

# Contents

|  |     |
|--|-----|
| <b>Preface</b> . . . . .                                     | vii |
| <b>1 Introduction</b> . . . . .                              | 1   |
| 1.1 Stochastic partial differential equations . . . . .      | 1   |
| 1.2 Singular SPDEs . . . . .                                 | 3   |
| 1.3 Regularity structures . . . . .                          | 6   |
| 1.4 About this book . . . . .                                | 7   |
| <b>2 A system of interacting diffusions</b> . . . . .        | 13  |
| 2.1 Deterministic dynamics . . . . .                         | 13  |
| 2.2 Existence and uniqueness of solutions . . . . .          | 16  |
| 2.3 Invariant measure and reversibility . . . . .            | 17  |
| 2.4 Large deviations . . . . .                               | 19  |
| 2.5 Metastability . . . . .                                  | 20  |
| Bibliographical notes . . . . .                              | 31  |
| <b>3 Allen–Cahn SPDE in one space dimension</b> . . . . .    | 33  |
| 3.1 Deterministic dynamics . . . . .                         | 34  |
| 3.2 Space-time white noise . . . . .                         | 37  |
| 3.3 The stochastic heat equation . . . . .                   | 41  |
| 3.4 Existence and uniqueness of solutions . . . . .          | 47  |
| 3.5 Invariant measure . . . . .                              | 50  |
| 3.6 Large deviations . . . . .                               | 54  |
| 3.7 Metastability . . . . .                                  | 55  |
| Bibliographical notes . . . . .                              | 61  |
| <b>4 Allen–Cahn SPDE in two space dimensions</b> . . . . .   | 63  |
| 4.1 The need for renormalisation . . . . .                   | 63  |
| 4.2 Wick calculus . . . . .                                  | 64  |
| 4.3 Existence and uniqueness of solutions . . . . .          | 74  |
| 4.4 Invariant measure . . . . .                              | 75  |
| 4.5 Large deviations . . . . .                               | 79  |
| 4.6 Metastability . . . . .                                  | 80  |
| Bibliographical notes . . . . .                              | 84  |
| <b>5 Allen–Cahn SPDE in three space dimensions</b> . . . . . | 87  |
| 5.1 Failure of the previous approaches . . . . .             | 87  |

|                   |  |     |
|-------------------|--|-----|
| 5.2               | Perturbation theory for the Gibbs measure . . . . .            | 89  |
| 5.3               | Regularity structures: Introduction . . . . .                  | 99  |
| 5.4               | Regularity structures: Algebraic aspects . . . . .             | 102 |
| 5.5               | Regularity structures: Analytic aspects . . . . .              | 118 |
| 5.6               | Existence and uniqueness of solutions . . . . .                | 146 |
| 5.7               | Large deviations . . . . .                                     | 151 |
|                   | Bibliographical notes . . . . .                                | 155 |
| <b>A</b>          | <b>Potential theory for reversible Markov chains</b> . . . . . | 157 |
| A.1               | First-hitting time and capacity . . . . .                      | 157 |
| A.2               | Dirichlet principle . . . . .                                  | 160 |
| A.3               | Thomson principle . . . . .                                    | 161 |
|                   | Bibliographical notes . . . . .                                | 163 |
| <b>B</b>          | <b>Hopf algebras</b> . . . . .                                 | 165 |
| B.1               | Definition and examples . . . . .                              | 165 |
| B.2               | Group action . . . . .   | 169 |
| B.3               | Comodules . . . . .  | 173 |
| <b>C</b>          | <b>BPHZ renormalisation</b> . . . . .                          | 175 |
| C.1               | Hepp sectors . . . . .   | 176 |
| C.2               | Subdivergences . . . . .                                       | 178 |
| C.3               | Zimmermann's forest formula . . . . .                          | 180 |
| C.4               | A renormalisation result . . . . .                             | 182 |
| C.5               | Decorated graphs . . . . .                                     | 184 |
| C.6               | Other bounds involving singular kernels . . . . .              | 185 |
| <b>D</b>          | <b>Wiener chaos expansion</b> . . . . .                        | 189 |
| D.1               | Hermite polynomials revisited . . . . .                        | 189 |
| D.2               | Wiener chaos . . . . .   | 190 |
| D.3               | Wick product . . . . .   | 192 |
| <b>E</b>          | <b>Hints and solutions for some exercises</b> . . . . .        | 195 |
| E.1               | Exercises of Chapter 2 . . . . .                               | 195 |
| E.2               | Exercises of Chapter 3 . . . . .                               | 196 |
| E.3               | Exercises of Chapter 4 . . . . .                               | 198 |
| E.4               | Exercises of Chapter 5 . . . . .                               | 199 |
| <b>References</b> | . . . . .  | 205 |
| <b>Index</b>      | . . . . .  | 217 |