Chapter 1

Introduction

1.1 Stochastic partial differential equations

Stochastic partial differential equations (SPDEs) describe the time-evolution of spatially extended systems subject to a random driving. There is a large variety of such equations, but this book will focus on parabolic SPDEs forced by space-time white noise. Their general form is

$$\partial_t \phi(t,x) = \Delta\phi(t,x) + F\big(\phi(t,x), \nabla\phi(t,x)\big) + \sqrt{2\varepsilon}\,\xi(t,x), \tag{1.1.1}$$

where Δ denotes the Laplacian acting on a spatial variable x in the d-dimensional torus, and F is a non-linearity that may in principle depend on both the solution ϕ and its gradient $\nabla\phi$, though later on in this monograph only ϕ-dependent non-linearities will be considered. The symbol ξ denotes space-time white noise, which can be intuitively understood as a Gaussian random forcing acting independently at different points in space and time. The parameter $\sqrt{2\varepsilon}$ measures the noise intensity, and may be either small or of order 1, depending on the type of phenomenon one wants to consider.

Let us briefly discuss some particular examples of non-linearities F that occur in various applications. A first example is the dynamic Φ^4 model, which is formally given by

$$\partial_t \phi(t,x) = \Delta\phi(t,x) - m^2\phi(t,x) - \phi(t,x)^3 + \xi(t,x) \tag{1.1.2}$$

with $m \geqslant 0$. This model, also called stochastic quantisation equation, was introduced as a way to analyse the Φ^4 measure in bosonic quantum field theory [2, 109, 135]. This measure is formally given by

$$\pi(\mathrm{d}\phi) = \frac{1}{\mathscr{Z}} \exp\biggl\{-\int_{\mathbb{T}^d} \biggl[\frac{1}{2}\|\nabla\phi(x)\|^2 + \frac{1}{2}m^2\phi(x)^2 + \frac{1}{4}\phi(x)^4\biggr]\mathrm{d}x\biggr\}\,\mathrm{d}\phi, \tag{1.1.3}$$

where the term in $\|\nabla\phi(x)\|^2$ describes the kinetic energy, while the other two terms represent a rotation-invariant potential energy, with m having the interpretation of a mass. As such, this measure is not well-defined, because there is no Lebesgue measure $\mathrm{d}\phi$ on the infinite-dimensional space of functions $\phi : \mathbb{T}^d \to \mathbb{R}$ (say in L^2). One can however try to construct the Φ^4 measure by starting with a lattice approximation, in which the lattice spacing, called ultraviolet cut-off, is sent to zero. While this works in dimension $d = 1$, dimensions 2 and 3 require a so-called renormalisation procedure, meaning that the mass parameter m has to depend on the lattice spacing in an appropriate way to yield a well-defined limit – in fact, it has to diverge as the lattice spacing goes to zero.

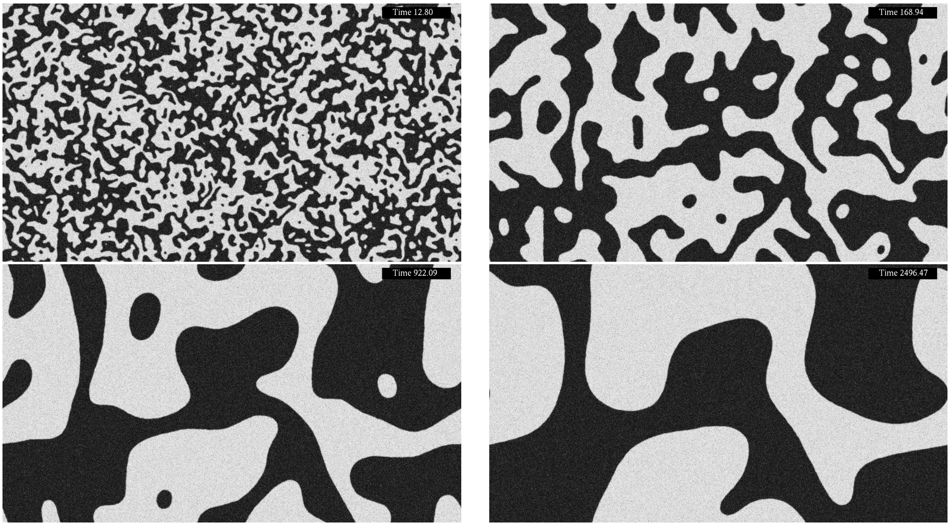

Figure 1.1. A solution of the Allen–Cahn equation (1.1.4) on a two-dimensional torus. The initial condition is random, and the field is shown at four successive times.

The idea proposed in [135] by Giorgio Parisi and Yong Shi Wu was to interpret the Gibbs measure (1.1.3) as the invariant measure of the dynamic equation (1.1.2), which could be used as in a Monte–Carlo method to sample the measure. It turns out that giving a mathematical sense to the dynamical model is not easier than for the static measure, but ultimately this approach has led to new insights into the theory. Another reason why the dynamic equation (1.1.2) is interesting is that it is related to the Ising model with Glauber dynamics near its critical temperature [31, 83].

Our second example, which will be the main focus of this book, is the stochastic Allen–Cahn equation

$$\partial_t \phi(t,x) = \Delta\phi(t,x) + \phi(t,x) - \phi(t,x)^3 + \sqrt{2\varepsilon}\,\xi(t,x). \qquad (1.1.4)$$

While this equation differs from the dynamic Φ^4 model (1.1.2) only by a sign, its solutions behave quite differently. The deterministic version of (1.1.4) was used by John W. Cahn and Sam Allen in [4] to model phase separation in multi-component alloys, and by Nathaniel Chafee and Ettore Infante in [55] to study bifurcations in PDEs. One particularity of the Allen–Cahn equation (1.1.4) with $\varepsilon = 0$ is that it admits exactly two stable solutions, which are constant in space, with ϕ equal to either 1 or -1. These can be considered as pure states in an alloy or mixture of two different fluids, such as oil and water. The deterministic equation also admits unstable stationary solutions in which ϕ changes sign, which become more numerous the larger the domain is. The sign changes of these solutions are interpreted as interfaces between different phases. Non-stationary solutions of the equation that start with an initial condition having several interfaces show an interesting dynamics, including interfaces colliding with each other and annihilating in the process (Figure 1.1).

The main new feature of the stochastic version (1.1.4) of the Allen–Cahn equation, that makes it qualitatively different from the dynamic Φ^4 model (1.1.2), is metastability. This property means that for small values of $\varepsilon > 0$, solutions tend to spend long time spans in the neighbourhood of each deterministically stable state, with occasional transitions between neighbourhoods. As a consequence, while the invariant measure of the system is still a Gibbs measure analogous to (1.1.3), convergence towards this measure is much slower than in the case of the dynamic Φ^4 model.

Another important model in the theory of parabolic SPDEs is the KPZ equation, given formally by

$$\partial_t \phi(t,x) = \partial_{xx}\phi(t,x) + \big(\partial_x \phi(t,x)\big)^2 + \xi(t,x) \tag{1.1.5}$$

in dimension $d = 1$. This model was introduced by Meran Kardar, Giorgio Parisi, and Yi-Cheng Zhang in [111] in order to describe the height of a growing interface. Here the second spatial derivative and the noise term describe the random deposition of molecules on a flat interface, while the non-linear term takes into account the fact that the growth does not occur in the vertical direction, but rather in the direction normal to the interface. It is obtained by a Taylor expansion of the term $\sqrt{1 + (\partial_x \phi(t,x))^2}$ that one gets by projecting the growth velocity on the vertical axis. The KPZ equation is expected to describe the large-scale behaviour of many different interface growth models that are not invariant under time reversal, and which form the so-called KPZ universality class (see for instance [66]).

As a last example of SPDE, let us mention the continuous parabolic Anderson model, given by

$$\partial_t \phi(t,x) = \Delta\phi(t,x) + \phi(t,x)\xi(x), \tag{1.1.6}$$

where $\xi(x)$ denotes spatial white noise that is constant in time. This equation is used to describe a random motion in a random environment, such as an alloy or a glass containing randomly distributed impurities [51]. It is related to the phenomenon of Anderson localisation, that is, the fact that waves in a random potential tend to be localised in space.

1.2 Singular SPDEs

Many of the examples of SPDEs we have discussed in the previous section share the problem that they are, in fact, mathematically ill-defined. This is due to space-time white noise being a very irregular object, which should be viewed as a random Schwartz distribution, instead of a random function.

A standard way of trying to construct solutions to SPDEs of the form (1.1.1) is to use the Duhamel formula (also called variation of constants formula) to obtain the fixed-point equation

$$\phi(t) = \mathrm{e}^{t\Delta}\,\phi_0 + \int_0^t \mathrm{e}^{(t-s)\Delta}\,F(\phi(s), \nabla\phi(s))\,\mathrm{d}s + \sqrt{2\varepsilon}\int_0^t \mathrm{e}^{(t-s)\Delta}\,\mathrm{d}\xi(s), \quad (1.2.1)$$

where $\mathrm{e}^{t\Delta}$ stands for convolution with the heat kernel P, given by

$$\big(\mathrm{e}^{t\Delta}\,f\big)(x) = \int_{\mathbb{T}^d} P(t, x-y) f(y)\,\mathrm{d}y.$$

A solution of the fixed-point equation (1.2.1) is called a mild solution of the SPDE (1.1.1) with initial condition ϕ_0.

To be able to obtain existence of a solution via Banach's fixed-point theorem, one needs a function space that remains invariant under the right-hand side of (1.2.1). This can be problematic, because the second integral in (1.2.1), called stochastic convolution, is not necessarily a function.

This issue can be quantified by introducing a scale of the so-called parabolic Besov–Hölder spaces $\mathscr{C}^\alpha_\mathfrak{s}$. For $\alpha \in (0,1)$, these spaces are defined similarly to the usual Hölder spaces by a condition on the increments, except that distances are measured in a way that treats time and space differently. For $\alpha > 1$, they are defined recursively by requiring that the derivatives of a function in $\mathscr{C}^\alpha_\mathfrak{s}$ belong to a space of lower exponent. For $\alpha < 0$, elements of $\mathscr{C}^\alpha_\mathfrak{s}$ are in general distributions, that fulfil a specific scaling condition when tested against approximations of the Dirac distribution (see Section 3.2 for details). One can then show that space-time white noise on the d-dimensional torus $\mathbb{T}^d$ belongs to $\mathscr{C}^\alpha_\mathfrak{s}$ for any $\alpha < -\frac{d+2}{2}$. Furthermore, convolution with the heat kernel improves the Hölder regularity of a distribution or function by two units, showing that the stochastic convolution belongs to $\mathscr{C}^\alpha_\mathfrak{s}$ for any $\alpha < \frac{2-d}{2}$.

Consider the case of the KPZ equation (1.1.5). Since it is defined on the one-dimensional torus, the stochastic convolution belongs to $\mathscr{C}^\alpha_\mathfrak{s}$ for any $\alpha < \frac{1}{2}$. Therefore, it is a well-defined function. However, its derivative (in the sense of distributions) will only belong to $\mathscr{C}^\alpha_\mathfrak{s}$ for any $\alpha < -\frac{1}{2}$, which poses a problem when trying to evaluate the non-linearity F after plugging the stochastic convolution into the right-hand side of (1.2.1), because there is no canonical way of defining the square of a distribution. More precisely, it is known that given two distributions $f \in \mathscr{C}^\alpha_\mathfrak{s}$ and $g \in \mathscr{C}^\beta_\mathfrak{s}$, the product fg can be defined in a consistent manner (that is, as a bilinear form giving the pointwise product if f and g are functions) if and only if $\alpha + \beta > 0$ (see Theorem 4.3.1).

A way around this difficulty was found by Lorenzo Bertini and Giambattista Giacomin in [30], based on the so-called Cole–Hopf transformation. This consists in formally setting $\phi(t,x) = \log Z(t,x)$. Disregarding the second-order term in Ito's formula, one obtains that $Z(t,x)$ satisfies the linear multiplicative heat equation

$$\mathrm{d}Z(t,x) = \partial_{xx} Z(t,x)\,\mathrm{d}t + Z(t,x)\,\mathrm{d}W(t,x),$$

where $Z(t,x)\,\mathrm{d}W(t,x)$ is interpreted as an Ito integral. It is known [68] that this equation admits a unique mild solution in a suitable function space, which can then be used as a definition of solution to the KPZ equation.

This approach turns out to be compatible with another way of constructing solutions to singular SPDEs, that consists in replacing space-time white noise ξ by a *spatially* regularised version ξ^δ, obtained by testing ξ against a spatial approximation at scale δ of the Dirac distribution. The solutions of the family of equations

$$\partial_t \phi_\delta(t,x) = \partial_{xx}\phi_\delta(t,x) + \big(\partial_x \phi_\delta(t,x)\big)^2 - C_\delta + \xi^\delta(t,x)$$

then indeed converge to the Cole–Hopf solution as δ decreases to zero, provided one chooses the so-called counterterm C_δ diverging like δ^{-1}, with an appropriate prefactor. This counterterm can be viewed as playing the role of the Ito correction that was neglected in the formal derivation of the Cole–Hopf solution, and provides another example of renormalisation procedure of a singular SPDE. Nevertheless, this approach is not entirely satisfactory, because it is not robust under changes of the approximation procedure. For instance, it does not work if one uses a space-time mollification of the noise term, instead of only a spatial one, cf. [91, Section 1].

Similar difficulties arise for both the dynamic Φ^4 model (1.1.2) and the Allen–Cahn equation (1.1.4) as soon as the space dimension d exceeds 1. Indeed, we have just seen that the stochastic convolution belongs to the Besov–Hölder space $\mathscr{C}^\alpha_{\mathfrak{s}}$ for any $\alpha < \frac{2-d}{2}$. Thus for $d = 1$, the stochastic convolution is still a function, and one can evaluate its third power, yielding a well-posed fixed-point problem. However, as soon as $d \geqslant 2$, we again encounter the problem that the stochastic convolution is a genuine distribution, for which no canonical definition of the third power exists.

A way around this difficulty in dimension $d = 2$ was found by Giuseppe Da Prato and Arnaud Debussche in the seminal work [67]. They consider renormalised equations of the form

$$\partial_t \phi_\delta(t,x) = \Delta\phi_\delta(t,x) - \big(\phi_\delta(t,x)^3 - 3C_\delta\phi_\delta(t,x)\big) + \xi^\delta(t,x) \tag{1.2.2}$$

where ξ^δ now denotes a spatio-temporal regularisation of space-time white noise, and C_δ is a logarithmically divergent constant. The main result is that as δ decreases to zero, solutions of (1.2.2) converge to a limit in any Besov–Hölder space of strictly negative exponent. In fact, the difference $\phi_\delta(t,x)^3 - 3C_\delta\phi_\delta(t,x)$ converges to the so-called third Wick power of $\phi(t,x)$, and the results in [67] extend to more general non-linearities that are renormalised in Wick's sense.

The main new idea in the Da Prato–Debussche approach is to write an equation for the difference ψ_δ between ϕ_δ and the stochastic convolution. This difference actually remains a function as δ decreases to zero, showing that while the stochastic convolution is a genuine distribution in the two-dimensional case, the solution of the Φ^4 equation differs from that distribution by a smoother object. Unfortunately, this argument no longer works in the three-dimensional case. Indeed, the stochastic convolution is then a distribution in $\mathscr{C}^{\alpha}_{\mathfrak{s}}$ only for $\alpha < -\frac{1}{2}$. Writing as before ψ_δ for the difference between ϕ_δ and the stochastic convolution, one obtains for ψ_δ an equation containing products that are not well-defined as $\delta \to 0$.

In the case of the parabolic Anderson model (1.1.6), the noise term ξ is only spatial, and one can show that it belongs to $\mathscr{C}^{\alpha}_{\mathfrak{s}}$ for any $\alpha < -\frac{d}{2}$. Therefore, in dimension 1, the stochastic convolution belongs to $\mathscr{C}^{\alpha}_{\mathfrak{s}}$ for any $\alpha < \frac{3}{2}$, and the fixed-point equation (1.2.1) is again well-defined. The equation is indeed well-posed, as shown for instance in [108]. In dimension $d = 2$, however, space-time white noise and the stochastic convolution are in Besov–Hölder spaces of exponent $\alpha < -1$ and $\alpha < 1$ respectively, so that their product is not well-defined.

1.3 Regularity structures

A major breakthrough in the theory of singular SPDEs was achieved by Martin Hairer in the works [91] and [92]. The article [91] provides a solution theory for the KPZ equation (1.1.5), while the article [92] introduces the concept of regularity structure, and associated function spaces that allow to solve a rather large family of previously ill-defined singular SPDEs. The method also requires a renormalisation procedure that was worked out in detail for the three-dimensional Φ^4 model and the two-dimensional parabolic Anderson model in [92]. The theory of regularity structures will be the main focus of Chapter 5 of this monograph. At this stage, let us just mention that one of the main ideas is to construct spaces making the following diagram commute:

$$
\begin{array}{ccc}
(\phi_0, Z^\delta) & \xmapsto{\ \mathscr{S}\ } & \Phi \\
\Psi \Big\uparrow & & \Big\downarrow \mathscr{R} \\
(\phi_0, \xi^\delta) & \xmapsto[\ \bar{\mathscr{S}}\ ]{} & \phi_\delta .
\end{array}
$$

Here $\bar{\mathscr{S}}$ is the classical solution map, associating to the initial condition ϕ_0 and a regularised realisation ξ^δ of space-time white noise a local solution ϕ_δ. The symbol Z^δ represents a so-called model, which gathers information on ξ^δ, the stochastic convolution, and several other iterated integrals involving the noise term. The map $\mathscr{S}$ associates with the initial condition and the model an element Φ of an abstract function space, called a space of modelled distributions. Finally, the so-called reconstruction operator $\mathscr{R}$ maps Φ to the solution ϕ_δ, and one has the relation

$$
\mathscr{R} \circ \mathscr{S} \circ \Psi = \bar{\mathscr{S}} .
$$

The interest of this approach is that the renormalisation procedure can be encoded in a modification of the model Z^δ, yielding a suitable modification of Φ, whose image under the reconstruction operator $\mathscr{R}$ converges to a well-defined distribution as δ decreases to zero.

While the article [92] provided a general theory of regularity structures and associated spaces of modelled distributions, the renormalisation part was restricted to the Φ^4 and parabolic Anderson models. This was remedied in a series of works by Yvain Bruned, Ajay Chandra, Ilya Chevyrev, Martin Hairer, and Lorenzo Zambotti [43, 45, 57], that provide a general theory allowing to systematically construct solutions to a broad class of singular SPDEs satisfying a so-called local subcriticality condition. This condition is also known as super-renormalisability in quantum field theory. In four space dimensions, the Φ^4 model is no longer locally subcritical, and no non-trivial concept of solution is expected to exist. See in particular [1] for recent advances in this direction.

The theory of regularity structures is not the only player in the recent developments in the theory of singular SPDEs. Another approach called paracontrolled calculus, developed by Massimiliano Gubinelli, Peter Imkeller, and Nicolas Perkowski in [89], allows to give a meaning to renormalised products of distributions in certain situations. It was used by Rémi Catellier and Khalil Chouk to solve the three-dimensional Φ^4 model in [53]. Yet another proof of existence of solutions to the Φ^4 model, based on the Wilsonian approach to the renormalisation group, was obtained by Antti Kupiainen in [112]. Depending on the kind of result one aims at, each of these methods has its advantages and drawbacks, so that it may often be useful to combine them when working with a specific example. In the works [9, 10], Ismaël Bailleul and Masato Hoshino have provided a dictionary between regularity structures and paracontrolled calculus, showing that the approaches are in a sense equivalent.

1.4 About this book

The purpose of this book is to provide a rather gentle introduction to SPDEs, singular SPDEs, and the theory of regularity structures, by focusing on a specific example, the Allen–Cahn equation and its lattice approximation. It thus does not develop a general theory for all non-singular or subcritical SPDEs, but rather illustrates how the theory works in the specific chosen example. Many aspects developed in this particular case can however rather easily be extended to other equations.

Another particularity of this monograph is that it does not focus on proving existence and uniqueness of solutions alone, but includes qualitative and quantitative results on how these solutions behave over long time spans. This includes the question of existence and uniqueness of an invariant probability measure, estimates on the speed of convergence to this measure, large-deviation techniques, and metastable properties, that describe situations where the convergence to the invariant measure is very slow.

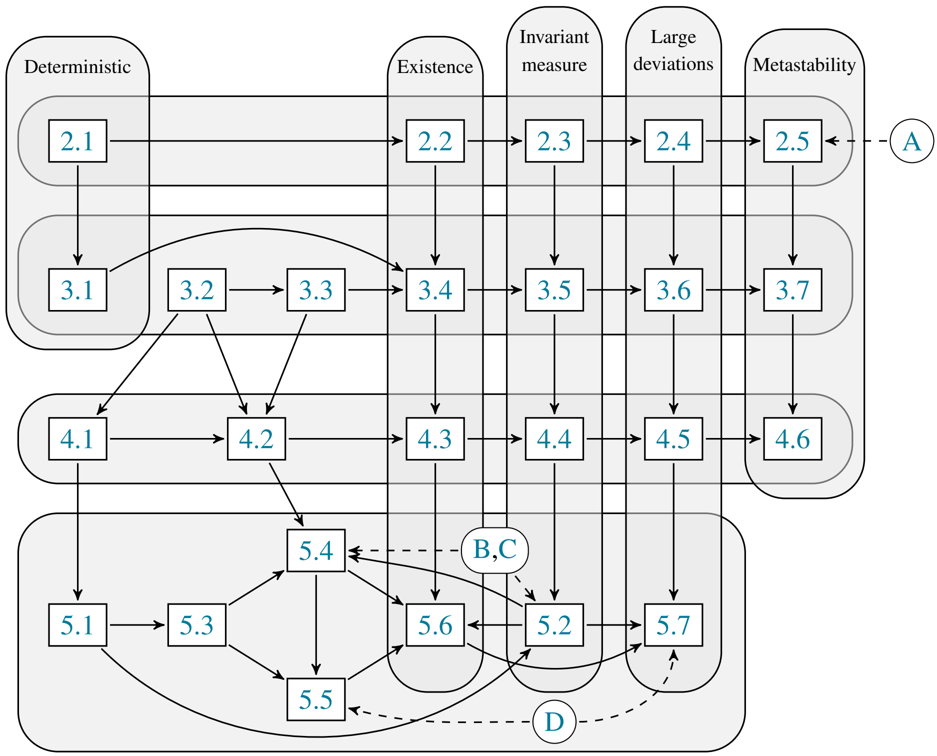

Figure 1.2. Schematic organisation of the sections of the book. An arrow indicates which sections should preferably be read before others.

Figure 1.2 gives an overview on the relations between the different sections. It is of course possible to read this book in a linear way, one chapter at a time. This will give a presentation of the models by increasing degree of difficulty, each chapter introducing some new aspects of the theory. However, it is also possible to skip certain sections to get more quickly to the most advanced parts of the theory. For instance, readers not immediately interested in large deviations and metastability may safely ignore the corresponding sections in Chapters 2, 3 and 4.

We provide now a more detailed account of the content of each chapter.

- In Chapter 2, we introduce a system of coupled stochastic differential equations (SDEs) obtained by discretising in space the one-dimensional Allen–Cahn equation. This gives us the opportunity to recall some basic properties of SDEs, regarding existence and uniqueness of solutions, invariant measures, and large-deviation estimates. An important part of the chapter is dedicated to quantifying the phenomenon of metastability that occurs for weak noise intensity. In particular, it provides a derivation of the Eyring–Kramers formula, that gives sharp asymptotics on transition times between two metastable equilibrium states of the system.

- Chapter 3 is dedicated to the one-dimensional Allen–Cahn SPDE. It contains a precise definition and several properties of space-time white noise, as well as a discussion of the stochastic heat equation. The solution of this equation, called stochastic convolution, and its stationary state, the Gaussian free field, are two important objects for constructing solutions to the Allen–Cahn equation. The chapter then proceeds with results on existence and uniqueness of solutions, invariant measures, large deviations and metastability.
- In Chapter 4, we turn to the Allen–Cahn equation on the two-dimensional torus. As we have seen, what is new in that case is that the equation is not well-posed as such, and requires a renormalisation procedure to make sense. This can be done with the help of Wick calculus for jointly Gaussian random variables, that we present before again turning to existence and uniqueness of solutions, invariant measures, large deviations and metastability.
- The final and longest Chapter 5 concerns the three-dimensional Allen–Cahn equation and the theory of regularity structures. Here we depart somewhat from previous chapters, by first developing perturbation theory for the invariant measure of the equation. This allows us to present several important ideas related to renormalisation in a somewhat simpler setting. The next three sections give a general presentation of the theory of regularity structures, where mainly algebraic and mainly analytic aspects of the theory have been collected respectively in Section 5.4 and Section 5.5. The chapter is then concluded with properties on existence and uniqueness of solutions and large deviations. Metastable properties have not been included, since they are not yet understood on the same level of sharpness as in lower-dimensional situations.

The main part of the monograph is complemented by five appendices, that give more details on specific mathematical aspects of the theory:

- Appendix A gives a summary of the potential-theoretic approach to metastability in the case of Markov chains. While this theory is not directly needed for the metastability results presented in this book, it provides an alternative and slightly simpler view on the methods used in Section 2.5 to derive Eyring–Kramers estimates for SDEs.
- Appendix B contains definitions and examples of Hopf algebras, that play a central role in the algebraic theory of regularity structures presented in Section 5.4. Hopf algebras are also used in the perturbative renormalisation of the invariant measure outlined in Section 5.2.
- In Appendix C, we give a short account of BPHZ renormalisation, named after Nikolay Bogoliubov, Ostap Parasyuk, Klaus Hepp, and Wolfhart Zimmermann,

Symbol	Meaning	Introduced in
$\lvert x\rvert$	absolute value, ℓ^1 norm	
$\lVert x\rVert$	Euclidean norm	
$\lVert z\rVert_{\mathfrak{s}}$	parabolic norm	Section 3.2
$\lVert f\rVert_{\mathscr{C}^0}$	supremum norm	Section 3.4
$\lVert f\rVert_{\mathscr{C}^\alpha_{\mathfrak{s}}(\mathfrak{K})}$	parabolic Hölder norm	Definitions 3.2.6, 3.2.7, and 3.3.7
$\lVert f\rVert_{\mathrm{TV}}$	total variation norm	Section 4.4
$\lVert\tau\rVert_\alpha$	norm on sector $\mathscr{T}_\alpha$	Section 5.5.2
$\lVert\Pi\rVert_{\gamma;\mathfrak{K}}$, $\lVert\Gamma\rVert_{\gamma;\mathfrak{K}}$	norms on models	Definition 5.5.1
$\lVert\!\vert f\rVert\!\vert_{\gamma;\mathfrak{K}}$	norm on $\mathscr{D}^\gamma$	Definition 5.5.9
$\mathscr{H} = L^2(\Lambda)$	square integrable $f : \Lambda \to \mathbb{R}$	
$\mathscr{S}'(\mathscr{H})$	Schwartz distributions	Section 3.2
$\mathscr{C}^\alpha_{\mathfrak{s}}(\mathfrak{K})$	parabolic Hölder–Besov space	Definitions 3.2.6, 3.2.7, and 3.3.7
H^s	fractional Sobolev space	Definition 3.3.2
$\mathscr{C}^0$	continuous functions	Section 3.4
$\mathscr{D}^\gamma$, $\mathscr{D}^{\gamma,\eta}$	modelled distributions	Definition 5.5.9
$x\cdot y$, $\langle x, y\rangle$	inner product on $\mathbb{R}^d$ or $\mathbb{R}^{\mathbb{Z}/N\mathbb{Z}}$	
$\langle f, g\rangle_{L^2(\Lambda)}$	$\int_\Lambda \overline{f(x)}g(x)\,\mathrm{d}x$	Section 2.3
$\langle f, g\rangle_\pi$	$\int_\Lambda \pi(x)\overline{f(x)}g(x)\,\mathrm{d}x$	Section 2.3
$\langle \xi, \varphi\rangle$	testing a distribution	Section 3.2
$\langle g, \tau\rangle$	action of $\mathscr{H}^*$ on $\mathscr{H}$	Section 5.5.5
$a\wedge b$, $a\vee b$	minimum, maximum	
$\mathbb{1}$, Id	identity operator	Sections 3.7 and 5.4
$\mathbb{1}_D$	indicator function	
$\mathbb{P}^x\{\cdot\}$	probability with initial value	
$\mathbb{E}^x[\cdot]$	expectation with initial value	
$f(x) = \mathcal{O}(g(x))$	Landau symbol	
$f(x) \lesssim g(x)$	bounded up to a constant	

Table 1.1. Frequently used notations.

that plays a role in the more analytic and combinatorial aspects of Sections 5.2 and 5.4. This includes in particular the notion of "Hepp sectors" and Zimmermann's forest formula.

- Appendix D contains a short presentation of the theory of Wiener chaos expansion, which builds on the theory of Wick renormalisation introduced in Section 4.2. This theory is used both in the construction of renormalised models in Section 5.5 and in the proof of the large-deviation principle in Section 5.7.
- Finally, Appendix E contains hints or solutions of some of the exercises included in Chapters 2 to 5.

Table 1.1 collects some notations frequently used in this monograph, grouped by norms or norm-like quantities, function spaces, inner products and linear forms, and other notations.