightarrow P(V)$, we want to classify *affine* and *projective $\varrho$-pairs*. The former objects are pairs $(\mathscr{P}, \sigma)$ which consist of a principal $G$-bundle $\mathscr{P}$ on $X$ and a section $\sigma\colon X \longrightarrow \mathscr{P}_\varrho$ and the latter objects are pairs $(\mathscr{P},\beta)$ composed of a principal $G$-bundle $\mathscr{P}$ on $X$ and a section $\beta\colon X \longrightarrow P(\mathscr{P}_\varrho) := (\mathscr{P}_\varrho \setminus \{\text{zero section}\})/\mathbb{C}^\star$. These objects may be viewed as families of points $v \in V$ and $x \in P(V)$ varying over $X$ in the way that vector bundles on $X$ may be considered as families of vector spaces varying over $X$: The bundle $\mathscr{G} := \mathscr{A}ut(\mathscr{P}) \longrightarrow X$ of (local) automorphisms of $\mathscr{P} \longrightarrow X$ is a group scheme over $X$, i.e., there are maps $e_X\colon X \longrightarrow \mathscr{G}$, the *neutral section*, $i_X\colon \mathscr{G} \longrightarrow \mathscr{G}$, the *inversion map*, and $m_X\colon \mathscr{G} \times_X \mathscr{G} \longrightarrow \mathscr{G}$, the *multiplication map*, of **varieties over** $X$, such that the diagrams expressing the group axioms for these operations do commute. The fibers of $\mathscr{G}$ over $X$ are affine algebraic groups which are isomorphic to $G$. Furthermore, there are the actions

$$\varkappa_X\colon \mathscr{G} \underset{X}{\times} \mathscr{P}_\varrho \longrightarrow \mathscr{P}_\varrho$$

and

$$\overline{\varkappa}_X\colon \mathscr{G} \underset{X}{\times} P(\mathscr{P}_\varrho) \longrightarrow P(\mathscr{P}_\varrho).$$

There are obvious isomorphy relations on the classes of affine and projective $\varrho$-pairs. As usual, there are some natural discrete data to be considered. To this end, we look at $X$ as a compact Riemann surface, i.e., as a complex manifold and eventually as an oriented topological manifold. Denote by $\Pi(G)$ the set of isomorphy classes of topological principal $G$-bundles on $X$. (If $G$ is connected, then $\Pi(G)$ can be identified with the fundamental group $\pi_1(G)$.) Then, to each (algebraic) principal $G$-bundle can be assigned a class in $\Pi(G)$, its *topological type*. If we fix $\vartheta \in \Pi(G)$, then the topological vector bundle $\mathscr{P}_\varrho$ does not depend on the principal $G$-bundle $\mathscr{P}$ of topological type $\vartheta$. Thus, it makes sense to speak about the cohomology class $[\beta(X)] \in H^\star(P(\mathscr{P}_\varrho), \mathbb{Z})$ of the section $\beta$. This class naturally identifies with an integer $l \in \mathbb{Z}$.

The program which will be carried out in the present book is the following:

- We first formulate a notion of semistability for affine and projective $\varrho$-pairs (which will depend on several parameters). This is, so to say, the Hilbert–Mumford criterion for $\varrho$-pairs.

- Having fixed the topological background data $\vartheta \in \Pi(G)$ and $l \in \mathbb{Z}$ as well as the stability parameters, we construct the moduli space $\mathscr{M}(\varrho, \vartheta, l)$ for semistable projective $\varrho$-bumps[2] of topological type $(\vartheta, l)$ as a projective scheme over $\mathbb{C}$.

- For fixed $\vartheta \in \Pi(G)$ and chosen stability parameters, we construct the moduli space $\mathscr{M}(\varrho, \vartheta)$ for semistable affine $\varrho$-pairs $(\mathscr{P}, \sigma)$, where $\mathscr{P}$ has topological type $\vartheta$, as a quasi-projective scheme of finite type over $\mathbb{C}$. It comes with a projective map to an affine space.

[2] Certain generalizations of $\varrho$-pairs needed for getting projective, i.e., compact moduli spaces.

These results are clearly formal generalizations of the results of GIT which we have considered above: In fact, replacing the base manifold $X$ by a point, we recover the results which we had declared before to be the most interesting ones. Note the important difference that we have to define a priori what the semistable objects are. The reason is that there is no scheme of finite type over $\mathbb{C}$ which parameterizes the isomorphy classes of affine or projective $\varrho$-pairs, even if we fix the topological background data. This "unboundedness" phenomenon is familiar from the theory of vector bundles. Thus, we first have to single out a bounded family of affine or projective $\varrho$-pairs. This is what the notion of semistability first does for us. Having a bounded family, standard methods may be applied to construct a parameter space $\mathfrak{B}$ which parameterizes isomorphy classes of affine or projective $\varrho$-pairs in such a way that any isomorphy class from the bounded family under inspection does correspond to a point in $\mathfrak{B}$. The parameter space comes also with an action of a general linear group $\mathrm{GL}(Y)$. Thus, we are in the setting of GIT as we have described before. The final point to be checked is that the Hilbert–Mumford criterion that comes from GIT agrees with our "Hilbert–Mumford criterion", i.e., the notion of semistability. The problem here is that our notion is intrinsic in terms of the $\varrho$-pair whereas the notion coming from GIT depends on many unnatural choices. The hard part of the work really is to arrange everything in such a way that this last step works out nicely: Whereas it is comparably easy to establish for $\mathrm{GL}_n(\mathbb{C})$ and projective $\varrho$-pairs (restricting to homogeneous representations), more and more elaborate tricks are necessary to pass via $\mathrm{GL}_{n_1}(\mathbb{C}) \times \cdots \times \mathrm{GL}_{n_t}(\mathbb{C})$ to arbitrary reductive groups and from projective to affine $\varrho$-pairs. It is the main aim of these notes to present these methods in a fairly self-contained way. It should be noted that these results seem to be completely new for reductive groups other than products of general linear groups and, in the case of affine $\varrho$-pairs in the above generality, also for $\mathrm{GL}_n(\mathbb{C})$. In the end, we see that our results are certainly a formal generalization of GIT but also an application of it.

After this outline of the main achievements of this monograph, we will look at potential applications. Let us have a brief glance at the case of principal bundles without extra structures. The best known reductive linear algebraic groups are automorphism groups of certain algebraic or geometric structures. The general linear group $\mathrm{GL}_n(\mathbb{C})$ is the group of linear automorphisms of $\mathbb{C}^n$. This observation makes the notion of a principal $\mathrm{GL}_n(\mathbb{C})$-bundle equivalent to the more familiar notion of a vector bundle of rank $n$. Likewise, the fact that $\mathrm{PGL}_{n+1}(\mathbb{C})$ is the group of automorphisms of the projective space $\mathbb{P}_n$ shows that the notion of a principal $\mathrm{PGL}_{n+1}(\mathbb{C})$-bundle is equivalent to the notion of a $\mathbb{P}_n$-bundle over $X$. Now, $\mathbb{P}_n$-bundles over $X$ are examples of smooth projective manifolds, and their classification up to isomorphy over the base manifold $X$ is equivalent to the classification of principal $\mathrm{PGL}_{n+1}(\mathbb{C})$-bundles over $X$. Thus, this special case relates to the above guiding problem in Algebraic Geometry. (Admittedly, since any projective bundle over the curve $X$ is the projectivization of a vector bundle, the classification of $\mathbb{P}_n$-bundles over $X$ can be expressed in terms of vector bundles as, e.g., in Section V.2 of [96]. The formalism of $\mathrm{PGL}_{n+1}(\mathbb{C})$-bundles is, however, the right framework when thinking about higher dimensional base varieties.) In a similar spirit, one can treat the classification of divisors in projective bundles, with respect to isomorphy which respects the embedding into the projective bundle and the map onto $X$, as the classification of certain projective $\varrho$-pairs for the group $\mathrm{GL}_n(\mathbb{C})$.

There is, however, a more exiting construction which gives a whole new horizon of applications: Treating $X$ again as a topological manifold, we define its fundamental group $\pi_1(X)$. We may equip $\pi_1(X)$ with the discrete topology and view it as a complex Lie group. In this way, the universal covering $\widetilde{X} \longrightarrow X$ becomes a holomorphic principal $\pi_1(X)$-bundle. Choosing an appropriate open covering of $X$ in the strong topology, $\widetilde{X}$ is thus determined by a cocycle with values in $\pi_1(X)$. (Note that this cocycle is locally constant.) Next, we may give ourselves a representation $\psi: \pi_1(X) \longrightarrow G$. Then, we use $\psi$ to transfer the cocycle from $\pi_1(X)$ to $G$. This new cocycle is the gluing datum for a holomorphic principal $G$-bundle $\mathscr{P}^\psi$ on $X$. By Serre's GAGA theorems (see [194] and [195]), it is associated to an algebraic principal $G$-bundle which we also call $\mathscr{P}^\psi$. A classical theorem of Narasimhan and Seshadri [158] asserts that the assignment $\psi \longmapsto \mathscr{P}^\psi$ induces a bijection between the set of equivalence classes of irreducible representations $\psi: \pi_1(X) \longrightarrow U_n(\mathbb{C})$ and isomorphy classes of stable vector bundles of degree 0. It even yields a homeomorphism between the corresponding moduli spaces. The representations are classified by a real analytic moduli space whereas the moduli space of stable bundles is a smooth quasi-projective variety. Thus, we have found an algebro geometric model for a topological space which has been defined in terms of the topology of $X$ only. The tools of Algebraic Geometry become in this way available to study questions of purely topological nature. The generalization of this theorem to other reductive groups is due to Ramanathan [174]. Donaldson has interpreted these results in the framework of the Kobayashi–Hitchin correspondence [58]. Now, the Kobayashi–Hitchin correspondence has been widely extended (see, e.g., [13], [156], and [138]). It relates, among other things, semistable affine and projective $\varrho$-pairs to solutions of certain vortex type equations. The moduli spaces which we have obtained will be again models for some gauge theoretically defined topological spaces. Thus, our results fill an apparent gap in the literature. The work of Hitchin, Donaldson, Simpson, and Corlette also brings us back to studying representations (the introduction to [38] gives a better account of this and precise references): Representations of the fundamental group $\pi_1(X)$ in (non-compact) real forms of a reductive linear algebraic group $G$, such as $U(p,q)$ for $\mathrm{GL}_{p+q}(\mathbb{C})$, lead to interesting algebro geometric objects, and our results provide moduli spaces. In various situations, where moduli spaces were known before, these constructions were exploited to gather information on the moduli spaces of representations (topological spaces defined in terms of the topology of $X$) by studying their algebro geometric models (see, e.g., [38] and [39]). Looking at Table 3 in [40], we see that some other moduli spaces among those which we construct here for the first time will become important, too.

### Detailed Content of the Book

The first chapter is an introduction to Geometric Invariant Theory (GIT) as developed by Mumford in his famous book [155]. As mentioned earlier in the introduction, we will mainly deal with actions of a reductive linear algebraic group $G$ on a vector space $V$ or a projective space $P(V)$ by means of a representation $\varrho: G \longrightarrow \mathrm{GL}(V)$. In order to be reasonably self-contained, we will first review the theory of linear algebraic groups and their representations on finite dimensional vector spaces. The crucial notions are irreducible and completely reducible representations. Since, in this book, we will always be working over the field of complex numbers, we may introduce reductive linear algebraic groups as those groups for which all finite dimensional representations are completely reducible. As a first illustration, we verify that finite groups and tori, e.g., $\mathbb{C}^\star$, are reductive linear algebraic groups. In the appendix to Section 1.1, we present a method due to Hermann Weyl which allows to check that the classical groups such as $\mathrm{GL}_n(\mathbb{C})$ and $\mathrm{SL}_n(\mathbb{C})$ are reductive. Then, it also follows that $\mathrm{GL}_{n_1}(\mathbb{C}) \times \cdots \times \mathrm{GL}_{n_t}(\mathbb{C})$ is reductive. This is a very important fact, because almost all the actions at which we will look in the applications are induced by representations of products of general linear groups. Subsections 1.1.5 and 1.1.6 discuss some more specific topics on representations which are mainly technical tools for the moduli space constructions in the second chapter. The main references for this section are the books by Borel [30] and Kraft [123].

In Section 1.2, we start to look at the problem of forming the quotient of a vector space by the action of an algebraic group $G$ via a representation $\varrho\colon G \longrightarrow \mathrm{GL}(V)$. First, we observe that we cannot parameterize the $G$-orbits in $V$ in any useful way by an algebraic variety, if there are non-closed orbits. Then, we derive how the globally defined regular functions on any quotient should look like. A classical theorem by Hilbert states that the ring of these functions is finitely generated, if the group $G$ is reductive. Thus, we get an affine algebraic variety as potential quotient. We formulate the basic properties of this quotient: Basically it is as good as it can be (keep in mind that $G$ must be reductive). Finally, we present the important notions of stable and semistable points and of nullforms. The nullforms are precisely those points in $V$ which map to the same point in the quotient as $0 \in V$. (The nullforms have to be "thrown away", if one wants to form the quotient of $P(V)$!) The Hilbert–Mumford criterion tells us how to detect the stable and semistable points (whence also the nullforms).

Before proceeding to the proofs of the fundamental theorems stated in Section 1.2, we would like to see them in action. This happens in Section 1.3. We will speak there about some specific representations which were studied in the classical literature on invariant theory. The most prominent one is the representation of $\mathrm{SL}_n(\mathbb{C})$ on algebraic forms of degree $d$ in $n$ variables. In a more geometric language, one studies here the classification of projective hypersurfaces of degree $d$ in the projective space $\mathbb{P}_{n-1}$ up to projective equivalence. We will evaluate the Hilbert–Mumford criterion in several examples and also compute the invariant rings in some situations, or, equivalently describe the resulting quotient. Another instructive example is the action of $\mathrm{GL}_n(\mathbb{C})$ on $(n \times n)$-matrices by conjugation. Here, one can explicitly determine the quotient and compare it with the Jordan normal form. The reader should study this example very carefully and reflect what it tells us about properties of the quotient. Interesting generalizations arise when studying the action of $\mathrm{GL}_n(\mathbb{C})$ on tuples of matrices or, more generally, quiver representations. Here, it is in general impossible to obtain a complete list of normal forms. Consequently, it is more interesting to find out as much about the quotient as possible.

Section 1.4 is devoted to the fundamental concepts of GIT. In order to correctly appreciate GIT quotients, we first define good and geometric quotients according to Mumford. The defining properties are certain natural requirements on a quotient which have two important consequences: First, a good quotient is also a quotient in the categorical sense. Second, a good quotient can be patched up from affine quotients. Hence, we first study the theory of affine quotients. This means we will first prove Hilbert's theorem on the finite generation of the ring of invariants. Then, we check that the quotient we thus obtain is really a good one in the sense of Mumford. Here, we have followed the exposition in the book by Dieudonné and Carrell [53]. If $\varrho\colon G \longrightarrow \mathrm{GL}(V)$ is a representation of the reductive group $G$ and if $V^{\mathrm{ss}}$ is the set of semistable points in $V$, then we get the $G$-invariant (open) subset $P(V)^{\mathrm{ss}} := V^{\mathrm{ss}}/\mathbb{C}^\star \subseteq P(V)$. The result of Hilbert grants that we may form the quotient $\pi\colon P(V)^{\mathrm{ss}} \longrightarrow P(V)/\!\!/_\varrho G$. The properties of good quotients imply that this is also a good quotient. The salient feature here is that $P(V)/\!\!/_\varrho G$ is again a projective variety. In order to apply these findings to a wider range of examples, one has to use linearizations. So, assume that $G$ is a reductive linear algebraic group, $X$ is a projective variety, and $\overline{\sigma}\colon G \times X \longrightarrow X$ is an action of $G$ on $X$. A linearization of $\overline{\sigma}$ consists of a representation $\varrho\colon G \longrightarrow \mathrm{GL}(V)$ and a $G$-equivariant closed embedding $\iota\colon X \hookrightarrow P(V)$. We may define $X^{\mathrm{ss}} := P(V)^{\mathrm{ss}} \cap X$. Then, we obtain the following commutative diagram

$$\begin{array}{ccc} X^{\mathrm{ss}} & \hookrightarrow & P(V)^{\mathrm{ss}} \\ {\scriptstyle \pi_X}\downarrow & & \downarrow{\scriptstyle \pi} \\ X/\!\!/_{(\varrho,\iota)}G & \hookrightarrow & P(V)/\!\!/_\varrho G, \end{array}$$

and $X/\!\!/_{(\varrho,\iota)}G$ is a good quotient of $X^{\mathrm{ss}}$ and a projective variety. The choice of a linearization is a parameter in the theory. Note that any given linearization $(\varrho, \iota)$ of $\overline{\sigma}$ may be multiplied by a character $\chi$ of $G$: The linearization $\chi \cdot (\varrho, \iota) := (\varrho_\chi, \iota)$ features the representation $\varrho_\chi\colon G \longrightarrow \mathrm{GL}(V)$, $g \longmapsto \chi(g) \cdot \varrho(g)$. We use this to study (all possible) linearizations of a $\mathbb{C}^\star$-action on a projective space: Let $\lambda\colon \mathbb{C}^\star \longrightarrow \mathrm{GL}(V)$ be a representation. It leads to an action $\overline{\lambda}\colon \mathbb{C}^\star \times P(V) \longrightarrow P(V)$. Of course, $(\lambda, \mathrm{id}_{P(V)})$ is a linearization of $\overline{\lambda}$. Next, we can form $(\lambda^k, \nu_k)$ where $\lambda^k$ is the $k$-th symmetric power of $\lambda$ and $\nu_k$ is the $k$-th Veronese embedding. If $\chi_d\colon \mathbb{C}^\star \longrightarrow \mathbb{C}^\star$ denotes the character $z \longmapsto z^{-d}$, we thus get the family $\chi_d \cdot (\lambda^k, \nu_k)$, $k \in \mathbb{Z}_{>0}$, $d \in \mathbb{Z}$, of linearizations. It is easy to verify that the quotient depends only on the ratio $d/k \in \mathbb{Q}$. A priori, we get an infinite family of possible quotients. However, we can easily determine the semistable points for each linearization and check that we get only a finite number of open subsets which arise as subsets of semistable points with respect to different linearizations and consequently also only finitely many possible quotients. Moreover, it is possible to understand the relationship between different quotients. Although this seems to be only a peculiar example, it is a very important one. Indeed, we will see in Section 1.6 that it has far reaching consequences. For the basic formalism of GIT, we have used the sources [155] and [160].

Before we proceed to Section 1.6, we will prove and study in Section 1.5 the Hilbert–Mumford criterion. It is the main reason for the success of GIT in applications. Note that, so far, we have only an abstract formalism which attaches to a group action on a projective variety and a linearization of that action an open subset of semistable points for which the good quotient exists as a projective variety. With the definition of semistability, it is almost impossible to find the semistable points. On the other hand, using the Hilbert–Mumford criterion, one often gets nice intrinsic characterizations of semistable points. (Recall that we have already studied meaningful examples in Section 1.3.) After presenting Richardson's proof of the Hilbert–Mumford criterion from [24], we will evaluate it in several attractive examples. Notably, we investigate King's example of $\chi$-semistable quiver representations [120]. The last two subsections contain some refined semistability criteria which are useful in some special problems.

In Section 1.6, we address the issue of the linearization as a parameter more seriously. First, we show that, given $G$, $X$, and $\overline{\sigma}: G \times X \longrightarrow X$ as before, there are only finitely many $G$-invariant open subsets which occur as open subsets associated to a linearization of $\overline{\sigma}$. This interesting and important fact was independently obtained by Dolgachev and Hu [57] and by Białynicki-Birula [21]. We give a transcription of Białynicki-Birula's approach to the GIT setting, originally published in [185]. Next, we would like to understand how the quotients to two different linearizations of a given action are related. To this end, we discuss the master space construction of Thaddeus [214] and use the semistability criteria from Section 1.5.3 to reduce to the case of a $\mathbb{C}^\star$-action with which we are already familiar. This is a simplified version of the results in [57] and [214].

The final section of the first chapter is devoted to a certain refinement of the Hilbert–Mumford criterion: Look again at a representation $\varrho: G \longrightarrow \mathrm{GL}(V)$ and at a nullform $v \in V$. By the Hilbert–Mumford criterion, there is a one parameter subgroup $\lambda: \mathbb{C}^\star \longrightarrow G$, such that $\lim_{z\to\infty} \lambda(z) \cdot v = 0$. The question is whether one can find a one parameter subgroup with this property, such that the convergence to zero is the "fastest possible" and "how unique" this one parameter subgroup is. The solution is due to Bogomolov [29], Hesselink [105], [106], Kempf [118], and Rousseau [181]. Their uniqueness result is that the parabolic subgroup $Q$ associated to $\lambda$ is unique. An essential consequence of this result is that the Hilbert–Mumford criterion remains true over non-algebraically closed fields of characteristic zero. Another useful application is due to Ramanan and Ramanathan [173] who associate to the nullform $v$ a point $[v_\infty] \in P(V)$ which is semistable for the action of a Levi subgroup $L$ of the instability parabolic subgroup $Q$ for the canonical linearization modified by a certain character $\chi$. All these results are exposed with almost no proof, following the paper [173]. Finally, we mention a result by the author [189] on the instability one parameter subgroup for an unstable point in a product $P(V_1) \times P(V_2)$. The content of Section 1.8 is crucial for many of the constructions of Chapter 2.

We have written Chapter 1 entirely in the language of complex algebraic varieties in order to make it accessible to a large audience. As prerequisite, a good acquaintance with Chapter I of Hartshorne's "Algebraic Geometry" [96] should suffice. The reader who is familiar with the theory of schemes will have no trouble in extending all the results to the setting of schemes of finite type over $\mathbb{C}$. Indeed, the results will be used in that framework in the second chapter. The foundations of GIT were, of course, put down in Mumford's book. His book is, however, considered to be rather technical. More user friendly accounts have been given since, including [160], [170], [56], and [153]. The reader may replace or complement some sections with these references. The main novelty of our exposition is the elementary discussion of the finiteness of the number of different quotients for the same action and the variation of the quotients. Moreover, the results of Section 1.8 do not seem to have been included in text books either. Finally, we have tried to highlight some phenomena or facts which have counterparts in the theory of moduli spaces which will be developed in the second chapter.

The second chapter is more in the style of a research monograph. The reader will need here some familiarity with the theory of schemes. Again, Hartshorne's book will amply suffice. However, it would be very useful, if the reader had some ideas about the concept of moduli spaces or spaces representing certain functors. A basic example of such a moduli space is projective $n$-space or more generally the projectivization of a vector bundle. Its universal property is given in [96], Chapter II, Proposition 7.12. An important generalization of this example are Graßmannians (see Lecture 6 in [95]). The reader who has mastered the example of Graßmannians is well-prepared for all kinds of parameter and moduli spaces which he or she will encounter in our book. Additional introductions to the concept of a moduli space are contained in [160] and Lecture 4 and 21 of [95]. Together with Chapter 1, these prerequisites should be sufficient for attacking Chapter 2.

Section 2.1 introduces the classification problem whose solution will occupy the rest of Chapter 2. In order to properly state it, we need the basic notions of the theory of principal bundles. Since there is no standard textbook which covers this theory, we will give a brief account of this theory, following an exposition of Serre [195].

In Section 2.2, we discuss or review the theory of vector bundles on complex algebraic curves. The reader should be aware that there are several excellent introductions to this topic, including [160], [135], [116], and, for bundles of rank two, [153]. To begin with, we will present the classification of topological vector bundles on a smooth projective curve. Then, we state the Riemann–Roch theorem for coherent sheaves and reduce it to the familiar case of line bundles. Section 2.2.3 discusses the crucial notion of a bounded family of vector bundles. Boundedness is a necessary condition for constructing moduli spaces. Unfortunately, the family of vector bundles of fixed topological type is **not** bounded. We will give numerical extra conditions which ensure boundedness. One of the possible conditions is the famous semistability. In that section, we will also present Grothendieck's quot scheme. Using it, we reduce the classification problem for semistable vector bundles of given topological type to the problem of forming the quotient of a certain quasi-projective variety by the action of a reductive linear algebraic group. In Section 2.2.4, we will sketch the exact procedure how the techniques from Chapter 1 are applied.

The hard work will begin in Section 2.3. We address the solution of the classification problem in case the structure group is a general linear group and the representation is a homogeneous one. This is the core also for the subsequent constructions: We will devise various tricks and methods in order to reduce everything to it. After reviewing the classification problem, we give an example how, for a concrete choice of the representation, the general problem specializes to the classification of interesting algebraic varieties. Then, we give the notion of semistability. This time, it will depend on a parameter, namely, a positive rational number. We have included two examples where the semistability concept comes in an easier form. Afterwards, the construction of the moduli space begins. We first check the boundedness of semistable objects, using the criterion from Section 2.2.3. Then, we construct the parameter space with its group action and evaluate the Hilbert–Mumford criterion. The latter is the hardest part. It origins from our paper [187] and is a refinement of techniques developed by Simpson [204] and Huybrechts and Lehn [114], [115]. After having constructed the moduli spaces, we describe (without proof) two basic geometric properties they enjoy under certain conditions. Section 2.3 concludes with the chain of moduli spaces. This parallels the finiteness of GIT quotients from Chapter 1. We briefly mention the work of Thaddeus [213] (without explicitly addressing the Verlinde formula).

The following section on moduli spaces of principal $G$-bundles should be regarded as a first highlight of this monograph. There, we give a full construction of the moduli space of semistable principal $G$-bundles with connected reductive structure group. Although these moduli spaces were constructed over 30 years ago by Ramanathan [175], [176], they haven't been treated in textbooks, so far. The approach we will present origins from the papers [186] and [188]. The basic idea is to use the results on decorated vector bundles. In order to do so, one has to describe principal $G$-bundles as vector bundles with additional structures. In Section 2.4.1, we will take the first steps in that direction. Moreover, we will discuss the notion of semistability, the concept of S-equivalence and polystability, and state the main theorem on the existence of moduli spaces. In the subsequent section, we study some GIT problems which naturally appear in our approach. Afterwards, we introduce pseudo $G$-bundles. These are certain generalizations of principal $G$-bundles. The advantage is that one can easily associate to a pseudo $G$-bundle a decorated vector bundle. This gives a notion of semistability which depends on a parameter $\delta$ and makes the construction of the moduli space for $\delta$-semistable pseudo $G$-bundles as a **projective** variety fairly easy. The "miracle" is that the moduli space thus obtained is the moduli space of semistable principal $G$-bundles. After presenting the proofs of these assertions, we give a brief survey on the literature on moduli spaces of principal $G$-bundles. In three appendices, we have collected some remarks on the moduli stack, a sketch of the construction of the moduli space for non-semisimple reductive structure groups, and a verification of the fact that our notion of semistability indeed coincides with Ramanathan's.

Section 2.5 is dedicated to the structure group $G := \mathrm{GL}_{r_1}(\mathbb{C}) \times \cdots \times \mathrm{GL}_{r_1}(\mathbb{C})$. There are two novel and important aspects here: a) The group $G$ has lots of characters which, of course, enter the definition of semistability and b) we have to choose a faithful representation $\varkappa\colon G \longrightarrow \mathrm{GL}(W)$. This introduces even more parameters into the theory. In the first two sections, we conceive some tools in the representation and invariant theory of $G$ which help us develop an efficient formalism for the complicated objects we are dealing with and prepares us for the construction of the moduli spaces. After presenting the moduli problem and the main result on the existence of moduli spaces in Section 2.5.3, we proceed to the construction of the moduli spaces. The faithful representation $\varkappa$ allows us to reduce to the case of decorated vector bundles. The details of the construction are rather tricky and technical. After having constructed the moduli spaces, we study the asymptotic behavior of the semistability concept. Again, the fact that $G$ has many characters gives us various directions in which we can look at the asymptotics. To get from the very general and abstract results to more concrete situations, we discuss our results in the special case of quiver representations. We will see that we obtain a generalization of King's result on moduli spaces of quiver representations to the setting of vector bundles on curves. Specializing even further, we give elements of the theory of holomorphic chains from the paper [2]. In that setting, we can easily describe and study several important phenomena which conjecturally extend to more general quiver problems. In view of the importance of quiver representations in representation theory, we hope that this special case of our construction will have interesting applications in the future. An immediate one concerns the determination of the Betti numbers of some representation spaces of the fundamental group $\pi_1(X)$ of the Riemann surface $X$.

If we want to treat classification problems involving principal $G$-bundles with reductive but non-semisimple structure group, then the techniques of Section 2.4 are not perfectly suited. The approach of that section is based on embeddings of the structure group $G$ into a special linear group $\mathrm{SL}(W)$. Now, $G$ has non-trivial characters but $\mathrm{SL}(W)$ does not. Hence, we cannot extend the characters from $G$ to $\mathrm{SL}(W)$. On the other hand, the characters of $G$ are important parameters for the semistability concept. The way out is to embed $G$ into a group of the shape $H := \mathrm{GL}_{r_1}(\mathbb{C}) \times \cdots \times \mathrm{GL}_{r_1}(\mathbb{C})$. Indeed, one can arrange that any character of $G$ extends to a character of $H$ (at least up to positive multiples). In this way, we may use the results of Section 2.5 in order to treat classification problems for principal $G$-bundles with arbitrary reductive structure group $G$. The set-up will be explained in Section 2.6. Most of the arguments are straightforward generalizations of their counterparts in Section 2.4. One can use these methods to construct moduli spaces for principal $G$-bundles without additional structures (see [80]). Since the details are rather awkward and we already have constructed the moduli spaces in Section 2.4 with different methods, this application is omitted here.

In the following section, we come to the real novelties of this volume. We are now able to cover the classification of principal $G$-bundles, $G$ a (not necessarily connected nor semisimple) reductive linear algebraic group, together with sections in the projective bundle that is associated via a previously fixed representation $\varrho: G \longrightarrow \mathrm{GL}(V)$. Here, there are still some technical assumptions on $\varrho$. In Section 2.7.1, we will define the notion of semistability for the objects under consideration and formulate the main result on the existence of moduli spaces. Afterwards, we will introduce some more general objects, called *decorated pseudo G-bundles*, and define a crude notion of semistability for them. The benefit is that we can harvest the projective moduli spaces for semistable decorated pseudo $G$-bundles from Section 2.5 and Section 2.6. In order to derive the main result of Section 2.7, we have to carefully analyze the behavior of semistability for decorated pseudo $G$-bundles when certain parameters become large. This is done in Section 2.7.2. The conclusion of this analysis is that the moduli spaces of semistable decorated principal $G$-bundles are special examples of the moduli spaces of decorated pseudo $G$-bundles for certain stability parameters. Next, we will also study the asymptotic behavior of the semistability concept for decorated principal $G$-bundles. This is crucial for the results in Section 2.8. As a first application, we will explain in Section 2.7.4 how we obtain the moduli space of semistable Higgs bundles together with a natural compactification as an example of the general result. In Section 2.7.5, we address the subtle point of representations $\varrho: G \longrightarrow \mathrm{GL}(V)$ whose kernel contains a positive dimensional central torus. Here, the notion of semistability may be relaxed a bit.

Section 2.8 finally presents the main result of this monograph: The semistability concept and the moduli spaces for principal $G$-bundles which are decorated by a section in the vector bundle that is associated via a previously fixed representation $\varrho: G \longrightarrow \mathrm{GL}(V)$. Here, there is absolutely no restriction on $\varrho$. Again, we will first introduce the notion of semistability and state the result on the existence of moduli spaces. This result will be reduced to the main technical result from Section 2.7. In order to do so, we have to cook up a homogeneous representation $\widetilde{\varrho}: G \longrightarrow \mathrm{GL}(\widetilde{V})$ from $\varrho$. Then, we can associate, for any principal $G$-bundle $\mathscr{P}$, to a section $\sigma: X \longrightarrow \mathscr{P}_\varrho$ a section $\beta: X \longrightarrow P(\mathscr{P}_{\widetilde{\varrho}})$. We check that this assignment is finite-to-one on the isomorphy classes and compatible with semistability. Therefore, we can use this assignment to construct the moduli spaces for semistable affine $\varrho$-pairs $(\mathscr{P}, \sigma)$ from those for semistable projective $\widetilde{\varrho}$-pairs $(\mathscr{P},\beta)$. An important point here is that we do not expect our moduli spaces to be projective in general. Instead, they should be projective over an affine variety which depends on the GIT-quotient $V/\!\!/_\varrho G$ via a generalized Hitchin map. After the construction of the moduli spaces, we will discuss in Section 2.8.4 some extensions and examples of our general results. In particular, we will show how we can remove the technical assumption in Section 2.7, how we recover the moduli space of Bradlow pairs, and how we get, as a new example, moduli spaces of Higgs bundles for **real** reductive groups. At the end, we will again discuss representations $\varrho: G \longrightarrow \mathrm{GL}(V)$ whose kernel contains a positive dimensional central torus.

The proofs leading to the main results of Chapter 2 are already very technical and lengthy. Still, one might ask for even more general results. The following two directions of generalization seem very natural: a) Extend the results to base fields of positive characteristic; b) Extend the results to base varieties of higher dimensions. A more specialized extension c) asks for equipping the vector and principal bundles with parabolic structures. Objects of this kind arise in connection with the investigation of representations of the fundamental group of an **open** Riemann surface in a connected real or complex reductive group. In Section 2.9, we will explain what we know about these potential extensions. In positive characteristic, the business becomes very complicated, if principal $G$-bundles with non-classical structure groups are involved and there the theory is still in its beginnings. Over smooth higher dimensional base varieties over a field of characteristic zero, the results are as general as over curves. There are only some fine points to be observed. Finally, on a curve over the field $\mathbb{C}$, the introduction of parabolic structures poses no problem at all.

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### Notation and Conventions

**General. —** We have tried to follow the standard terminology of Algebraic Geometry such as in Hartshorne’s book [96]. The ground field will be $\mathbb{C}$ unless otherwise specified. In the first chapter, the reader may think of varieties in the classical sense, i.e., in the one defined in Chapter I of [96]. We write $\mathbb{E}_n$ for the unit $(n \times n)$-matrix.

**Sets. —** If $S$ is a set and $n$ a positive integer, we write $S^{\times n}$ for the $n$-fold cartesian product $S \times \cdots \times S$. We write {pt} for a set containing exactly one element.

**Categories. —** A *groupoid* is a category in which all morphisms are isomorphisms.

**Schemes and varieties. —** A *scheme* will be a scheme of finite type over the complex numbers. A *variety* is a scheme which is reduced and irreducible. For a cartesian product $X \times Y = X \times_{\operatorname{Spec}(\mathbb{C})} Y$ of schemes, we let $\pi_X : X \times Y \longrightarrow X$ and $\pi_Y : X \times Y \longrightarrow Y$ be the projections onto the first and the second factor, respectively. If $X$ is a scheme or a variety and $W$ is any **subset**, then $\overline{W}$ stands for its closure in the Zariski topology (with its induced reduced scheme structure). If $X$ is a projective scheme and $\mathscr{F}$ is a coherent $\mathscr{O}_X$-module, the cohomology groups of $\mathscr{F}$ are finite dimensional $\mathbb{C}$-vector spaces, and we set $h^i(\mathscr{F}) := h^i(X, \mathscr{F}) := \dim_{\mathbb{C}}(H^i(X, \mathscr{F}))$, $i \geq 0$. An open subset $U \subset X$ of the variety $X$ is said to be *big*, if the complement $X \setminus U$ has codimension at least two in $X$.

**Algebraic groups. —** In the standard reference [30], the theory of reductive groups is developed only for **connected** groups. We will slightly deviate from this: A reductive group need not be connected. (This allows to include the orthogonal groups.) However, we require a **semisimple group** to be connected, so that a semisimple group does not have any non-trivial character.

**Vector bundles . —** By a standard abuse of language, we do not distinguish between vector bundles (geometric objects) and locally free sheaves (see [96], Exercise II.V.18): A geometric vector bundle is identified with its sheaf of sections. Recall that the projective bundle $\mathbb{P}(E)$ associated to a vector bundle $E$ on the variety $X$ is $\mathscr{P}roj(\mathscr{S}ym^\star(E))$, i.e., it is the bundle of hyperplanes in the fibers of $E$ or, equivalently, lines in the fibers of the dual vector bundle $E^\vee$. This applies, in particular, to vector spaces, so that $\mathbb{P}(V)$ stands for $(V^\vee \setminus \{0\})/\mathbb{C}^\star$, $V$ being a finite dimensional $\mathbb{C}$-vector space. Occasionally, we will also use $P(E) := (E \setminus \text{zero section})/\mathbb{C}^\star \cong \mathbb{P}(E^\vee)$ for the projective bundle of lines in the fibers of $E$.

If $E$ is a vector bundle on a curve and $\mathscr{F} \subset E$ is a **subsheaf**, then

$$F := \ker\Big(E \longrightarrow (E/\mathscr{F})/\mathrm{Tors}(E/\mathscr{F})\Big)$$

is a **subbundle** of $E$ which coincides with $\mathscr{F}$ in all but finitely many points. We will refer to $F$ as the *subbundle generated by* $\mathscr{F}$. (Note that we have $\deg(F) \geq \deg(\mathscr{F})$.)

**Semistability conditions .** — If a certain object such as a one parameter subgroup or a subbundle occurs in a definition of semistability, we will always assume that it is non-trivial. (This prevents us from defining stable objects in a way that they will never exist.)