Preface

As a student of physics, I was naturally attracted to theoretical physics, including constructive quantum field theory. What could, indeed, be more exciting than working on cutting-edge theories dealing with the fundamental laws of the universe? From a mathematical point of view, however, I soon got the impression that the situation was rather unsatisfactory. Many recent theories appeared to have a mathematical justification that was shady at best, and there were even result, jokingly referred to as "destructive quantum field theory", that showed that some of these models could not possibly make any sense mathematically.

Eventually, I focused on another favourite subject of my student years, namely the theory of dynamical systems. One very appealing aspect of that theory is that it comes with a kind of road map, providing well-defined first steps that can be applied to nearly any dynamical system, be it an ordinary differential equation or an iterated map. Since existence of solutions is usually ensured by classical results, one can immediately proceed to a qualitative analysis of the solutions. Usually, one starts by searching stationary points and periodic orbits of the system. Linearising the equation around these solutions yields information on their stability and other aspects of the local behaviour. Then there exists a whole set of tools to analyse the effect of nonlinear terms, including normal forms, the center-manifold theorem, and bifurcation theory. Beyond these first steps in the analysis, for most systems there remain many questions open to further study by more refined methods.

After my Ph.D, I started working on stochastic differential equations, which allow to model the effect of noise on dynamical systems. While there were also many tools for analysing these systems, partly similar to those available for ordinary differential equations, they were not yet as well developed, leaving numerous opportunities for improvements. Eventually, this led me to the field of stochastic partial differential equations (SPDEs), for which general methods yielding a good understanding of the long-term dynamics were even less advanced than for stochastic differential equations. SPDEs are of course also interesting because of their many applications, in various fields such as climate modelling, neuroscience, and statistical physics.

It was only when Martin Hairer obtained the Fields medal in 2014 for his theory of regularity structures that I realised that some SPDEs actually have strong connections with quantum field theory. Indeed, one of the examples worked out in detail by Martin in his landmark paper "A theory of regularity structures" is the Φ^4 model that originated in bosonic quantum field theory. Hence these exciting new developments suddenly revealed a host of new possibilities in making mathematically rigourous many models that were previously ill-defined. In truth, this recent progress did not come out of the blue: it built on a long series of previous results, involving the theory of renormalisation, Feynman diagrams, calculus in spaces of distributions, and the theory of rough paths. In fact, some researchers had never stopped working on the mathematical foundations of quantum field theory, but this work took time to bear fruit.

In the last few years, the theory of so-called singular SPDEs (that is, those that require a renormalisation procedure to make sense mathematically) has experienced tremendous progress. Once results on well-posedness and local existence of solutions had been obtained, new, more quantitative results on global existence, invariant measures, and convergence towards such measures started appearing as well, gradually making the theory of singular SPDEs quite complete. Still, some aspects of the theory may appear daunting to the non-specialist, as they combine tools from different fields such as probability theory, stochastic analysis, functional analysis, and algebraic methods.

The aim of this monograph is to present a rather gentle introduction to the subject, by focusing on a specific example, the Allen–Cahn equation on a torus in dimensions 1, 2 and 3. Each dimension presents new challenges for the analysis, which are introduced in an incremental way. This text also includes a number of results going beyond existence and uniqueness of solutions, discussing invariant measures, convergence to these measures, and non-equilibrium properties such as metastability.

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