Chapter 1

Introduction

This book is devoted to the study of dispersive estimates for small perturbations of a stationary solution (the "kink") of a cubic wave equation of the form

$$(\partial_t^2 - \partial_x^2)\phi = \phi - \phi^3,$$

in one space dimension. Before discussing that equation and stating our results, we shall give a general presentation of the framework in which this study lies.

1.1 Long time existence for perturbed evolution equations

The question of long time (or global) existence of solutions to nonlinear dispersive equations, like the wave equation, has been a major line of research for at least the last fifty years. Let us start from the following simple model that encompasses several equations

$$\big(D_t - p(D_x)\big)u = N(u), \tag{1.1}$$

where $u : (t,x) \mapsto u(t,x)$ is a function defined on $I \times \mathbb{R}^d$, with I interval of $\mathbb{R}$, with values in $\mathbb{C}$, where $D_t = \frac{1}{i}\frac{\partial}{\partial t}$, $p(D_x) = \mathcal{F}^{-1}(p(\xi)\hat{u}(\xi))$, $\mathcal{F}^{-1}$ denoting inverse Fourier transform, and where $N(u)$ is some nonlinearity. The function $p(\xi)$ may be equal to

- $p(\xi) = |\xi|$, in which case (1.1) is an half-wave equation,
- $p(\xi) = \sqrt{1+|\xi|^2}$, corresponding to a half-Klein–Gordon equation,
- $p(\xi) = \frac{1}{2}|\xi|^2$ in the case of a Schrödinger equation.

The right-hand side in (1.1) is a nonlinear expression, that we denote by $N(u)$, though it may contain also factors like $\frac{D_x}{|D_x|}u$, $\frac{D_x}{\langle D_x\rangle}u$, or their conjugates, or even first-order derivatives of u in general. For instance, a Klein–Gordon equation of the form

$$\big(\partial_t^2 - \Delta + 1\big)\phi = F(\phi, \partial_t\phi, \nabla_x\phi) \tag{1.2}$$

with real-valued ϕ, will be reduced to (1.1) defining $u = (D_t + \sqrt{1+|D_x|^2})\phi$, so that

$$\partial_t\phi = \frac{i}{2}(u-\bar{u}), \quad \nabla_x\phi = \frac{1}{2}\nabla_x\big(1+|D_x|^2\big)^{-\frac{1}{2}}(u+\bar{u}),$$

and setting

$$N(u) = F\left(\frac{1}{2}\big(1+|D_x|^2\big)^{-\frac{1}{2}}(u+\bar{u}), \frac{i(u-\bar{u})}{2}, \frac{1}{2}\nabla_x\big(1+|D_x|^2\big)^{-\frac{1}{2}}(u+\bar{u})\right), \tag{1.3}$$

which is a non-local nonlinearity. One may proceed in the same way for a quasi-linear version of (1.2), i.e. equations where the right-hand side of (1.2) contains second-order derivatives, and is linear in these second-order derivatives. Then $N(u)$ depends also on first-order derivatives of $(u, \bar{u})$.

When one wants to study long time existence for solutions of equations like (1.1) or (1.2), one of the possible ways is to try to perturb initial data corresponding to a stationary solution, and to show that this perturbation gives rise to a global solution that will remain, for long or all times, close to the stationary solution. Of course, the simplest stationary solution that one may consider is the zero one, in which case one is led to study (1.1) with small initial data. Since the right-hand side vanishes at least at order two at zero, one may hope that it might be considered as an higher-order perturbation.

This framework has been considered by many authors since the mid-seventies, starting with problems of the form (1.1) in higher space dimensions. Let us explain why the question is easier in high space dimensions describing some classical results.

1.2 The use of dispersion

A key point in the study of equations of the form (1.1) is the use of dispersion. Consider first the linear equation $(D_t - p(D_x))u = 0$. Assuming that $p(\xi)$ is real valued, $p(D_x)$ is self-adjoint when acting on L^2 or on Sobolev spaces, so that one has preservation of the Sobolev norms of u along the evolution: $\|u(t,\cdot)\|_{H^s} = \|u(0,\cdot)\|_{H^s}$ for any t. If one considers instead equation (1.1), a Sobolev energy estimate gives just that, as long as the solution exists, one has for any $t \geq 0$,

$$\|u(t,\cdot)\|_{H^s} \leq \|u(0,\cdot)\|_{H^s} + \int_0^t \|N(u)(\tau,\cdot)\|_{H^s}\, d\tau, \qquad (1.4)$$

so that one needs, in order to control uniformly the left-hand side, to be able to estimate the integral term on the right-hand side. If one considers a simple model where $N(u)$ is given by $N(u) = P(u, \bar{u})$, where P is an homogeneous polynomial of order $r \geq 2$, one has, for $s > \frac{d}{2}$ where d is the space dimension, a bound

$$\|N(u)\|_{H^s} \leq C\|u\|_{L^\infty}^{r-1}\|u\|_{H^s},$$

so that (1.4) implies

$$\|u(t,\cdot)\|_{H^s} \leq \|u(0,\cdot)\|_{H^s} + C\int_0^t \|u(\tau,\cdot)\|_{L^\infty}^{r-1}\|u(t,\cdot)\|_{H^s}\, d\tau. \qquad (1.5)$$

As a consequence, by Gronwall's lemma,

$$\|u(t,\cdot)\|_{H^s} \leq \|u(0,\cdot)\|_{H^s} \exp\left(C\int_0^t \|u(\tau,\cdot)\|_{L^\infty}^{r-1}\, d\tau\right). \qquad (1.6)$$

One thus sees that, if we want to get a control of $\|u(t,\cdot)\|_{H^s}$ for large t, one needs to obtain as well a priori estimates for $\|u(\tau,\cdot)\|_{L^\infty}$. In particular, to get a uniform global bounds in (1.6), one would need the right-hand side of this inequality to be bounded, i.e. $\int_0^{+\infty}\|u(\tau,\cdot)\|_{L^\infty}^{r-1}\,d\tau < +\infty$.

One may try to guess what are the best estimates one may expect for $\|u(\tau,\cdot)\|_{L^\infty}$ from those holding true for solutions to the linear equation $(D_t - p(D_x))u = 0$. As the solution is given by

$$u(t,x) = \frac{1}{(2\pi)^d}\int e^{itp(\xi)+ix\xi}\hat{u}_0(\xi)\,d\xi \tag{1.7}$$

where $u_0 = u(0,\cdot)$, one sees from the stationary phase formula that if u_0 is smooth enough and has enough decay at infinity, $\|u(t,\cdot)\|_{L^\infty} = O(t^{-\frac{\kappa}{2}})$, where κ depends on the rank of the Hessian of $p(\xi)$. In the case of the wave equation $p(\xi) = |\xi|$, one has $\kappa = d-1$, while for Schrödinger or Klein–Gordon equations (i.e. $p(\xi) = \frac{1}{2}|\xi|^2$ or $p(\xi) = \sqrt{1+|\xi|^2}$), $\kappa = d$. Conjecturing that the same decay will hold for solutions of the nonlinear equation, we would get that the integral on the right-hand side of (1.6) will converge if $\frac{\kappa}{2}(r-1) > 1$, so that if $\frac{d-1}{2}(r-1) > 1$ for the wave equation and $\frac{d}{2}(r-1) > 1$ for the Klein–Gordon or Schrödinger ones.

1.3 Vector fields methods and global solutions

The above heuristics turn out to give a correct answer for nonlinear wave equations if one considers general nonlinearities: actually, in this case, smooth enough decaying initial data of small size give rise to global solutions when $d \geq 4$ if the nonlinearity does not depend on u and is at least quadratic (i.e. $r \geq 2$) as it has been proved by Klainerman [50], Shatah [75], including for quasi-linear nonlinearities. In the same way, for Klein–Gordon equations with quadratic nonlinearities, global existence holds if $d \geq 3$ (see Klainerman [49], Shatah [76]). Moreover, the solutions scatter, i.e. have the same long time asymptotics as the solution of a linear equation.

Let us recall the "Klainerman vector fields method" that provides a powerful way of proving that type of properties. We consider an equation of the form

$$\Box u = f(\partial_t u, \nabla_x u), \tag{1.8}$$

where u is a function of (t,x) in $\mathbb{R}\times\mathbb{R}^d$, $\Box = \partial_t^2 - \Delta_x$ and f is a smooth function vanishing at least at order 2 at the origin. Instead of $\Box$ in the linear part of (1.8), one may more generally take the operator $\sum_{j,k} g^{jk}(\partial_t u, \nabla_x u)\partial_j\partial_k$, where $x_0 = t$ and the coefficients g^{jk} are smooth and satisfy $\sum_{j,k} g^{jk}(0,0)\partial_j\partial_k = \Box$, so that the method is not limited to semilinear equations, but works as well for quasi-linear ones, that is one of its main interests. For the sake of simplification, we shall just discuss (1.8), referring to the original paper of Klainerman [51] and to the book of Hörmander [42] for the more general case. The Sobolev energy inequality applied to (1.8) together with nonlinear estimates for the right-hand side imply that, if $s > \frac{d}{2}$, the energy $E_s(t) = \|\partial_t u(t,\cdot)\|_{H^s}^2 + \|\nabla_x u(t,\cdot)\|_{H^s}^2$ satisfies, as long as $\|u'(\tau,\cdot)\|_{L^\infty}$ is bounded,

$$E_s(t)^{\frac{1}{2}} \leq E_s(0)^{\frac{1}{2}} + C\int_0^t \|u'(\tau,\cdot)\|_{L^\infty} E_s(\tau)^{\frac{1}{2}}\,d\tau, \tag{1.9}$$

where we set u' for $(\partial_t u, \nabla_x u)$. This is the analogous of (1.5) for the solution of (1.8) and in order to exploit this estimate, one needs to show that $t \mapsto \|u'(t,\cdot)\|_{L^\infty}$ is integrable. The Klainerman vector fields method allows one to deduce such a property from L^2 estimates for the action of convenient vector fields on u. More precisely, one introduces the Lie algebra of vector fields tangent to the wave cone $t^2 = |x|^2$, generated by

$$\begin{aligned} &t\partial_{x_j} + x_j\partial_t, && j = 1,\dots,d,\\ &x_i\partial_{x_j} - x_j\partial_{x_i}, && 1 \leq i < j \leq d,\\ &t\partial_t + \sum_{j=1}^d x_j\partial_{x_j} \end{aligned} \tag{1.10}$$

and if one denotes by $(Z_i)_{i\in\mathcal{I}}$ the family of fields given by (1.10) or by the usual derivatives $\partial_t, \partial_{x_j}$, $j = 1,\dots,d$, we set, for $I = \{i_1,\dots,i_p\} \subset \mathcal{I}^p$, $Z^I = Z_{i_1}\cdots Z_{i_p}$ and $|I| = p$. Then, as Z^I commutes to $\Box$ by construction (up to a multiple of the equation), one gets from (1.8) essentially

$$\Box Z^I u = Z^I f(\partial_t u, \nabla_x u) \tag{1.11}$$

from which it follows that, if $t \geq 0$,

$$\|Z^I u(t,\cdot)\|_{L^2} \leq \|Z^I u(0,\cdot)\|_{L^2} + \int_0^t \|Z^I f(\partial_t u, \nabla_x u)(\tau,\cdot)\|_{L^2}\,d\tau. \tag{1.12}$$

Using that Z^I is a composition of vector fields, one deduces from Leibniz rule that, setting $u'_N = (Z^I u')_{|I|\leq N}$,

$$\begin{aligned} \|u'_N(t,\cdot)\|_{L^2} \leq \|u'_N(0,\cdot)\|_{L^2} + \int_0^t & C\big(\|u'_{N/2}(\tau,\cdot)\|_{L^\infty}\big)\\ &\times \|u'_{N/2}(\tau,\cdot)\|_{L^\infty}\|u'_N(\tau,\cdot)\|_{L^2}\,d\tau. \end{aligned} \tag{1.13}$$

This is thus an inequality of the form (1.9), and in order to deduce from it an a priori bound for the left-hand side of (1.13), one again needs a dispersive estimate for $\|u'_{N/2}(\tau,\cdot)\|_{L^\infty}$ in $O(\tau^{-\frac{d-1}{2}})$. This estimate follows from the Klainerman–Sobolev inequality

$$(1+|t|+|x|)^{d-1}\big(1+\big||t|-|x|\big|\big)|w(t,x)|^2 \leq C\sum_{|I|\leq\frac{d+2}{2}} \|Z^I w(t,\dots)\|_{L^2} \tag{1.14}$$

for the proof of which we refer for instance to [42, Proposition 6.5.1]. This implies in particular that, if we take N large enough so that $\frac{N}{2} + \frac{d+2}{2} \leq N$, one has for $t \geq 0$,

$$\|u'_{N/2}(t,\cdot)\|_{L^\infty} \leq C(1+t)^{-\frac{d-1}{2}}\|u'_N(t,\cdot)\|_{L^2}. \tag{1.15}$$

One deduces from (1.13) and (1.15) a priori bounds of the form

$$\|u'_N(t,\cdot)\|_{L^2} \le A\varepsilon, \tag{1.16}$$

$$\|u'_{N/2}(t,\cdot)\|_{L^\infty} \le B\varepsilon(1+t)^{-\frac{d-1}{2}} \tag{1.17}$$

by a bootstrap argument when $d \ge 4$: If one assumes that (1.16) and (1.17) hold for t in some interval $[0, T]$, one shows that if A, B have been taken large enough in function of the initial data, and if ε is small enough, then (1.16) and (1.17) hold on the same interval with (A, B) replaced by $(\frac{A}{2}, \frac{B}{2})$. One has just to plug (1.16) and (1.17) in (1.13), and to use that $(1+t)^{-\frac{d-1}{2}}$ is integrable in order to prove (1.16) with A replaced by $\frac{A}{2}$. Concerning (1.17) with B replaced by $\frac{B}{2}$, it follows from (1.15) and (1.16) if B is taken large enough with respect to A. Combining these a priori bounds with local existence theory for smooth data shows that solutions are global, for ε small enough, and satisfy (1.16) and (1.17) for any time.

The same type of arguments works more generally when f in (1.8) vanishes at order $r \ge 2$ at zero and $\frac{(d-1)}{2}(r-1) > 1$.

Of special interest is the limiting case of long range nonlinearities when

$$\frac{d-1}{2}(r-1) = 1.$$

This happens in particular if $d = 3, r = 2$, i.e. for quadratic nonlinearities in three space dimension. In this case, one gets in general that data of size $\varepsilon > 0$ give rise to solutions existing over a time interval of length at least $e^{\frac{c}{\varepsilon}}$ for some $c > 0$, but finite time blow-up may occur. Nevertheless, if the solution satisfies a special structure, the so-called "null condition", global existence holds true (see Klainerman [51]). We again refer to the book of Hörmander [42] and references therein for more discussion of long time existence for wave equations, in particular in two space dimension, and to Alinhac [2] for the study of blow-up phenomena when solutions are not global. We also refer to Christodoulou and Klainerman [11] and to Lindblad and Rodnianski [62] for applications to general relativity.

In Section 1.4 we discuss the case of long range nonlinearities for Schrödinger and Klein–Gordon equations in one space dimension, which is the relevant framework for the problem we study in this book. To conclude the present section, let us make some comments on another well known way of exploiting the dispersive character of wave (or other linear) equations, namely Strichartz estimates. The vector fields method that we described above has the advantage of providing explicit decay rates for the solution (and, combined with other arguments, may even furnish precise information on asymptotic behavior of solutions). Moreover, it applies to quasi-linear equations, even if we described it just on a simple semilinear case. On the other hand, it is limited to the study of equations with small and *decaying* data.

When one deals with semilinear equations, and wants to study solutions whose data do not have further decay than being in some Sobolev space, one may instead use Strichartz estimates. Recall that they are given, for a solution u to a linear wave equation,

$$
\begin{aligned}
(\partial_t^2 - \Delta)u &= F,\\
u(0,\cdot) = u_0, &\quad \partial_t u(0,\cdot) = u_1,
\end{aligned}
\tag{1.18}
$$

defined on $I \times \mathbb{R}^d$, where I is an interval containing 0, by

$$
\|u\|_{L_t^q L_x^r(I\times\mathbb{R}^d)} \le C\big(\|u_0\|_{L^2} + \|u_1\|_{\dot{H}^{-1}} + \|F\|_{L_t^{\tilde{q}'} L_x^{\tilde{r}'}(I\times\mathbb{R}^d)}\big), \tag{1.19}
$$

where the indices satisfy

$$
\begin{aligned}
&\frac{1}{\tilde{q}} + \frac{1}{\tilde{q}'} = 1, \quad \frac{1}{\tilde{r}} + \frac{1}{\tilde{r}'} = 1, \quad \frac{1}{q} + \frac{d}{r} = \frac{d}{2}, \quad \frac{1}{\tilde{q}'} + \frac{d}{\tilde{r}'} = \frac{d}{2} + 2,\\
&\frac{1}{q} + \frac{d-1}{2r} \le \frac{d-1}{4}, \quad \frac{1}{\tilde{q}} + \frac{d-1}{2\tilde{r}} \le \frac{d-1}{4},\\
&(q,r,d) \ne (2,\infty,3), \quad q,r \ge 2,\ r < \infty\\
&(\tilde{q},\tilde{r},d) \ne (2,\infty,3), \quad \tilde{q},\tilde{r} \ge 2,\ \tilde{r} < \infty.
\end{aligned}
\tag{1.20}
$$

We refer to the book of Tao [83] and references therein for the proof. These estimates express both a smoothing and a time decay property of the solution. Because of that, they are useful both in the study of local existence with non-smooth initial data or for global existence and scattering problems in the semilinear case, including for large data. We shall not pursue here on that matter, as this is not the kind of methods we shall use below, since we are more interested in explicit decay rates of solutions. We refer to [83] for some of the many applications of these Strichartz estimates.

1.4 Klainerman–Sobolev estimates in one dimension

The preceding section was devoted to the use of Klainerman vector fields in the framework of wave equations in higher space dimensions. In the present section, we shall focus on the case of (half-)Klein–Gordon or Schrödinger equations in dimension one, as this is the closest framework to our main theorem. As a prerequisite, we shall describe first how (a variant of) the method of Klainerman vector fields allows one to get dispersive decay estimates for solutions when the nonlinearity vanishes at high enough order at initial time. We start with the simplest model of gauge invariant nonlinearities, to which more general equations may be in any case reduces by the normal forms methods we shall discuss later. Denote thus for ξ in $\mathbb{R}$, $p(\xi) = \sqrt{1+\xi^2}$ or $p(\xi) = \frac{\xi^2}{2}$ and consider equation (1.1) with $N(u) = |u|^{2p}u$ with $p \in \mathbb{N}^*$, i.e.

$$
\begin{aligned}
\big(D_t - p(D_x)\big)u &= \alpha|u|^{2p}u,\\
u|_{t=1} &= u_0,
\end{aligned}
\tag{1.21}
$$

where for convenience of notation we take the initial data at time $t = 1$, α is a complex number and u_0 will be given in a convenient space. One has the following statement.

Theorem 1.4.1. *Let p be larger than or equal to 2 in (1.21). There are s_0, ρ_0 in $\mathbb{N}$ such that, for any $s \geq s_0$, there are $\varepsilon_0 > 0, C > 0$ and for any $\varepsilon \in]0, \varepsilon_0]$, any $u \in H^s(\mathbb{R})$ satisfying*

$$\|u_0\|_{H^s} + \|xu_0\|_{L^2} \leq \varepsilon, \tag{1.22}$$

the solution to (1.21) is global and satisfies for any $t \geq 1$,

$$\|u(t,\cdot)\|_{H^s} \leq C\varepsilon, \quad \|u(t,\cdot)\|_{W^{\rho_0,\infty}} \leq C\frac{\varepsilon}{\sqrt{t}}, \tag{1.23}$$

where $\|w\|_{W^{\rho_0,\infty}} = \|\langle D_x\rangle^{\rho_0} w\|_{L^\infty}$.

We shall present the proof following arguments due to Hayashi and Tsutsumi [40] in the case of Schrödinger equations. For Klein–Gordon equations, the first proof of such a result is due to Klainerman and Ponce [52] and Shatah [75], using a different method. We shall describe here a unified approach for both equations. Notice also that for Klein–Gordon equations, global existence result hold for much more general nonlinearities. We shall give references to that in the forthcoming sections.

Idea of proof of Theorem 1.4.1. We apply the Klainerman vector fields idea, except that instead of using true vector fields, we make use of the operator

$$L_+ = x + tp'(D_x). \tag{1.24}$$

This operator commutes to the linear part of the equation, $[L_+, D_t - p(D_x)] = 0$. Moreover, because the nonlinearity is gauge invariant, a Leibniz rule holds. Actually, in the case of Schrödinger equations, one has a bound

$$\|L_+(|u|^{2p}u)\|_{L^2} \leq C\|u\|_{L^\infty}^{2p}\|L_+u\|_{L^2} \tag{1.25}$$

that follows using that if $p(\xi) = \frac{\xi^2}{2}$, then $L_+ = x + tD_x$ and then

$$\begin{aligned} L_+(|u|^{2p}u) &= L_+(u^{p+1}\bar{u}^p) \\ &= (p+1)(L_+u)|u|^{2p} - pu^{p+1}\bar{u}^{p-1}\overline{L_+u}. \end{aligned}$$

When $p(\xi) = \sqrt{1+\xi^2}$, one has an estimate similar to (1.25) up to replacing the L^∞ norm by a $W^{\rho_0,\infty}$ one, for some large enough ρ_0, and up to some remainders that do not affect the argument below (see [20]). We shall pursue here the argument in the Schrödinger case. Applying L_+ to (1.21) and using the commutation property seen above and (1.25), we obtain

$$\big(D_t - p(D_x)\big)(L_+u) = O_{L^2}\big(\|u\|_{L^\infty}^{2p}\|L_+u\|_{L^2}\big) \tag{1.26}$$

so that one has by L^2 energy inequality

$$\|L_+u(t,\cdot)\|_{L^2} \leq \|L_+u(1,\cdot)\|_{L^2} + C\int_1^t \|u(\tau,\cdot)\|_{L^\infty}^{2p}\|L_+u(\tau,\cdot)\|_{L^2}\,d\tau. \tag{1.27}$$

The proof of the theorem now proceeds with a bootstrap argument: One wants to find constants $A > 0$, $B > 0$ such that

$$\begin{aligned} \|u(t,\cdot)\|_{H^s} &\le A\varepsilon, \\ \|L_+u(t,\cdot)\|_{L^2} &\le A\varepsilon, \\ \|u(t,\cdot)\|_{L^\infty} &\le B\frac{\varepsilon}{\sqrt{t}} \end{aligned} \tag{1.28}$$

for any $t \ge 1$, as long as $\varepsilon > 0$ is small enough. Assume that these inequalities hold true for t in some interval $[1, T]$. Then, it is enough to show, using equation (1.21), that for t in the same interval $[1, T]$, one has in fact the better estimates

$$\begin{aligned} \|u(t,\cdot)\|_{H^s} &\le \frac{A}{2}\varepsilon, \\ \|L_+u(t,\cdot)\|_{L^2} &\le \frac{A}{2}\varepsilon, \\ \|u(t,\cdot)\|_{L^\infty} &\le \frac{B}{2}\frac{\varepsilon}{\sqrt{t}}. \end{aligned} \tag{1.29}$$

Actually, estimates (1.28) hold on some interval $[1, T]$ if one has taken A, B large enough, because of assumptions (1.22) made on the initial data, and of Sobolev embedding in order to get the L^∞ bound.

To show that (1.28) implies the first two estimates (1.29), one uses (1.5) (with r replaced by $2p + 1$) and (1.27). Plugging there the a priori bounds (1.28), one gets for any t in $[1, T]$,

$$\begin{aligned} \|u(t,\cdot)\|_{H^s} &\le \|u_0\|_{H^s} + CB^{2p}A\varepsilon^{2p+1}\int_1^t \tau^{-p}\,d\tau, \\ \|L_+u(t,\cdot)\|_{L^2} &\le \|L_+u(1,\cdot)\|_{H^s} + CB^{2p}A\varepsilon^{2p+1}\int_1^t \tau^{-p}\,d\tau \end{aligned} \tag{1.30}$$

with $p > 1$. Consequently, using assumption (1.22), taking A large enough and ε small enough, one gets the first two inequalities (1.29). To obtain the last one, one uses Klainerman–Sobolev estimates, that allow one to recover an L^∞ bound *with the right time decay* from an L^2 one for L_+u. In the case we are treating $p(\xi) = \frac{\xi^2}{2}$, this is very easy: one writes, by the usual Sobolev embedding

$$\|w\|_{L^\infty} \le C\|w\|_{L^2}^{\frac{1}{2}}\|D_x w\|_{L^2}^{\frac{1}{2}}.$$

Applying this with $w = e^{i\frac{x^2}{2t}}u(t,\cdot)$, one gets

$$\|u(t,\cdot)\|_{L^\infty} \le \frac{C}{\sqrt{t}}\|u(t,\cdot)\|_{L^2}^{\frac{1}{2}}\|L_+u(t,\cdot)\|_{L^2}^{\frac{1}{2}}. \tag{1.31}$$

Plugging the first two inequalities (1.28) inside the right-hand side, one gets

$$\|u(t,\cdot)\|_{L^\infty} \le \frac{\varepsilon}{\sqrt{t}}CA,$$

which gives the last bound (1.29) if B is chosen large enough relatively to A and concludes the proof. ∎

1.5 The case of long range nonlinearities

In equation (1.21) we limited ourselves to the case $p > 1$, which may be considered as a short range case: actually, if we consider $|u|^{2p}$ as a potential, the time decay of $\|u(t,\cdot)\|_{L^\infty}$ in $t^{-\frac{1}{2}}$ shows that $\||u(t,\cdot)|^{2p}\|_{L^\infty}$ is time integrable at infinity. This played an essential role in order to bound the integrals on the right-hand side of (1.30). Thought, a variant of Theorem 1.4.1 holds as well when $p = 1$:

Theorem 1.5.1. *Let $p(\xi) = \sqrt{1+\xi^2}$ or $p(\xi) = \frac{\xi^2}{2}$ in one space dimension, α a* real *constant. There are s_0, ρ_0 in $\mathbb{N}$, $\delta > 0$ such that for any $s \geq s_0$, there are $\varepsilon_0 > 0$, $C > 0$ so that, for any $\varepsilon \in]0, \varepsilon_0]$, any u_0 in $H^s(\mathbb{R})$ satisfying (1.22), the solution of*

$$\begin{aligned} \big(D_t - p(D_x)\big)u &= \alpha |u|^2 u, \\ u|_{t=1} &= u_0 \end{aligned} \tag{1.32}$$

is defined for any $t \geq 1$ and satisfies there

$$\|u(t,\cdot)\|_{H^s} \leq C\varepsilon t^\delta, \quad \|u(t,\cdot)\|_{W^{\rho_0,\infty}} \leq C\frac{\varepsilon}{\sqrt{t}}. \tag{1.33}$$

Remarks. We make the following observations.

- A difference between the conclusion of Theorem 1.4.1 and the above statement is that the Sobolev estimate is not uniform: a slight growth in t^δ is possible. Actually, δ may be taken of the form $C\varepsilon^2$ for some constant C.
- The form of the nonlinearity is important, at the difference with the short range case of the preceding section. For instance, one cannot take on the right-hand side of (1.32) for α an arbitrary complex number. The fact that α should be real is an example of a null condition that has to be imposed in order to get global solutions.
- The proof of the theorem provides also modified scattering for u as t goes to infinity.

Let us give some references. For the Schrödinger case, a first proof of Theorem 1.5.1 and of modified scattering of solutions is due to Hayashi and Naumkin [38]. See also Katayama and Tsutsumi [46] and, more recently, Lindblad and Soffer [65], Kato and Pusateri [47] and Ifrim and Tataru [45]. In the case of Klein–Gordon equations, including in the case of quasi-linear nonlinearities satisfying a null condition, we refer to Moriyama, Tonegawa and Tsutsumi [71], Moriyama [70], Delort [18–20], Lindblad and Soffer [63], Lindblad [64] and Stingo [82]. See also Hani, Pausader, Tzvetkov and Visciglia [37] for some further applications.

Before explaining the general strategy of proof of Theorem 1.5.1, let us describe informally how the dispersive estimate in (1.33) will be proved, using an auxiliary

ODE deduced from (1.32). We make this derivation in the case $p(\xi) = \frac{1}{2}\xi^2$, deferring to next paragraph the case of general p. Denote by $\varphi(x) = -\frac{1}{2}x^2$ and look for a solution to (1.32) under the form

$$u(t,x) = \frac{e^{it\varphi(\frac{x}{t})}}{\sqrt{t}} A\Big(t, \frac{x}{t}\Big), \tag{1.34}$$

where $A(t, y)$ is a smooth function. Plugging this Ansatz inside equation (1.32) with $p(D_x) = \frac{1}{2}D_x^2$, one gets

$$D_t A(t,y) = \frac{\alpha}{t}|A(t,y)|^2 A(t,y) + \frac{1}{2t^2} D_y^2 A(t,y). \tag{1.35}$$

If one ignores the last term (that will be proved a posteriori to be a time integrable remainder), one gets that A solves the ODE

$$D_t A(t,y) = \frac{\alpha}{t}|A(t,y)|^2 A(t,y) \tag{1.36}$$

from which follows, as α is real, that $|A(t,y)| = |A(1,y)|$ for all $t \geq 1$, whence

$$A(t,y) = A(1,y)\exp\big(i\alpha|A(1,y)|^2 \log t\big).$$

One thus gets a uniform bound for A, and also discovers that the phase of oscillation of (1.34) involves a logarithmic modification that reflects modified scattering, i.e. one gets when time goes to infinity

$$u(t,x) \sim \frac{1}{\sqrt{t}} A_0\Big(\frac{x}{t}\Big) \exp\Big(-i\frac{x^2}{2t} + i\alpha\Big|A_0\Big(\frac{x}{t}\Big)\Big|^2 \log t\Big)$$

for some function A_0. Of course, to establish this rigorously, one has to show that the last term in (1.35) is really a remainder whose addition to the right-hand side of (1.36) does not modify the analysis of asymptotic behavior of solutions.

One may perform such a derivation in a rigorous way using a wave-packets analysis as in Ifrim and Tataru [45] or using a semiclassical approach as we do here. The idea is the following: because of formula (1.34), u appears naturally as a function of t and $\frac{x}{t}$, so that it is natural to write it in terms of a new unknown v by

$$u(t,x) = \frac{1}{\sqrt{t}} v\Big(t, \frac{x}{t}\Big), \tag{1.37}$$

where v will satisfy an equation

$$D_t v - \frac{1}{2t}(x \cdot D_x + D_x \cdot x)v - p\Big(\frac{D_x}{t}\Big)v = \frac{\alpha}{t}|v|^2 v. \tag{1.38}$$

By (1.34), we expect $v(t,x)$ to oscillate like $e^{it\varphi(x)}$. We compute for any smooth function $a(t,x)$,

$$p\Big(\frac{D_x}{t}\Big)\big(e^{it\varphi(x)}a(t,x)\big) = \big(p(\partial_x\varphi(x))a(t,x) + O(t^{-1})\big)e^{it\varphi(x)}.$$

One expects thus that the main contribution to the left-hand side of (1.38) will be obtained replacing $\frac{D_x}{t}$ by $\partial_x \varphi$. This gives an ODE which is nothing but (1.35) if we replace v by $e^{it\varphi(x)}A(t,x)$. In other words, we obtain an ODE allowing us to describe the asymptotics of the solution starting from the quantum problem given by the PDE (1.36) and reducing it to the classical problem obtained making in (1.38) the substitution $\frac{D_x}{t} \mapsto \partial_x \varphi$. We explain below, in the strategy of proof of Theorem 1.5.1, the rigorous way of doing so controlling the errors.

Strategy of proof of Theorem 1.5.1. The starting point of the proof is the same as for Theorem 1.4.1, except that the inequalities to be bootstrapped read now as

$$\begin{aligned} \|u(t,\cdot)\|_{H^s} &\leq A\varepsilon t^{\delta}, \\ \|L_+ u(t,\cdot)\|_{L^2} &\leq A\varepsilon t^{\delta}, \\ \|u(t,\cdot)\|_{W^{\rho_0,\infty}} &\leq B\frac{\varepsilon}{\sqrt{t}} \end{aligned} \tag{1.39}$$

instead of (1.28), with $\delta > 0$ a small number. Again, one has (1.30) with $p = 1$ and the integral term replaced by $\int_1^t \tau^{-1+\delta}\, d\tau \leq \delta^{-1} t^{\delta}$. If $\varepsilon^2\delta^{-1}$ is small enough, one deduces from (1.30) that the first two inequalities in (1.39) actually hold with A replaced by $\frac{A}{2}$. On the other hand, one cannot deduce the L^∞ estimate in (1.39) from the Sobolev and L^2 ones using (1.31), as the lack of uniformity in the estimate of $\|L_+ u(t,\cdot)\|_{L^2}$ would just provide a bound in $O(t^{-\frac{1}{2}+0})$ instead of $O(t^{-\frac{1}{2}})$. On thus needs an extra argument to obtain the L^∞ estimates (since the L^2 ones cannot be expected to be improved). There have been several approaches to do so, that all rely on the derivation from the PDE (1.32) of an ODE, that may be used in order to get the optimal L^∞ decay (and the asymptotics of the solution). That ODE may be written either on the solution itself or on its Fourier transform (actually on the profile $e^{itp(\xi)}\hat{u}(t,\xi)$ of the Fourier transform). As indicated in the preceding paragraph, the method we shall use in this book, inspired in part from the approach of Ifrim and Tataru [45] based on wave packets, relies on a semiclassical version of the equation satisfied by a rescaled unknown.

We introduce as a semiclassical parameter $h = \frac{1}{t} \in]0,1]$ and define from the unknown u the new unknown v through (1.37). If we denote $\|v\|_{H_h^s} = \|\langle hD_x\rangle^s v\|_{L^2}$, then $\|u(t,\cdot)\|_{H^s} = \|v(t,\cdot)\|_{H_h^s}$. The last estimate in (1.39) is equivalent to getting an $O(\varepsilon)$ bound for $\|\langle hD_x\rangle^{\rho_0} v(t,\cdot)\|_{L^\infty}$. Plugging (1.37) inside (1.32), one gets

$$\big(D_t - \mathrm{Op}_h^{\mathrm{W}}(x\xi + p(\xi))\big)v = h\alpha|v|^2 v, \tag{1.40}$$

where the semiclassical Weyl quantization $\mathrm{Op}_h^{\mathrm{W}}$ associates to a “symbol” $a(x,\xi)$ the operator

$$v \mapsto \mathrm{Op}_h^{\mathrm{W}}(a)v = \frac{1}{2\pi h}\int e^{i(x-y)\frac{\xi}{h}} a\Big(\frac{x+y}{2},\xi\Big)v(y)\,dy\,d\xi. \tag{1.41}$$

The above formula makes sense for more general functions a than the one

$$a(x,\xi) = x\xi + p(\xi)$$

appearing in (1.40). We do not give here these precise assumptions, referring to Appendix D below. Let us just remark that one may translate the action of operator L_+ on u by

$$L_+u(t,x) = \frac{1}{\sqrt{t}}(\mathcal{L}_+v)\Big(t,\frac{x}{t}\Big) \tag{1.42}$$

with

$$\mathcal{L}_+ = \frac{1}{h}\mathrm{Op}_h^{\mathrm{W}}(x + p'(\xi)) \tag{1.43}$$

so that the second a priori assumption (1.39) may be translated as

$$\|\mathcal{L}_+v\|_{L^2} = O(\varepsilon h^{-\delta}). \tag{1.44}$$

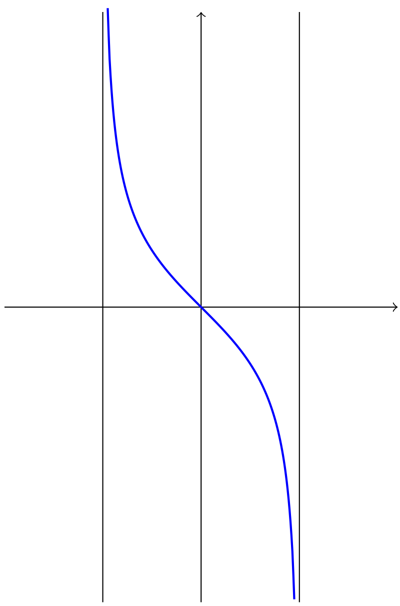
This brings us to introduce the submanifold

$$\Lambda = \{(x,\xi) \in \mathbb{R}\times\mathbb{R} : x + p'(\xi) = 0\} \tag{1.45}$$

that is actually the graph

$$\Lambda = \{(x, d\varphi(x)) : x \in]-1,1[\} \quad \text{with } \varphi(x) = \sqrt{1-x^2} \tag{1.46}$$

given by the following picture.



The idea is to deduce from (1.40) an ODE restricting the symbol $x\xi + p(\xi)$ to Λ. By (1.46) and a direct computation, $(x\xi + p(\xi))|_\Lambda = \varphi(x)$, so that we would want to deduce from (1.40) an ODE of the form

$$\big(D_t - \varphi(x)\big)w = h\alpha|w|^2w + R, \tag{1.47}$$

where w should be conveniently related to v and R being a remainder such that

$$\int_1^{+\infty} \|R(t,\cdot)\|_{W_h^{\rho_0,\infty}}\, dt = O(\varepsilon).$$

We notice first that the a priori bound (1.44) provides a uniform estimate for v cut-off outside a $\sqrt{h}$-neighborhood of Λ. The idea is as follows:

First, contributions to v cut-off for high frequencies have nice bounds if we assume the first a priori estimate (1.39): actually, it implies

$$\|\langle hD_x\rangle^s v(t,\cdot)\|_{L^2} = O(\varepsilon h^{-\delta}),$$

so that if $\chi \in C_0^\infty(\mathbb{R})$ is equal to one close to zero, $\beta > 0$ is small and $s_0 > \frac{1}{2}$, one gets by semiclassical Sobolev estimate

$$\begin{aligned}\|\mathrm{Op}_h^{\mathrm{W}}\big(\chi(h^\beta\xi)\big)v\|_{L^\infty} &\le Ch^{-\frac{1}{2}}\|\langle hD_x\rangle^{s_0}\mathrm{Op}_h^{\mathrm{W}}\big(\chi(h^\beta\xi)\big)v\|_{L^2}\\ &\le Ch^{-\frac{1}{2}+\beta(s-s_0)}\|\langle hD_x\rangle^s v\|_{L^2}\\ &\le C\varepsilon h^{-\frac{1}{2}-\delta+\beta(s-s_0)}.\end{aligned} \tag{1.48}$$

Consequently, for any fixed N in $\mathbb{N}$, if $s\beta$ is large enough, we get an $O(\varepsilon h^N)$ bound for estimate (1.48). This shows that one may assume essentially that $\hat{v}$ is supported for $h^\beta|\xi| \le C$ for some constant, some small $\beta > 0$. In the rest of this section, in order to avoid technicalities, we shall argue as if we had actually $|\xi| \le C$. The case $h^\beta|\xi| \le C$ may be treated similarly, up to an extra loss $h^{-\beta'}$ in the estimates of the remainders, $\beta' > 0$ being as small as we want. This extra loss does not affect the general pattern of the reasoning.

Take γ in $C_0^\infty(\mathbb{R})$, equal to one close to zero, with small enough support, and decompose

$$v = \underline{v}_\Lambda + \underline{v}_{\Lambda^c}, \tag{1.49}$$

where

$$\underline{v}_\Lambda = \mathrm{Op}_h^{\mathrm{W}}\Big(\gamma\Big(\frac{x+p'(\xi)}{\sqrt{h}}\Big)\Big)v,\ \underline{v}_{\Lambda^c} = \mathrm{Op}_h^{\mathrm{W}}\Big((1-\gamma)\Big(\frac{x+p'(\xi)}{\sqrt{h}}\Big)\Big)v, \tag{1.50}$$

i.e. $\underline{v}_\Lambda$ (resp. $\underline{v}_{\Lambda^c}$) is the contribution to v that is microlocally located inside (resp. outside) a $\sqrt{h}$-neighborhood of Λ. Then $\underline{v}_{\Lambda^c}$ satisfies, as a consequence of the L^2 estimate (1.44), a uniform L^∞ bound: define $\gamma_1(z) = \frac{(1-\gamma)(z)}{z}$ and write

$$\begin{aligned}\underline{v}_{\Lambda^c} &= \mathrm{Op}_h^{\mathrm{W}}\Big(\gamma_1\Big(\frac{x+p'(\xi)}{\sqrt{h}}\Big)\Big(\frac{x+p'(\xi)}{\sqrt{h}}\Big)\Big)v\\ &= h^{\frac{1}{2}}\mathrm{Op}_h^{\mathrm{W}}\Big(\gamma_1\Big(\frac{x+p'(\xi)}{\sqrt{h}}\Big)\Big)(\mathcal{L}_+v) + \text{remainder}.\end{aligned} \tag{1.51}$$

Since, at fixed x, $\xi \mapsto \gamma_1((x+p'(\xi))/\sqrt{h})$ is supported inside an interval of length $O(\sqrt{h})$, one may show that the L^∞ norm of the first term on the right-hand side of (1.51) is essentially bounded from above by $h^{-\frac{1}{4}}$ times its L^2 norm, i.e.

$$\|\underline{v}_{\Lambda^c}\|_{L^\infty} \leq Ch^{\frac{1}{4}}\|\mathcal{L}_+ v\|_{L^2}. \tag{1.52}$$

(Actually, if one takes into account the fact that on the support of $\hat{v}$ one has $|\xi| \leq ch^{-\beta}$ instead of $|\xi| \leq C$, one would get a power $h^{\frac{1}{4}-\beta'}$ instead of $h^{\frac{1}{4}}$, for some $0 < \beta' \ll 1$ in (1.52), that would not change the estimates below). In any case, combining with (1.44), we get an estimate

$$\|\underline{v}_{\Lambda^c}\|_{L^\infty} = O(\varepsilon h^{\frac{1}{4}-\delta'}), \quad \delta' > 0 \text{ small}. \tag{1.53}$$

If we assume a uniform a priori bound for v (that follows from the third inequality (1.39) and from (1.37)), we see that (1.53) implies that the difference $|v|^2 v - |\underline{v}_\Lambda|^2 \underline{v}_\Lambda$ will be $O(\varepsilon^3 h^{\frac{1}{4}-\delta'})$, so that replacing on the right-hand side of equation (1.40) $h|v|^2 v$ by $h|\underline{v}_\Lambda|^2 \underline{v}_\Lambda$ induces an error of the form of R in (1.47), i.e. we have

$$\big(D_t - \mathrm{Op}_h^{\mathrm{W}}(x\xi + p(\xi))\big)v = h\alpha|\underline{v}_\Lambda|^2\underline{v}_\Lambda + R. \tag{1.54}$$

We make act next $\mathrm{Op}_h^{\mathrm{W}}(\gamma((x+p'(\xi))/\sqrt{h}))$ on that equality. We get at the left-hand side $(D_t - \mathrm{Op}_h^{\mathrm{W}}(x\xi + p(\xi)))\underline{v}_\Lambda$ and a commutator whose principal contribution may be written as

$$-\frac{h^{\frac{3}{2}}}{i}\mathrm{Op}_h^{\mathrm{W}}\Big(\gamma'\Big(\frac{x+p'(\xi)}{\sqrt{h}}\Big)\Big)(\mathcal{L}_+ v). \tag{1.55}$$

This is of the same form as (1.51), up to an extra h factor, so that, arguing as in (1.52) and (1.53), we bound the L^∞ norm of (1.55) by $C\varepsilon h^{\frac{5}{4}-\delta'} = C\varepsilon t^{-\frac{5}{4}+\delta'}$. As $\delta' > 0$ is small, this is an integrable quantity that may enter in the remainders on the right-hand side of (1.47). As the action of $\mathrm{Op}_h^{\mathrm{W}}(\gamma((x+p'(\xi))/\sqrt{h}))$ on the right-hand side of (1.54) may be written under the same form, up to a modification of the remainder, we get

$$\big(D_t - \mathrm{Op}_h^{\mathrm{W}}(x\xi + p(\xi))\big)\underline{v}_\Lambda = h\alpha|\underline{v}_\Lambda|^2\underline{v}_\Lambda + R. \tag{1.56}$$

We make now a Taylor expansion of $x\xi + p(\xi)$ on Λ given by (1.45) and (1.46). As $\frac{d}{d\xi}(x\xi + p(\xi))|_\Lambda = 0$, we get

$$x\xi + p(\xi) = \varphi(x) + O\big((x + p'(\xi))^2\big). \tag{1.57}$$

The action of $\mathrm{Op}_h^{\mathrm{W}}((x+p'(\xi))^2)$ on $\underline{v}_\Lambda$ may be written essentially as (1.55), so provides again a contribution to R in (1.56). Finally, plugging (1.57) inside (1.56), we see that we get an equation of the form (1.47) for $w = \underline{v}_\Lambda$. This implies in particular that $\frac{\partial}{\partial t}|\underline{v}_\Lambda(t,\cdot)|^2$ is time integrable (since the coefficient α in (1.56) is real) and thus that $\|\underline{v}_\Lambda(t,\cdot)\|_{L^\infty}$ is bounded. Coming back to the expression (1.37) of u in terms of $v = \underline{v}_\Lambda + \underline{v}_{\Lambda^c}$, remembering (1.53) and adjusting constants, one gets that the a priori assumptions (1.39) imply that the last inequality in these formulas holds true with B replaced by $\frac{B}{2}$ (the reasoning for $W^{\rho_0,\infty}$ norms instead of L^∞ ones being similar). This shows that the bootstrap argument holds. Moreover, the ODE (1.47) may be used also in order to get asymptotics for u when times goes to infinity. ■

1.6 More general nonlinearities and normal forms

In model (1.32), we considered only a special case of nonlinearity namely $\alpha|u|^2u$. We used this special structure in order to get a Leibniz type rule (see (1.25)). However, we know that we should be able to obtain global solutions even for (some) cubic or quadratic nonlinearities that have a more general form. This is done in [18, 19] for quasi-linear Klein–Gordon equations with a nonlinearity satisfying a null condition (see also Stingo [82]). One makes use of "real" Klainerman vector fields instead of the operator L_+ above. On the other hand, for other equations like Schrödinger ones, the natural operator to be used in order to exploit dispersion is an operator like L_+, that is not a vector field. It is possible to reconcile both points of view using normal forms. Moreover, the use of the latter allows also one to treat quadratic nonlinearities. Consider as a model

$$\begin{aligned} \big(D_t - p(D_x)\big)u &= \alpha_0 u^2 + \alpha|u|^2u, \\ u|_{t=1} &= u_0, \end{aligned} \tag{1.58}$$

where $p(\xi) = \sqrt{1+\xi^2}$, α_0 is a complex number and α a real one. We would like to prove the analogous of Theorem 1.5.1, namely:

Theorem 1.6.1. *There are s_0, ρ_0 in $\mathbb{N}$, $\delta > 0$ such that, for any $s \geq s_0$, there are $\varepsilon_0 > 0$, $C > 0$ so that, for any $\varepsilon \in]0, \varepsilon_0]$, any u_0 in $H^s(\mathbb{R})$ satisfying* (1.22)*, the solution of* (1.58) *is global and satisfies for any $t \geq 1$,*

$$\|u(t,\cdot)\|_{H^s} \leq C\varepsilon t^{\delta}, \quad \|u(t,\cdot)\|_{W^{\rho_0,\infty}} \leq C\frac{\varepsilon}{\sqrt{t}}. \tag{1.59}$$

Remarks. We make the following observations.

- Again, one can obtain also the asymptotics of the solution when t goes to infinity, and in particular show modified scattering, and not just the dispersive estimate (1.59).
- One may consider more general quadratic and cubic nonlinearities than on the right-hand side of the first equation in model (1.58), as soon as they satisfy the null condition (see [18, 19, 82]).

The key idea of the proof is essentially to reduce (1.58) to (1.32) by normal forms. One cannot expect to get directly energy estimates on (1.58): for instance, the quadratic part of the nonlinearity has Sobolev norm bounded from above by $C\|u(t,\cdot)\|_{L^\infty}\|u(t,\cdot)\|_{H^s}$, so taking into account the a priori L^∞ estimate in (1.39), by $(C\varepsilon/\sqrt{t})\|u(t,\cdot)\|_{H^s}$. One thus would get an inequality of the form (1.6) with $r = 2$, which would give only exponential time control. Though, as shown first by Shatah [76] and Simon and Taflin [77], one may easily reduce the quadratic nonlinearity in (1.58) to a cubic one.

Lemma 1.6.2. *Define*

$$m(\xi_1, \xi_2) = \Big(\sqrt{1+\xi_1^2} + \sqrt{1+\xi_2^2} - \sqrt{1+(\xi_1+\xi_2)^2}\Big)^{-1}.$$

Then $m(\xi_1, \xi_2)$ is well defined,

$$|m(\xi_1, \xi_2)| \leq C\big(1 + \min(|\xi_1|, |\xi_2|)\big) \tag{1.60}$$

and if one sets

$$\mathrm{Op}(m)(u_1, u_2) = \frac{1}{(2\pi)^2} \int e^{ix(\xi_1+\xi_2)} m(\xi_1, \xi_2) \hat{u}_1(\xi_1) \hat{u}_2(\xi_2)\, d\xi_1\, d\xi_2, \tag{1.61}$$

one has for a fixed ρ_0 and any large enough s,

$$\|\mathrm{Op}(m)(u_1, u_2)\|_{H^s} \leq C\big(\|u_1\|_{W^{\rho_0,\infty}} \|u_2\|_{H^s} + \|u_1\|_{H^s} \|u_2\|_{W^{\rho_0,\infty}}\big). \tag{1.62}$$

Moreover, the map given by $u \mapsto u - \mathrm{Op}(m)(u, u)$ is a diffeomorphism from the open set $H^s \cap \{u \in W^{\rho_0,\infty} : \|u\|_{W^{\rho_0,\infty}} < r\}$ to its image, for small enough r, and if u is in that set, and solves equation (1.58), then $w = u - \mathrm{Op}(m)(u, u)$ solves

$$\big(D_t - p(D_x)\big)w = \alpha|w|^2 w - 2\alpha_0 \mathrm{Op}(m)(w^2, w) + R(w), \tag{1.63}$$

where R is a sum of contributions of degree of homogeneity larger than or equal to 4.

Proof. Estimate (1.60) follows by an immediate computation. It implies that one does not lose derivatives when applying $\mathrm{Op}(m)$ to a couple (u_1, u_2), i.e. that (1.62) holds without losing on s on the right-hand side. This allows one to construct the local diffeomorphism $u \mapsto w$. When one makes act $D_t - p(D_x)$ on $\mathrm{Op}(m)(u, u)$, one gets using equation (1.58), on the one hand

$$\mathrm{Op}(m)\big(p(D_x)u, u\big) + \mathrm{Op}(m)\big(u, p(D_x)u\big) - p(D_x)\mathrm{Op}(m)(u, u) \tag{1.64}$$

which, because of the definition of m is equal to u^2, and on the other hand contributions of the form

$$\mathrm{Op}(m)\big(\alpha_0 u^2 + \alpha|u|^2 u, u\big),\ \mathrm{Op}(m)\big(u, \alpha_0 u^2 + \alpha|u|^2 u\big). \tag{1.65}$$

If we compute the left-hand side of (1.63), we thus see that (1.64) compensates the quadratic term, and that we are left on the right-hand side with the $|u|^2 u$ term and expressions of the form (1.65). If we express u in terms of $w = u - \mathrm{Op}(m)(u, u)$, we shall get the cubic terms on the right-hand side of equation (1.63), and higher-order terms $R(w)$. These higher-order contributions are essentially of the form

$$R_k = \mathrm{Op}(m_k)(w, \dots, w, \bar{w}, \dots, \bar{w})$$

with $k \geq 4$, $m_k = m_k(\xi_1, \dots, \xi_k)$ a smooth function satisfying convenient estimates, and R_k defined as in (1.61) from

$$\begin{aligned}\mathrm{Op}(m_k)(u_1, \dots, u_k) = \frac{1}{(2\pi)^k} \int & e^{ix(\xi_1+\cdots+\xi_k)} m_k(\xi_1, \dots, \xi_k) \\ & \times \hat{u}_1(\xi_1) \dots \hat{u}_k(\xi_k)\, d\xi_1 \cdots d\xi_k.\end{aligned} \tag{1.66}$$

Then $R(w)$ satisfies estimates of the form

$$\|R(w)\|_{H^s} \leq C\|w\|^3_{W^{\rho_0,\infty}}\|w\|_{H^s} \tag{1.67}$$

if w stays in some ball of $W^{\rho_0,\infty}$, i.e. plays the role of a perturbation that is at least quartic. ∎

The preceding lemma thus reduces the case of a quadratic nonlinearity to a cubic one. Of course, the cubic term on the right-hand side of (1.63) is non-local, but this is not a real extra difficulty. Because of that, in order not no be disturbed by unessential technicalities, we shall pursue the reasoning considering a simple variant of (1.63), namely

$$\big(D_t - p(D_x)\big)u = \alpha|u|^2u + \alpha_1 u^3 + \alpha_2 u^2\bar{u}^2 \tag{1.68}$$

with α real, α_1, α_2 complex, forgetting any contribution homogeneous of order larger than or equal to 5 that is in any case easier to treat. Moreover, the special structure of the nonlinear terms on the right-hand side does not matter *except* the fact that α is real.

We have already noticed that a term like u^3 is not compatible with the action of L_+ on the right-hand side. The same holds for $u^2\bar{u}^2$. In order to get around that difficulty, one may try to perform a normal form in order to get rid of cubic or quartic terms. Nevertheless, unlike the quadratic case, one my not eliminate all these terms. Actually, to get rid of $u^2\bar{u}^2$ for instance, one would have to introduce a new unknown of the form $u - \mathrm{Op}(m_4)(u, u, \bar{u}, \bar{u})$, where m_4 would be the inverse of

$$-\sqrt{1+\xi_1^2} - \sqrt{1+\xi_2^2} + \sqrt{1+\xi_3^2} + \sqrt{1+\xi_4^2} - \sqrt{1+(\xi_1+\cdots+\xi_4)^2}. \tag{1.69}$$

But such a quantity vanishes for some values of $(\xi_1, \ldots, \xi_4)$. The idea to overcome that difficulty is to use "space-time normal forms" introduced by Germain, Masmoudi and Shatah in [29–32], and Germain and Masmoudi [28] (see also the review paper of Lannes [58] and the works of Hu and Masmoudi [44], Deng, Ionescu, Pausader and Pusateri [21], Wang [84] and Deng and Pusateri [22] for further applications and extensions of the method). These authors define and use space-time normal forms on the profiles of the solutions, namely the functions $e^{-itp(D_x)}u$. Here, we present an equivalent approach based on u itself and on microlocal cut-offs similar to those introduced in (1.50), following [20]. Actually, introducing again from u the unknown v given by (1.37), we rewrite (1.68) as

$$\big(D_t - \mathrm{Op}_h^{\mathrm{W}}\big(x\xi + p(\xi)\big)\big)v = h\alpha|v|^2v + h\alpha_1 v^3 + h^{\frac{3}{2}}\alpha_2 v^2\bar{v}^2 \tag{1.70}$$

using notation (1.41). The idea of space-time normal forms may be described in a geometrical way as follows. As we have seen above, a term like $v^2\bar{v}^2$ in (1.70) may not be fully eliminated by a usual (time) normal form since (1.69) may vanish for some values of the arguments. Though, we have seen in (1.34) that v defined by (1.37) is expected to be a function oscillating as $e^{i\frac{\varphi(x)}{h}}$, which means that we expect that v is "concentrated" on the manifold Λ defined in (1.45), (1.46). This means that, up to remainders having better time decay, we should hope to be able to design a normal form eliminating the term $v^2\bar{v}^2$ of (1.70) as soon as (1.69) does not vanish when the frequencies ξ_1, ξ_2 (corresponding to v) are set equal to $d\varphi(x)$ (by characterization (1.46) of Λ) and ξ_3, ξ_4 (corresponding to $\bar{v}$) are set equal to $-d\varphi(x)$. Notice that restricted to this subset, (1.69) is just equal to -1, so does not vanish. Of course, in order to justify that, we need to explain how we may reduce ourselves to the fact that v may be replaced by a function that is frequency localized on Λ, up to convenient remainders, and show how this allows one to prove energy estimates for the solution of (1.70). Our goal will thus be to prove the following:

Proposition 1.6.3. *The solution v of (1.70) satisfies estimates of the form*

$$\|v(t,\cdot)\|_{H_h^s} \le \|v(1,\cdot)\|_{H^s} + C\int_1^t \|v(\tau,\cdot)\|^2_{W^{\rho_0,\infty}_{h(\tau)}}\big(1+\|v(\tau,\cdot)\|_{W^{\rho_0,\infty}_{h(\tau)}}\big) \times \|v(\tau,\cdot)\|_{H^s_{h(\tau)}}\frac{d\tau}{\tau} \tag{1.71}$$

and

$$\|\mathcal{L}_+ v(t,\cdot)\|_{L^2} \le \|\mathcal{L}_+ v(1,\cdot)\|_{H^s} + C\int_1^t \|v(\tau,\cdot)\|^2_{W^{\rho_0,\infty}_{h(\tau)}}\big(1+\|v(\tau,\cdot)\|_{W^{\rho_0,\infty}_{h(\tau)}}\big) \times \|\mathcal{L}_+ v(\tau,\cdot)\|_{L^2}\frac{d\tau}{\tau}, \tag{1.72}$$

where $h=\frac{1}{t}$, $h(\tau)=\frac{1}{\tau}$, $\|v\|_{H_h^s} = \|\langle hD_x\rangle^s v\|_{L^2}$, $\|v\|_{W_h^{\rho_0,\infty}} = \|\langle hD_x\rangle^{\rho_0} v\|_{L^\infty}$ and $\mathcal{L}_+$ is defined in (1.43).

Remark. These estimates are the translation on v of bounds of the form (1.5) and (1.27) on u according to (1.37). Consequently, if we prove them, we shall get, as in the proof of Theorem 1.5.1, that an a priori set of inequalities of the form (1.39) will imply that the first two of these bounds hold with A replaced by $\frac{A}{2}$.

Proof of the proposition. As indicated before the statement, in order to get (1.71) and (1.72), we have to perform a "space-time" normal form. More precisely, we shall decompose in the $v^3, v^2\bar{v}^2$ terms of (1.70) each factor v as

$$v = \underline{v}_\Lambda + \underline{v}_{\Lambda^c}, \tag{1.73}$$

where $\underline{v}_{\Lambda^c}$ will have better bounds than v, so that cubic or quartic terms involving at least one factor $\underline{v}_{\Lambda^c}$ will provide remainders. In a second step, we shall get rid of the remaining nonlinearities $\alpha_1\underline{v}_\Lambda^3$, $\alpha_2\underline{v}_\Lambda^2\bar{\underline{v}}_\Lambda^2$ by a normal form process. The function $\underline{v}_\Lambda$ in (1.73) will be defined as in (1.49), except that we cut-off around an $O(1)$-neighborhood of Λ instead of an $O(\sqrt{h})$ one, i.e. we define now

$$\underline{v}_\Lambda = \mathrm{Op}_h^{\mathrm{W}}\big(\gamma(x+p'(\xi))\big)v,\ \underline{v}_{\Lambda^c} = \mathrm{Op}_h^{\mathrm{W}}\big((1-\gamma)(x+p'(\xi))\big)v. \tag{1.74}$$

(Actually, the above definition is the correct one when the frequency ξ stays in a compact set. It should be adapted for large ξ, but we forget this technical detail in this introduction.) Then $\underline{v}_{\Lambda^c}$ will satisfy estimates with an $O(h)$ gain, as we may write it essentially under the form

$$\underline{v}_{\Lambda^c} = h\mathrm{Op}_h^{\mathrm{W}}\big(\gamma_1(x + p'(\xi))\big)(\mathcal{L}_+ v), \tag{1.75}$$

where $\gamma_1(z) = \frac{(1-\gamma)(z)}{z}$, so that

$$\|\underline{v}_{\Lambda^c}\|_{L^2} \leq Ch\|\mathcal{L}_+ v\|_{L^2}.$$

Decomposing on the right-hand side of (1.70) $v = \underline{v}_\Lambda + \underline{v}_{\Lambda^c}$, one has thus

$$\big(D_t - \mathrm{Op}_h^{\mathrm{W}}(x\xi + p(\xi))\big)v = h\alpha|v|^2 v + h\alpha_1(\underline{v}_\Lambda)^3 + h^{\frac{3}{2}}\alpha_2 \underline{v}_\Lambda^2 \bar{\underline{v}}_\Lambda^2 + h^2 S(v), \tag{1.76}$$

where $S(v)$, coming from monomials involving at least one factor $\underline{v}_{\Lambda^c}$, satisfies an estimate of the form

$$\|S(v)\|_{L^2} \leq C\|v\|_{L^\infty}^2 \|\mathcal{L}v\|_{L^2} \tag{1.77}$$

as long as $\|v\|_{L^\infty}$ stays bounded. Actually, one has to be more careful when making the above estimates, since Λ has a degeneracy when ξ goes to infinity. The preceding reasoning works for $|\xi|$ staying in a compact set , or equivalently for x staying in a compact subset of $]-1, 1[$. The general case is a little bit more involved, and in particular estimate (1.77) holds with $\|v\|_{L^\infty}$ replaced by $\|v\|_{W_h^{\rho_0,\infty}}$ for some ρ_0.

Since making act the operator $\mathcal{L}_+$ on S makes lose a factor h^{-1} (see the definition (1.43) of $\mathcal{L}_+$), we get that

$$h^2\|\mathcal{L}_+ S(v)\|_{L^2} \leq Ch\|v\|_{L^\infty}^2 \|\mathcal{L}_+ v\|_{L^2}, \tag{1.78}$$

which will be the kind of estimate we want for remainders. By (1.25) with $p = 1$, rewritten in terms of the unknown v, we have also

$$h\|\mathcal{L}_+(|v|^2 v)\|_{L^2} \leq Ch\|v\|_{L^\infty}^2 \|\mathcal{L}_+ v\|_{L^2}. \tag{1.79}$$

On the other hand, the remaining contributions on the right-hand side of (1.77) would not satisfy such estimates, but may now be eliminated by normal forms. Actually, take χ in $C_0^\infty(\mathbb{R})$, equal to one close to zero, and define

$$\begin{aligned} m_4(x, \xi_1, \dots, \xi_4) = {} & \prod_{j=1}^{2} \chi(x + p'(\xi_j)) \prod_{j=3}^{4} \chi(x - p'(\xi_j)) \\ & \times \Big[-\sqrt{1+\xi_1^2} - \sqrt{1+\xi_2^2} + \sqrt{1+\xi_3^2} \\ & + \sqrt{1+\xi_4^2} - \sqrt{1+(\xi_1+\cdots+\xi_4)^2}\Big]^{-1}. \end{aligned} \tag{1.80}$$

This function is well defined, as the term inside the bracket does not vanish on the support of the cut-off: actually (again forgetting what happens for large frequencies), on the support of the cut-off, $\xi_j = d\varphi(x) + o(1)$, $j = 1,2$, $\xi_j = -d\varphi(x) + o(1)$, $j = 3,4$, so that the term inside the bracket is equal to $-1 + o(1)$, and thus does not vanish. Consequently, if we define

$$\begin{aligned}\operatorname{Op}_h(m_4)&\big(\underline{v}_\Lambda, \underline{v}_\Lambda, \bar{\underline{v}}_\Lambda, \bar{\underline{v}}_\Lambda\big)\\ &= \frac{1}{(2\pi)^4}\int e^{ix(\xi_1+\cdots+\xi_4)} m_4(\xi_1,\ldots,\xi_4)\\ &\qquad\times \hat{\underline{v}}_\Lambda(\xi_1)\hat{\underline{v}}_\Lambda(\xi_2)\widehat{\bar{\underline{v}}}_\Lambda(\xi_3)\widehat{\bar{\underline{v}}}_\Lambda(\xi_4)\, d\xi_1\cdots d\xi_4,\end{aligned} \tag{1.81}$$

one obtains that

$$\big(D_t - \operatorname{Op}_h^{\mathrm{W}}(x\xi + \sqrt{1+\xi^2})\big)\big(\operatorname{Op}_h(m_4)(\underline{v}_\Lambda,\ldots,\bar{\underline{v}}_\Lambda)\big) = \underline{v}_\Lambda^2\bar{\underline{v}}_\Lambda^2 + \text{remainder},$$

where the remainder, coming from the nonlinearities of the equation, contains at least one h factor. Defining in the same way some cubic symbol m_3, in order to eliminate the $\underline{v}_\Lambda^3$ term in (1.76), one gets that

$$\begin{aligned}\big(D_t - \operatorname{Op}_h^{\mathrm{W}}(x\xi + \sqrt{1+\xi^2})\big)\big(v - h\operatorname{Op}_h(m_3)(\underline{v}_\Lambda, \underline{v}_\Lambda, \underline{v}_\Lambda)\\ - h^{\frac{3}{2}}\operatorname{Op}_h(m_4)(\underline{v}_\Lambda,\ldots,\bar{\underline{v}}_\Lambda)\big) = h^2 S(v) + h\alpha|v|^2 v\end{aligned} \tag{1.82}$$

for a new $S(v)$ satisfying (1.77).

In other words, we have reduced ourselves to an equation where the right-hand side has the same structure as in (1.7) (up to changing the unknown u to v by (1.37)), modulo a remainder $h^2 S(v)$ that has better time decay. Using estimates of the form (1.78)–(1.79), one thus gets, applying L^2 energy inequalities to (1.82) and denoting

$$w = v - h\operatorname{Op}_h(m_3)(\underline{v}_\Lambda, \underline{v}_\Lambda, \underline{v}_\Lambda) - h^{\frac{3}{2}}\operatorname{Op}_h(m_4)(\underline{v}_\Lambda,\ldots,\bar{\underline{v}}_\Lambda),$$

that

$$\|\mathcal{L}_+ w(t,\cdot)\|_{L^2} \le \|\mathcal{L}_+ w(1,\cdot)\|_{L^2} + \int_1^t \|v(\tau,\cdot)\|_{L^\infty}^2 \|\mathcal{L}_+ v(\tau,\cdot)\|_{L^2}\frac{d\tau}{\tau}. \tag{1.83}$$

As one may show that $\|\mathcal{L}_+ w(t,\cdot)\|_{L^2}$ is equivalent to $\|\mathcal{L}_+ v(t,\cdot)\|_{L^2}$, one does get an estimate of the form (1.72). ■

Remark. As already mentioned, in the proof of Proposition 1.6.3, we argued as if the frequencies were staying in a compact set. When one makes the reasoning taking into account what happens also for large frequencies, one gets a lower bound of the bracket in (1.80) computed for ξ_j in a convenient neighborhood of $\pm d\varphi(x)$ by a negative power of $\langle d\varphi(x)\rangle$. Since for all j, $\langle d\varphi(x)\rangle \sim \langle \xi_j\rangle$ if $(\xi_1,\ldots,\xi_4)$ is in the support of (1.80), one may write $\langle d\varphi(x)\rangle \sim 1 + \max_2(|\xi_1|,\ldots,|\xi_4|)$, and the bounds one gets in general for a symbol of the form m_4 is

$$|m_4(x,\xi_1,\ldots,\xi_4)| \le C\big(1 + \max_2(|\xi_1|,\ldots,|\xi_4|)\big)^{N_0} \tag{1.84}$$

for some N_0. Because of that, one gets bounds of type

$$\|\mathrm{Op}_h(m_4)(v, \dots, \bar{v})\|_{H_h^s} \le C\|v\|^3_{W_h^{\rho_0,\infty}}\|v\|_{H_h^s} \tag{1.85}$$

for any s and with ρ_0 depending only on N_0. In other words, coming back to the unknown u, one obtains an estimate similar to (1.62). These inequalities (1.84) and (1.85) explain why one gets in Proposition 1.6.3 upper bounds involving $W_h^{\rho_0,\infty}$ norms instead of L^∞ ones.

End of proof of Theorem 1.6.1. As for the proof of Theorem 1.5.1, one has just to bootstrap estimates (1.39), showing that if they hold on some time interval and A, B have been taken large enough and ε small enough, then they still hold with A, B replaced by $\frac{A}{2}, \frac{B}{2}$. We have seen after the statement of Proposition 1.6.3 that this holds for the first two inequalities (1.39). To show that the last one holds, with B replaced by $\frac{B}{2}$, one argues as in the proof of Theorem 1.5.1. Actually, in that proof, we did not really use the special form of the nonlinearity in (1.40) (except the fact that α is real), and the same arguments hold for an equation like (1.68). ■

1.7 Perturbations of non-zero stationary solution

Our main goal in this book is to study the perturbation of a non-zero stationary solution of a cubic wave equation in dimension one. In this section, we mention some results and references on that kind of problems. The first set of questions one may ask is the *orbital* stability of stationary solutions.

Let us mention first the result of Henry, Perez and Wreszinski [41] that will be very relevant for us. Consider U a C^2 function on an interval $[a_-, a_+]$ satisfying $U \ge 0$, $U(a_-) = U(a_+) = 0$, $U''(a_\pm) > 0$. Assume moreover that there is a smooth strictly increasing function $x \mapsto H(x)$ solving the equation

$$H''(x) = U'(H(x))$$

such that

$$\lim_{x\to\pm\infty} H(x) = a_\pm$$

and that

$$E_0 = \int_{\mathbb{R}} \Big(\frac{H'(x)^2}{2} + U(H(x))\Big)\, dx < +\infty.$$

Define for any function ϕ and any $q > 0$,

$$d_q(\phi) = \inf_{c\in\mathbb{R}} \int_{\mathbb{R}} \big((\phi'(x) - H'(x+c))^2 + q(\phi(x) - H(x+c))^2\big)\, dx.$$

One may state the main result of [41] as follows.

Theorem 1.7.1. *There are positive constants r, q, k such that if $(t, x) \mapsto \phi(t, x)$ is the solution of*

$$(\partial_t^2 - \partial_x^2)\phi + U'(\phi) = 0 \tag{1.86}$$

satisfying $\phi(0, \cdot) \in H^1_{\text{loc}}(\mathbb{R})$, $\partial_x\phi(0, \cdot), \partial_t\phi(0, \cdot) \in L^2(\mathbb{R})$, and

$$\begin{gathered} d_q(\phi(0,\cdot)) < r, \\ \int_{\mathbb{R}} \left(\frac{\partial_t\phi(0,x)^2}{2} + \frac{\partial_x\phi(0,x)^2}{2} + U(\phi(0,x)) \right) dx < E_0 + kr^2, \end{gathered} \tag{1.87}$$

then ϕ is globally defined and for any t

$$d_q(\phi(t, \cdot)) \leq r. \tag{1.88}$$

This theorem means that H is orbitally stable, in that sense that an initial data that is close enough to H gives rise to a solution that remains at any time close to a translation of H. It applies in particular to $U(\phi) = \frac{1}{4}(\phi^2 - 1)^2$, $H(x) = \tanh(\frac{x}{\sqrt{2}})$ and $a_\pm = \pm 1$, i.e. it shows the orbital stability of the "kink", that is the stationary solution $H(x) = \tanh(\frac{x}{\sqrt{2}})$ of the Φ^4 model given by the equation

$$(\partial_t^2 - \partial_x^2)\phi = \phi - \phi^3. \tag{1.89}$$

The question of orbital stability has been then widely studied for other equations. In particular, we refer to Weinstein [86] for orbital stability of Schrödinger or generalized KdV equations. References to earlier works on that topic may be found in the reference list of that paper.

Once orbital stability is established for a given equation, the next step is to study asymptotic stability. For Schrödinger equations, the first results are due to Buslaev and Perelman [5–7] in dimension one and to Soffer and Weinstein [78] in higher dimension. Buslaev and Perelman consider a one-dimensional Schrödinger equation, of the form

$$i\partial_t\psi = -\partial_x^2\psi + F(|\psi|^2)\psi. \tag{1.90}$$

Under convenient assumptions on F, one may construct soliton solutions of the equation, that have the form

$$e^{-i\beta_0 - it\omega_0 + \frac{i}{2}xv_0}\phi(x - b_0 - tv_0) \tag{1.91}$$

for constants $\beta_0, \omega_0, b_0, v_0$ and where ϕ is a smooth exponentially decaying function. The main result of the above references is that if one solves the initial value problem for (1.90), with initial condition close to the preceding soliton solution, then the solution may be written when time goes to infinity as a sum of a modified soliton, i.e. a function of the form (1.91) (with different values of the parameters $\beta_0, \dots, v_0$), of a solution to a linear Schrödinger equation and of a remainder that converges to zero in L^2.

In the work of Soffer and Weinstein, one introduces a potential in the linear part of the operator, i.e. one considers an equation of the form

$$i\partial_t \phi = -\Delta\phi + (V(x) + \lambda|\phi|^{m-1})\phi \tag{1.92}$$

in $d = 2$ or 3 space dimension, and for $1 < m < \frac{d+2}{d-2}$. They assume, among other things, that the operator $-\Delta + V(x)$ has exactly one eigenvalue, that is moreover strictly negative. They show that for E close to that eigenvalue, there is a solution of (1.92) of the form $e^{-iEt}\psi_E(x)$, with ψ_E smooth and decaying. Then, under some further assumption, they prove that, if one solves the Cauchy problem starting from an initial data that is close to $e^{i\gamma_0}\psi_{E_0}$, for given E_0 close to the eigenvalue, γ_0 real, then the solution may be written at any time t as $e(t)\psi_{E(t)} + R(t)$ where $E(t)$ is real, $e(t)$ is in the unit circle of $\mathbb{C}$ and $R(t)$ goes to zero in a weighted Sobolev space. We refer to [78] for a precise description of the asymptotics of $t \mapsto E(t), e(t)$ when time goes to infinity.

Following the above references, a lot of results concerning asymptotic stability for solutions to nonlinear Schrödinger equations or Gross–Pitaevsky ones have been obtained. Limiting ourselves to one-dimensional problems, and without trying to give an exhaustive list of references, one may cite Buslaev and Sulem [8], Bethuel, Gravejat and Smets [4], Gravejat and Smets [36], Germain, Pusateri and Rousset [35], Cuccagna and Pelinovski [16], Cuccagna and Jenkins [15], Gang and Sigal [25–27], Cuccagna, Georgiev and Visciglia [14]. Still in one space dimension, analogous results have been obtained for (generalized) KdV equations, by Pego and Weinstein [73], Germain, Pusateri and Rousset [34], Martel and Merle [67–69] and for Benjamin–Ono equation by Kenig and Martel [48]. Let us point out that for Schrödinger or gKdV equations, the perturbation of the initial data induces a non-zero translation speed on the stationary solution, so that the perturbed solution is the sum of a *progressive wave* and of a dispersive part. This will be in contrast with the results we shall obtain in this book, where the bound state that is perturbed will remain stationary.

Let us discuss now some results more closely related to our work, concerning nonlinear wave equations. A main breakthrough has been made by Soffer and Weinstein who in [79] consider an equation similar to (1.92), but where the Schrödinger operator is replaced by the wave (or Klein–Gordon) one in three space dimension, namely

$$\partial_t^2 \phi = (\Delta - V(x) - m^2)\phi + \lambda\phi^3, \tag{1.93}$$

where λ is some real constant, $m > 0$ and V is a smooth decaying potential. One assumes among other things that $-\Delta + V + m^2$ has $[m^2, +\infty[$ as continuous spectrum and that there is a unique positive eigenvalue $0 < \Omega^2 < m^2$. One denotes by φ a normalized eigenfunction associated to that eigenvalue, so that for any R, θ in $\mathbb{R}$, $(t, x) \mapsto R\cos(\Omega t + \theta)\varphi(x)$ is a solution to equation (1.93) when $\lambda = 0$. The main result of [79] asserts that if one solves (1.93) with small initial data in weighted Sobolev spaces of smooth enough and decaying enough functions, the solution at time t may be written under the form

$$\phi(t,x) = R(t)\cos(\Omega t + \theta(t))\varphi(x) + \eta(t,x), \tag{1.94}$$

where $R(t) = O(|t|^{-\frac{1}{4}})$ and $\|\eta(t,\cdot)\|_{L^8} = O(|t|^{-\frac{3}{4}})$ when t goes to $\pm\infty$. This result holds under a special non-resonance condition, Fermi's golden rule, that we shall further discuss below in the framework of our problem.

The above breakthrough has been at the origin of many other works. Let us mention in particular Bambusi and Cuccagna [3] who generalized the result of [80] to a wider framework, namely the case when the operator $-\Delta + V(x) + m^2$ has several eigenvalues instead of just one. Closer to our main result in this book, let us mention the work where Cuccagna [13] studies asymptotic stability of a kink solution in three space dimension. More precisely, one considers the solution H of (1.89) as a solution independent of two of the three space variables of the equation $(\partial_t^2 - \Delta)\phi = \phi - \phi^3$ on $\mathbb{R}^3$. The main result of [13] asserts that if one starts from initial data that are a small perturbation of $(H, 0)$ by a smooth compactly supported function on $\mathbb{R}^3$, then the solution of the evolution equation may be written as $H + \phi(t,\cdot)$, where $\phi(t,\cdot)$ is $O(|t|^{-\frac{1}{2}})$ in L^∞. The proof uses the fact that in three space dimension, one has much better dispersive decay than on the real line.

1.8 The kink problem. I

The main goal of this book is to study long time dispersion for small perturbations of the "kink" $H(x) = \tanh(\frac{x}{\sqrt{2}})$ that is a stationary solution of equation (1.89) that we recall below

$$(\partial_t^2 - \partial_x^2)\phi = \phi - \phi^3.$$

We have seen in the preceding section (see Theorem 1.7.1) that H is orbitally stable, and one wants to study its asymptotic stability. In order to do so, one writes ϕ under the form

$$\phi(t,x) = H(x) + \varphi(t\sqrt{2}, x\sqrt{2}) \tag{1.95}$$

and we aim at describing the asymptotics of φ, in particular its dispersive properties, when at initial time φ is small in a convenient weighted Sobolev space. By Theorem 1.7.1, we know that φ is globally defined. It satisfies by direct computation the equation

$$\big(D_t^2 - (D_x^2 + 1 + 2V(x))\big)\varphi = \kappa(x)\varphi^2 + \frac{1}{2}\varphi^3, \tag{1.96}$$

where

$$V(x) = -\frac{3}{4}\cosh^{-2}\Big(\frac{x}{2}\Big), \quad \kappa(x) = \frac{3}{2}\tanh\frac{x}{2}. \tag{1.97}$$

The fact that the linear part of equation (1.96) contains a non-zero potential has two consequences: first, as seen in the preceding section, the operator $D_x^2 + 1 + 2V(x)$ may have bound states (and it has for the potential given by (1.97)). Second, even in the absence of bound states, that operator does not have nice commutation properties with the operator L_+ that we used in order to get dispersion in Sections 1.5 and 1.6.

Let us first discuss some results that are known concerning equations of the form (1.96) either in the case of potentials without bound states, or for equations of that form with $V = 0$ but where the nonlinearities have coefficients that are non- constant functions of x, as on the right-hand side of (1.97). Such results have been proved by Kopylova [53] for linear Klein–Gordon equations in a moving frame and, in the nonlinear case, by Lindblad and Soffer [66], Lindblad, Lührmann and Soffer [60,61], Lindblad, Lührmann, Schlag and Soffer [59], Sterbenz [81]. Very recently, Germain and Pusateri [33] obtained the most general result in that framework. They consider a model version of (1.96) of the form

$$\big(\partial_t^2 - \partial_x^2 + V(x) + m^2\big)\varphi = a(x)\varphi^2, \tag{1.98}$$

where $a(x)$ is a function similar to κ on the right-hand side of (1.96), i.e. a smooth function that has finite limits at $\pm\infty$ and whose derivative is rapidly decaying. The potential V is assumed to be Schwartz and such that $-\partial_x^2 + V$ has no bound state. One of the results of [33] may be stated as follows:

Theorem 1.8.1. *Let V be a* generic *potential without bound state, $m > 0$. There is $\varepsilon_0 > 0$ such that for any $\varepsilon \in]0, \varepsilon_0]$, equation* (1.97) *has for any (φ_0, φ_1) satisfying*

$$\big\|\big(\sqrt{-\partial_x^2 + V + 1}\varphi_0, \varphi_1\big)\big\|_{H^4} + \big\|\langle x\rangle\big(\sqrt{-\partial_x^2 + V + 1}\varphi_0, \varphi_1\big)\big\|_{H^1} \le \varepsilon$$

a unique global solution corresponding to the initial data $\varphi|_{t=0} = \varphi_0$, $\partial_t\varphi|_{t=0} = \varphi_1$. Moreover, the dispersive estimate

$$\big\|\big(\sqrt{-\partial_x^2 + V + 1}\varphi_0, \varphi_1\big)\big\|_{L^\infty} \le C\varepsilon(1 + |t|)^{-\frac{1}{2}} \tag{1.99}$$

holds and for some small $\delta > 0$

$$\|\varphi(t,\cdot)\|_{H^5} + \|\partial_t\varphi(t,\cdot)\|_{H^4} \le C\varepsilon(1 + |t|)^{\delta}. \tag{1.100}$$

Finally, let us mention that for nonlinearities with coefficients that are *rapidly enough decaying* in x, Lindblad, Lührmann and Soffer [60] (in the case $V \equiv 0$) and Lindblad, Lührmann, Schlag and Soffer [59] (for generic potentials) could show that a dispersive bound like (1.99) *does not* hold in general, and has to be replaced by the product of the right-hand side with a logarithmic loss.

Remark. The assumption that V is generic is explained in Chapter 2 below. The result of [33] is actually more general than Theorem 1.8.1 above. It also applies to non-generic potentials if one makes in addition evenness/oddness assumptions. Let us also mention that the question of asymptotic stability estimates on a compact domain in space, when the linearized equation on the stationary solution has no bound state, has been addressed by Kowalczyk, Martel, Muñoz and Van Den Bosch [57] for some models of semilinear wave equations.

Let us explain the new difficulties one has to take into account to prove a result of the form above in comparison with the case $V \equiv 0$. Clearly, if one wanted to apply the operator

$$L_{+,m} = x + t\frac{D_x}{(m^2 + D_x^2)^{\frac{1}{2}}}$$

(or a "true" Klainerman vector field like $t\partial_x + x\partial_t$) to equation (1.97), its commutator with the potential V would generate a new term with coefficients growing like t, which makes the method inapplicable. In order to circumvent such a difficulty, two approaches are possible. The one implemented by Germain and Pusateri relies on the use of the "modified Fourier transform", which is a version of the Fourier transform well adapted to $-\Delta + V$ instead of being tailored to $-\Delta$. They introduce then the profile g of the solution by

$$g(t,x) = e^{it\sqrt{-\partial_x^2+V+m^2}}\Big(\partial_t - i\sqrt{-\partial_x^2 + V + m^2}\Big)\phi \tag{1.101}$$

and its modified Fourier transform $\tilde{g}(t,\xi)$. The analogue of what does work in the case $V \equiv 0$ would be to get estimates of $\|\partial_\xi \tilde{g}(t,\xi)\|_{L^2}$ (which is related to $\|L_{+,m}\phi\|_{L^2}$ when $V \equiv 0$). It turns out that, in order to get the most general statement of their paper, Germain and Pusateri have to introduce a bigger space than L^2 in which $\partial_\xi \tilde{g}$ has to be estimated, allowing for some degeneracy close to a special frequency. They have then to combine estimates in that space with normal forms constructed from the modified Fourier transform.

The approach we use in this book is the one of wave operators. Let us just say here that, when V is a potential in $\mathcal{S}(\mathbb{R})$, without bound states, one may construct a bounded operator W_+ on L^2 such that

$$W_+^* W_+ = \mathrm{Id}, \quad W_+ W_+^* = \mathrm{Id} \quad \text{and} \quad W_+^*(-\Delta + V)W_+ = -\Delta.$$

Applying W_+^* to (1.98), one thus gets

$$\big(\partial_t^2 - \partial_x^2 + m^2\big)W_+^*\varphi = W_+^*(a(x)\varphi^2).$$

If $w = W_+^*\varphi$, one is thus reduced to an equation of the form

$$\big(\partial_t^2 - \partial_x^2 + m^2\big)w = W_+^*(a(x)(W_+ w)^2), \tag{1.102}$$

i.e. to an equation for which the linear part has again constant coefficients, and thus has nice commutation properties relatively to $t\partial_x + x\partial_t$ or to $L_{+,m}$. Of course, the drawback is that the right-hand side of (1.102) is no longer a local nonlinearity, but involves the operators W_+, W_+^*. In the framework we shall be interested in, namely odd initial conditions and odd coefficient $a(x)$, it turns out that W_+, W_+^* may be expressed from pseudo-differential operators $b(x, D_x)$, with a symbol $b(x,\xi)$ such that $\frac{\partial b}{\partial x}(x,\xi)$ is rapidly decaying when $|x|$ tends to infinity. We shall explain in more detail in Chapter 2 how we treat an equation of the form (1.102). Let us just say now that if we had a cubic nonlinearity on the right-hand side, one could use directly vector fields methods on w. For a quadratic nonlinearity, one has to make use first of normal forms in order to reduce quadratic nonlinearities to cubic ones. The difference with Lemma 1.6.2 is that, because of the presence of $W_+, W_+^*, a(x)$ on the right-hand side of (1.102), one has to consider quadratic corrections of the form (1.61), but with a symbol $m(x, \xi_1, \xi_2)$ that depends also on x. This introduces new commutators, involving quadratic operators associated to the symbol $\frac{\partial m}{\partial x}(x, \xi_1, \xi_2)$. Though, as the latter is rapidly decaying in x, and since we limit ourselves to odd solutions, such terms form remainders that are not fully negligible, but that may be treated more easily than in the more general case considered by Germain and Pusateri [33] or Lindblad, Lührmann and Soffer [60].

1.9 The kink problem II. Coupling with the bound state

In the preceding section, we discussed an equation of the form (1.98) with a potential V that has no bound state. In this section, we go back to the kink problem (1.96), where the potential V given by (1.97) does have bound states, so that the preceding discussion does not apply.

Our starting point has been the paper [56] of Kowalczyk, Martel and Muñoz, where the authors study the asymptotics of solutions of (1.89) when one takes as an initial condition an odd perturbation of $(H, 0)$ that is small enough in the energy norm. They prove that the perturbation of the solution $(\varphi, \partial_t \varphi)$ may be decomposed under the form

$$(\varphi(t,x), \partial_t \varphi(t,x)) = (u_1(t,x), u_2(t,x)) + (z_1(t), z_2(t))Y(x), \tag{1.103}$$

where Y is in $\mathcal{S}(\mathbb{R})$ and is a normalized *odd* eigenfunction of $-\frac{1}{2}\partial_x^2 + V(x)$, $z_j(t)$ are scalar functions of time and $(u_1(t,x), u_2(t,x))$ is the dispersive part of the solution. The main result of [56] states that the functions $t \mapsto z_j(t)$ decay in time in the sense that

$$\int_{-\infty}^{+\infty} \big(|z_1(t)|^4 + |z_2(t)|^4\big)\, dt < +\infty$$

and that the *local* energy of (u_1, u_2) satisfies

$$\int_{-\infty}^{+\infty} \int_{\mathbb{R}} \big((\partial_x u_1)^2 + u_1^2 + u_2^2\big)(t,x) e^{-c_0|x|}\, dt\, dx < +\infty.$$

At the light of the discussion previously given in the case of small perturbations of the zero solution of nonlinear Klein–Gordon equations, or for (1.98) with a potential that has no bound state, the above inequalities raise the following questions: making eventually stronger assumptions on the smoothness/decay of the initial perturbation, could one get an explicit decay rate for the preceding quantities, instead of just integral inequalities? Moreover, could one obtain decay estimates for $\|u_j(t, \cdot)\|_{L^\infty}$ instead of just local in space decay?

A more long term objective might be to obtain for odd perturbations of the kink solution of (1.89) a description as precise as the one that holds when $V \equiv 0$ or when V is a potential without bound state. We are far from being able to achieve that in this paper, where as a first step we aim at describing the perturbed solution up to time ε^{-4} if ε is the small size of the smooth decaying perturbation of the kink at initial time. Recall that if we look for solutions of (1.89) under the form (1.95), we get that the perturbation φ satisfies (1.96), with notation (1.97). We already mentioned that the Schrödinger operator $-\partial_x^2 + 2V(x)$ has discrete spectrum: it has two negative eigenvalues -1 and $-\frac{1}{4}$ and absolutely continuous spectrum $[0, +\infty[$. Eigenvalue -1 will not be of interest to us as it is associated to an even eigenfunction, while we solve (1.96) for odd initial data. Consequently, restricting ourselves to odd solutions, one may decompose the solution of (1.96) as $\varphi = P_{\text{ac}}\varphi + \langle \varphi, Y \rangle Y$, where P_{ac} is the projector on the absolutely continuous spectrum $[0, +\infty[$ and Y is an (odd) normalized eigenfunction associated to eigenvalue $-\frac{1}{4}$. Setting $a(t) = \langle Y, \varphi \rangle$, one may deduce from (1.96) that $(a, P_{\text{ac}}\varphi)$ satisfies a coupled system of ODE/PDE (see (2.9) in Chapter 2).

Our main result asserts the following: Let $c > 0$ be given and consider (1.96) with initial data $\varphi|_{t=1} = \varepsilon\varphi_0$, $\partial_t\varphi|_{t=1} = \varepsilon\varphi_1$ with (φ_0, φ_1) satisfying for some large enough s,

$$\|\varphi_0\|_{H^{s+1}}^2 + \|\varphi_1\|_{H^s}^2 + \|x\varphi_0\|_{H^1}^2 + \|\varphi_1\|_{L^2}^2 \leq 1. \tag{1.104}$$

Then, if $\varepsilon < \varepsilon_0$ is small enough, the decomposition $\varphi(t, \cdot) = P_{\text{ac}}\varphi(t, \cdot) + a(t)Y$ of the solution of (1.96) satisfies

$$\begin{aligned} |a(t)| + |a'(t)| &= O(\varepsilon(1 + t\varepsilon^2)^{-\frac{1}{2}}), \\ \|P_{\text{ac}}\varphi(t, \cdot)\|_{L^\infty} &= O(t^{-\frac{1}{2}}(\varepsilon^2\sqrt{t})^{\theta'}), \end{aligned} \tag{1.105}$$

where $\theta' \in]0, \frac{1}{2}[$, *as long as* $t \leq \varepsilon^{-4+c}$. Let us mention that we limit our study to positive times (that does not reduce generality) and that, in order to simplify some notation, we take the Cauchy data at $t = 1$ instead of $t = 0$. Moreover, the statements we get in Theorem 2.1.1 below give more precise information that (1.105). We just stress here the fact that (1.105) provides the information we are looking for, namely an explicit decay rate for a and $P_{\text{ac}}\varphi$, up to time ε^{-4+c}.

We notice that the dispersive estimate obtained for $\|P_{\text{ac}}\varphi\|_{L^\infty}$ is pretty similar to the bound in $\varepsilon t^{-\frac{1}{2}}$ that holds for small solutions of equations $(\partial_t^2 - \partial_x^2 + 1)u = N(u)$. Here, when $t \leq \varepsilon^{-4+c}$, we get that

$$\|P_{\text{ac}}\varphi\|_{L^\infty} = O(\varepsilon^{\frac{c}{2}\theta'} t^{-\frac{1}{2}}),$$

i.e. an estimate in $c(\varepsilon)t^{-\frac{1}{2}}$, with $c(\varepsilon)$ going to zero with zero. Of course, if t goes close to ε^{-4}, the small factor in front of $t^{-\frac{1}{2}}$ in the second estimate (1.105) gets closer and closer to one, and this explains why our result is limited to times that are $O(\varepsilon^{-4+c})$. We shall comment more on that below.

Let us remark also that for dispersive estimates of the form (1.105), there is a "trivial" regime, corresponding to $t \leq c\varepsilon^{-2}$. For such times, the ODE satisfied by $a(t)$, from which we shall deduce the first bound (1.105), is in a small time regime, before any singularity could form. On the other hand, to reach a time of size ε^{-4+0}, one has to use the structure of that ODE, namely exploit Fermi's golden rule that we shall discuss in Chapter 2 below, in order to exclude blowing up in finite time, and prove the decay estimate (1.105).

Let us comment more on the limitation to times $t = O(\varepsilon^{-4+0})$ which contrasts with the fact that, when the potential has no bound state, one may obtain dispersive estimates up to infinity. The new difficulty, when bound states are present, comes from the fact that in (1.105), $a(t)$ and $a'(t)$ have a decay in $\frac{\varepsilon}{(1+t\varepsilon^2)^{1/2}}$, which is larger than the rate in $\frac{\varepsilon}{\sqrt{t}}$ that holds for dispersive bounds in the absence of eigenvalues. This has consequences on the estimates satisfied by the dispersive part of the solution $P_{\mathrm{ac}}\varphi(t,\cdot)$. Actually, applying P_{ac} to equation (1.96), one will get an equation that, at first glance, might seem pretty similar to (1.98), since on the range of P_{ac}, $-\partial_x^2 + 2V$ will have no bound state. Though, a major difference appears on the right-hand side: if, for instance, one plugs in the quadratic term of (1.96) the decomposition $\varphi(t,\cdot) = P_{\mathrm{ac}}\varphi(t,\cdot) + a(t)Y$, one might get a source term

$$a(t)^2 P_{\mathrm{ac}}(\kappa(x)Y^2), \tag{1.106}$$

where $a(t)$ has only an $O(\frac{1}{\sqrt{t}})$ decay for $t \gg \varepsilon^{-2}$ (ant not a $\frac{\varepsilon}{\sqrt{t}}$ bound). This has dramatic consequences on the solution to the equation itself. Actually, the solution $P_{\mathrm{ac}}\varphi$ will have to encompass the solution of the linear equation

$$\big(D_t^2 - (D_x^2 + 1 + 2V(x))\big)w = a(t)^2 P_{\mathrm{ac}}(\kappa(x)Y^2)$$

with zero initial data. We shall solve this equation, but will be able to obtain for its solution only a bound in $t^{-\frac{1}{2}}(\varepsilon^2\sqrt{t})^{\theta'}$ for $t \le \varepsilon^{-4+0}$ and some $\theta' > 0$. When doing so, we are not able to obtain $O(t^{-\frac{1}{2}})$ bounds for w along two lines

$$\frac{x}{t} = \pm\sqrt{\frac{2}{3}}$$

when $t \gg \varepsilon^{-4}$. Actually, one might expect a logarithmic loss along these two lines, similar to the ones in the work of Lindblad, Lührmann and Soffer [60] and Lindblad, Lührmann, Schlag and Soffer [59].

Let us also stress on the fact that, besides (1.106), other new terms appear in comparison to the case of potentials without bound states. For instance, a contribution like $P_{\mathrm{ac}}(\kappa(x)(P_{\mathrm{ac}}\varphi)a(t)Y)$ needs also a specific treatment, as it is not amenable to standard normal forms treatment. We describe that in more detail in Section 2.7 of Chapter 2.

To conclude this introduction, let us point out the results of Kopylova and Komech in [54, 55] concerning asymptotic stability of a (moving) kink for a modified version of (1.89). In their model, the Hamiltonian of the equation is tuned in such a way that the projection of equation (1.96) on the absolutely continuous spectrum has coefficients in the nonlinearity that decay when x goes to infinity (instead of converging to some constant) This allows the authors to obtain a description of the dispersive behavior of the corresponding solution for *any* time.

Finally, let us refer to the recent paper of Chen, Liu and Lu [10] concerning asymptotic stability of kinks for sine-Gordon equations. Using the integrability of that equation, they may prove soliton resolution for generic data and show the full asymptotic stability of kinks under space decaying perturbations (see Corollary 1.5 of their paper). In particular, the difference between the solution and the moving kink is shown to decompose, when time goes to infinity, as the sum of an $O(t^{-\frac{1}{2}})$ contribution that involves a logarithmic phase correction and of a more decaying remainder.