Chapter 2

The kink problem

2.1 Statement of the main result

Consider $\phi : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ a global solution to the nonlinear wave equation

$$(\partial_t^2 - \partial_x^2)\phi = \phi - \phi^3. \tag{2.1}$$

The function

$$H(x) = \tanh\Big(\frac{x}{\sqrt{2}}\Big) \tag{2.2}$$

is a stationary solution of (2.1), and we are interested in describing the dispersive behaviour in large time of solutions to (2.1) corresponding to initial data that are small, smooth, odd and decaying perturbations of the state H. It is known that this state is orbitally stable in the energy space by Henry, Perez and Wreszinski [41], and for odd perturbations in that space, asymptotic stability with space exponential weight is proved by Kowalczyk, Martel and Muñoz [56]. This result describes the dispersive behaviour of the perturbation on compact space domains, but does not give insight into its behaviour in the whole space time. Our goal is to obtain information when (t, x) describes $I_\varepsilon \times \mathbb{R}$, where I_ε is a time interval of length $O(\varepsilon^{-4+0})$, ε being the size of the initial data in a convenient space of smooth decaying functions.

We shall look for solutions to (2.1) under the form

$$\phi(t, x) = H(x) + \varphi\big(t\sqrt{2}, x\sqrt{2}\big). \tag{2.3}$$

We get for φ the equation

$$\big(D_t^2 - (D_x^2 + 1 + 2V(x))\big)\varphi = \kappa(x)\varphi^2 + \frac{1}{2}\varphi^3, \tag{2.4}$$

where $D_t = \frac{1}{i}\frac{\partial}{\partial t}$, $D_x = \frac{1}{i}\frac{\partial}{\partial x}$ and

$$V(x) = -\frac{3}{4}\cosh^{-2}\Big(\frac{x}{2}\Big), \quad \kappa(x) = \frac{3}{2}\tanh\Big(\frac{x}{2}\Big). \tag{2.5}$$

The operator $-\partial_x^2 + 2V$ has $[0, +\infty[$ as its continuous spectrum and has two eigenvalues -1 and $-\frac{1}{4}$. The first one is associated to an even eigenfunction, and the second one to the odd normalized eigenfunction

$$Y(x) = \frac{\sqrt{3}}{2}\tanh\Big(\frac{x}{2}\Big)\cosh^{-1}\Big(\frac{x}{2}\Big) \tag{2.6}$$

(see Nikiforov and Uvarov [72] and Kowalczyk, Martel and Muñoz [56]).

We denote by P_{ac} the spectral projector on the continuous spectrum, restricted to odd functions. The spectral projector on the eigenspace associated to the eigenvalue $-\frac{1}{4}$ is $\varphi \mapsto \langle \varphi, Y \rangle Y$ so that

$$P_{\mathrm{ac}}\varphi = \varphi - \langle \varphi, Y \rangle Y, \tag{2.7}$$

where $\langle \cdot, \cdot \rangle$ denotes the L^2 scalar product. If φ solves (2.4), we set

$$a(t) = \langle \varphi, Y \rangle \tag{2.8}$$

so that (2.4) may be written

$$\begin{aligned} \Big(D_t^2 - \frac{3}{4}\Big) a(t) &= \Big\langle Y, \kappa(x)\big(a(t)Y + P_{\mathrm{ac}}\varphi\big)^2 + \frac{1}{2}\big(a(t)Y + P_{\mathrm{ac}}\varphi\big)^3 \Big\rangle, \\ \big(D_t^2 - (D_x^2 + 1 + 2V(x))\big) P_{\mathrm{ac}}\varphi & \\ &= P_{\mathrm{ac}}\Big(\kappa(x)\big(a(t)Y + P_{\mathrm{ac}}\varphi\big)^2 + \frac{1}{2}\big(a(t)Y + P_{\mathrm{ac}}\varphi\big)^3\Big). \end{aligned} \tag{2.9}$$

Our main result asserts that, up to a time of order ε^{-4}, the dispersive part $P_{\mathrm{ac}}\varphi$ of (2.9) has a time decay in uniform norm of magnitude $t^{-\frac{1}{2}}$, and that the function $a(t)$ in (2.8) has some oscillatory behavior, with decay in $t^{-\frac{1}{2}}$. More precisely, we have:

Theorem 2.1.1. *There is $\rho_0 \in \mathbb{N}$ and for any $\rho \geq \rho_0$, any $c > 0$, any $\theta' \in]0, \frac{1}{2}[$, any large enough N in $\mathbb{N}$, any large enough s in $\mathbb{N}$, there are $\varepsilon_0 \in]0, 1[$, $C > 0$ such that for any couple (φ_0, φ_1) of real-valued* odd *functions in $H^{s+1}(\mathbb{R}) \times H^s(\mathbb{R})$ satisfying*

$$\|\varphi_0\|_{H^{s+1}}^2 + \|\varphi_1\|_{H^s}^2 + \|x\varphi_0\|_{H^1}^2 + \|x\varphi_1\|_{L^2}^2 \leq 1, \tag{2.10}$$

the global solution φ of

$$\begin{aligned} \big(D_t^2 - (D_x^2 + 1 + 2V(x))\big)\varphi &= \kappa(x)\varphi^2 + \frac{1}{2}\varphi^3, \\ \varphi|_{t=1} &= \varepsilon\varphi_0, \\ \partial_t\varphi|_{t=1} &= \varepsilon\varphi_1 \end{aligned} \tag{2.11}$$

satisfies when $\varepsilon \in]0, \varepsilon_0[$ the following bounds for any $t \in [1, \varepsilon^{-4+c}]$: The oscillatory part a of φ given by (2.8) may be written

$$a(t) = e^{it\frac{\sqrt{3}}{2}} g_+(t) - e^{-it\frac{\sqrt{3}}{2}} g_-(t), \tag{2.12}$$

where

$$|g_\pm(t)| \leq C\varepsilon(1 + t\varepsilon^2)^{-\frac{1}{2}}, \; |\partial_t g_\pm(t)| \leq C\varepsilon t^{-\frac{1}{2}}(1 + t\varepsilon^2)^{-\frac{1}{2}}. \tag{2.13}$$

The dispersive part $P_{\mathrm{ac}}\varphi(t, \cdot)$ satisfies

$$\begin{aligned} \|P_{\mathrm{ac}}\varphi(t, \cdot)\|_{W^{\rho,\infty}} &\leq C t^{-\frac{1}{2}}(\varepsilon^2\sqrt{t})^{\theta'}, \\ \|\langle x \rangle^{-2N} P_{\mathrm{ac}}\varphi(t, \cdot)\|_{W^{\rho,\infty}} &\leq C t^{-\frac{3}{4}}(\varepsilon^2\sqrt{t})^{\theta'}, \\ \|\langle x \rangle^{-2N} P_{\mathrm{ac}} D_t\varphi(t, \cdot)\|_{W^{\rho-1,\infty}} &\leq C t^{-\frac{3}{4}}(\varepsilon^2\sqrt{t})^{\theta'}, \end{aligned} \tag{2.14}$$

where $\|\psi\|_{W^{\rho,\infty}} = \|\langle D_x \rangle^\rho \psi\|_{L^\infty}$.

Remarks. We make the following observations.

- The first estimate (2.14) shows that, up to time essentially equal to ε^{-4}, the dispersive part of the solution decays like $t^{-\frac{1}{2}}$, which is the behavior of small global solutions to nonlinear Klein–Gordon equations (see [18,19,64,82]). Nevertheless, in that case, the upper bound is in $O(\varepsilon t^{-\frac{1}{2}})$, while in (2.14), we have a degeneracy of the factor multiplying $t^{-\frac{1}{2}}$ when t goes to ε^{-4}.
- We construct in the proof some approximate solutions that are $o(t^{-\frac{1}{2}})$ for times $t \leq \varepsilon^{-4+c}$ and ε small. To go past that time seems to require extra arguments – like devising more accurate approximate solutions – in order to get a useful pointwise control of $P_{\mathrm{ac}}\varphi$ for $t > \varepsilon^{-4}$.
- Our estimates are consistent with the ones of Kowalczyk, Martel and Muñoz [56] in time $O(\varepsilon^{-4})$. Actually, it follows from (2.12), (2.13) that if $p > 2$,
$$\int_1^{\varepsilon^{-4+c}} |a(t)|^p \, dt \leq C\varepsilon^{p-2}$$
and
$$\int_1^{\varepsilon^{-4+c}} \Big(\|\langle x\rangle^{-2N-1} P_{\mathrm{ac}}\varphi(t,\cdot)\|_{H^1}^2 + \|\langle x\rangle^{-2N-1} D_t P_{\mathrm{ac}}\varphi(t,\cdot)\|_{L^2}^2 \Big) \, dt \leq C\varepsilon^{4\theta'}$$
for large enough N. These estimates are in accordance with those proved in [56] (when $p = 4$ for the first one) (see Theorem 1.2 in that reference).

2.2 Reduced system

We shall conjugate the second equation (2.9) by the wave operator W_+ associated to $-\frac{1}{2}\partial_x^2 + V(x)$. We discuss in Appendix A.1 below the properties of such an operator. According to Proposition A.1.1 of that Appendix, it may be written, when acting on odd functions, under the form
$$W_+ = b(x, D_x) \circ c(D_x), \tag{2.15}$$
where $b(x,\xi)$ is a symbol of order zero satisfying estimates (A.8) and
$$c(\xi) = e^{i\theta(\xi)} \mathbb{1}_{\xi>0} + e^{-i\theta(\xi)} \mathbb{1}_{\xi<0}$$
for some odd smooth real-valued function θ. Moreover, if we set $A = -\frac{1}{2}\partial_x^2 + V(x)$, $A_0 = -\frac{1}{2}\partial_x^2$, one has by (A.6) and (A.7), for any Borel function m on $\mathbb{R}$,
$$\begin{aligned} m(A)P_{\mathrm{ac}} &= W_+ m(A_0) W_+^*, \quad & m(A_0) &= W_+^* m(A) W_+ \\ W_+ W_+^* &= P_{\mathrm{ac}}, & W_+^* W_+ &= \mathrm{Id}_{L^2} \end{aligned} \tag{2.16}$$
so that applying W_+^* on the second equation (2.9), we get
$$\begin{aligned} \big(D_t^2 - (D_x^2 + 1)\big)(W_+^* P_{\mathrm{ac}}\varphi) &= W_+^*\big(\kappa(x)(a(t)Y + P_{\mathrm{ac}}\varphi)^2\big) \\ &\quad + W_+^*\Big(\frac{1}{2}(a(t)Y + P_{\mathrm{ac}}\varphi)^3\Big). \end{aligned} \tag{2.17}$$

Let us define

$$w = b(x, D_x)^* P_{\mathrm{ac}}\varphi. \tag{2.18}$$

Since $P_{\mathrm{ac}}\varphi$ is real valued, and since because of the symmetry properties (A.9) of $b(x,\xi)$, $b(x, D_x)$ and $b(x, D_x)^*$ preserve the space of real (resp. even, resp. odd) functions, w is still a real-valued odd function. As $c(D_x) \circ c(D_x)^* = \mathrm{Id}$,

$$\begin{aligned} P_{\mathrm{ac}}\varphi = W_+ W_+^* P_{\mathrm{ac}}\varphi = b(x, D_x)w \\ c(D_x)W_+^* P_{\mathrm{ac}}\varphi = w, \end{aligned} \tag{2.19}$$

so that making act $c(D_x)$ on (2.17) we see that w solves

$$\begin{aligned} \big(D_t^2 - (D_x^2 + 1)\big)w = b(x, D_x)^*\big(\kappa(x)\big(a(t)Y + b(x, D_x)w\big)^2\big) \\ + \frac{1}{2} b(x, D_x)^*\big(a(t)Y + b(x, D_x)w\big)^3. \end{aligned} \tag{2.20}$$

We shall study from now on the system given by the first equation (2.9) and (2.20). We define

$$\begin{aligned} w_0 = b(x, D_x)^* P_{\mathrm{ac}}\varphi_0, \\ w_1 = b(x, D_x)^* P_{\mathrm{ac}}\varphi_1. \end{aligned} \tag{2.21}$$

Since by (2.15) and (2.16), $P_{\mathrm{ac}} = b(x, D_x) \circ b(x, D_x)^*$, and since $b(x, D_x)$ and $[x, b(x, D_x)]$ are bounded on Sobolev spaces, we get from (2.10) that

$$\|w_0\|_{H^{s+1}}^2 + \|w_1\|_{H^s}^2 + \|xw_0\|_{H^1}^2 + \|xw_1\|_{L^2}^2 \le C_0 \tag{2.22}$$

for some constant C_0. Denote by $p(D_x)$ the operator

$$p(D_x) = \sqrt{1 + D_x^2} \tag{2.23}$$

and introduce complex-valued odd unknowns

$$\begin{aligned} u_+ = \big(D_t + p(D_x)\big)w, \\ u_- = \big(D_t - p(D_x)\big)w = -\bar{u}_+. \end{aligned} \tag{2.24}$$

If $I = (i_1, \dots, i_p)$ is an element of $\{-, +\}^p$, we shall set

$$u_I = (u_{i_1}, \dots, u_{i_p}) \tag{2.25}$$

and we denote also $u_{I,j} = u_{i_j}$, so that equivalently

$$u_I = (u_{I,1}, \dots, u_{I,p}). \tag{2.26}$$

Let us write (2.20) under the equivalent form

$$\big(D_t - p(D_x)\big)u_+ = \sum_{j=0}^{2} F_j^2[a; u_+, u_-] + \sum_{j=0}^{3} F_j^3[a; u_+, u_-], \tag{2.27}$$

where F_j^2 (resp. F_j^3) will be made of terms that are $O(t^{-1})$ (resp. $O(t^{-\frac{3}{2}})$) in L^∞ if the bounds (2.12)–(2.14) hold true, and are given by the following:

- Contribution depending only on a and not on $u_\pm$ are

$$F_0^2[a;u_+,u_-] = F_0^2[a] = a(t)^2 b(x,D_x)^*(\kappa(x)Y^2),$$
$$F_0^3[a;u_+,u_-] = F_0^3[a] = \frac{1}{2}a(t)^3 b(x,D_x)^*(Y^3). \tag{2.28}$$

- Contributions that are homogeneous of degree $j>0$ in (u_+,u_-) are given by the following quantities, where if $|I| = (i_1,\dots,i_p)$, we set $|I| = p$ and $\varepsilon_I = i_1\cdots i_p$:

$$F_j^2[a;u_+,u_-] = a(t)^{2-j}\sum_{|I|=j} F_{j,I}^2[u_I], \quad j=1,2,$$
$$F_j^3[a;u_+,u_-] = a(t)^{3-j}\sum_{|I|=j} F_{j,I}^3[u_I], \quad j=1,2,3, \tag{2.29}$$

with linear terms in (u_+,u_-)

$$F_{1,I}^2[u_I] = \varepsilon_I b(x,D_x)^*\big(Y(x)\kappa(x)b(x,D_x)p(D_x)^{-1}u_I\big),$$
$$F_{1,I}^3[u_I] = \frac{3}{4}\varepsilon_I b(x,D_x)^*\big(Y(x)^2 b(x,D_x)p(D_x)^{-1}u_I\big), \tag{2.30}$$

quadratic terms in (u_+,u_-)

$$F_{2,I}^2[u_I] = \frac{1}{4}\varepsilon_I b(x,D_x)^*\left(\kappa(x)\prod_{\ell=1}^{2} b(x,D_x)p(D_x)^{-1}u_{I,\ell}\right),$$
$$F_{2,I}^3[u_I] = \frac{3}{8}\varepsilon_I b(x,D_x)^*\left(Y(x)\prod_{\ell=1}^{2} b(x,D_x)p(D_x)^{-1}u_{I,\ell}\right), \tag{2.31}$$

and a cubic term in (u_+,u_-)

$$F_{3,I}^3[u_I] = \frac{1}{16}\varepsilon_I b(x,D_x)^*\left(\prod_{\ell=1}^{3} b(x,D_x)p(D_x)^{-1}u_{I,\ell}\right). \tag{2.32}$$

Notice that since κ and Y are odd, as well as $u_\pm$, and $b(x,D_x)$ preserves odd functions, F_j^2, F_j^3 are odd functions.

Let us write now the first equation in (2.9) in terms of a, u_+, u_-. We define

$$a_+(t) = \Big(D_t + \frac{\sqrt{3}}{2}\Big)a, \quad a_-(t) = \Big(D_t - \frac{\sqrt{3}}{2}\Big)a = -\bar{a}_+ \tag{2.33}$$

so that $a = \frac{\sqrt{3}}{3}(a_+ - a_-)$ and we rewrite the first equation (2.9) as

$$\Big(D_t - \frac{\sqrt{3}}{2}\Big)a_+ = \sum_{j=0}^{2}(a_+ - a_-)^{2-j}\Phi_j[u_+,u_-] + \sum_{j=0}^{3}(a_+ - a_-)^{3-j}\Gamma_j[u_+,u_-], \tag{2.34}$$

where the terms independent of $u_\pm$ are

$$\begin{aligned}\Phi_0 &= \frac{1}{3}\langle Y, \kappa Y^2\rangle,\\ \Gamma_0 &= \frac{\sqrt{3}}{18}\langle Y, Y^3\rangle\end{aligned} \tag{2.35}$$

and for $j \geq 1$,

$$\begin{aligned}\Phi_j[u_+, u_-] &= \sum_{|I|=j} \Phi_{j,I}[u_I],\\ \Gamma_j[u_+, u_-] &= \sum_{|I|=j} \Gamma_{j,I}[u_I]\end{aligned} \tag{2.36}$$

with linear expressions

$$\begin{aligned}\Phi_{1,I}[u_I] &= \frac{\sqrt{3}}{3}\varepsilon_I\big\langle Y, Y\kappa b(x, D_x)p(D_x)^{-1}u_I\big\rangle,\\ \Gamma_{1,I}[u_I] &= \frac{1}{4}\varepsilon_I\big\langle Y, Y^2 b(x, D_x)p(D_x)^{-1}u_I\big\rangle,\end{aligned} \tag{2.37}$$

quadratic expressions

$$\begin{aligned}\Phi_{2,I}[u_I] &= \frac{1}{4}\varepsilon_I\Big\langle Y, \kappa \prod_{\ell=1}^{2} b(x, D_x)p(D_x)^{-1}u_{I,\ell}\Big\rangle,\\ \Gamma_{2,I}[u_I] &= \frac{\sqrt{3}}{8}\varepsilon_I\Big\langle Y, Y \prod_{\ell=1}^{2} b(x, D_x)p(D_x)^{-1}u_{I,\ell}\Big\rangle,\end{aligned} \tag{2.38}$$

and cubic quantities

$$\Gamma_{3,I}[u_I] = \frac{1}{16}\varepsilon_I\Big\langle Y, \prod_{\ell=1}^{3} b(x, D_x)p(D_x)^{-1}u_{I,\ell}\Big\rangle. \tag{2.39}$$

We shall study from now on system (2.27), (2.34) with initial data at $t = 1$. According to (2.24), (2.21), (2.22), (2.33) and the fact that by (2.8), $a(1) = \langle \varepsilon\varphi_0, Y\rangle$ and $\partial_t a(1) = \langle \varepsilon\varphi_1, Y\rangle$, with φ_0, φ_1 satisfying (2.10), we may assume

$$u_+|_{t=1} = \varepsilon u_{+,0}, \quad a_+|_{t=1} = \varepsilon a_{+,0}, \tag{2.40}$$

where $u_{+,0}$ is a complex-valued odd function in $H^s(\mathbb{R}, \mathbb{C})$ satisfying

$$\begin{aligned}\|u_{+,0}\|_{H^s}^2 + \|xu_{+,0}\|_{L^2}^2 &\leq C_0^2,\\ |a_{+,0}| &\leq C_0^2\end{aligned} \tag{2.41}$$

for some fixed constant C_0.

In the following sections, we shall describe the main steps of the method of proof of our main result.

2.3 Step 1: Writing of the system from multilinear operators

In Section 2.2, we have reduced (2.9) to the system made of equations (2.27) and (2.34). One may rewrite (2.27) on a more synthetic way as

$$\begin{aligned}\big(D_t - p(D_x)\big)u_+ = F_0^2[a] + F_0^3[a] + \sum_{2\le|I|\le 3} \mathrm{Op}(m_{0,I})[u_I] \\ + a(t)\sum_{1\le|I|\le 2} \mathrm{Op}(m'_{1,I})[u_I] \\ + a(t)^2\sum_{|I|=1} \mathrm{Op}(m'_{2,I})[u_I]\end{aligned} \tag{2.42}$$

with the following notation: The term $F_0^2[a]$ (resp. $F_0^3[a]$) is the quadratic (resp. cubic) contribution in a obtained setting $w = 0$ on the right-hand side of (2.27). It has structure $a(t)^2 Z_2$ (resp. $a(t)^3 Z_3$) for some $\mathcal{S}(\mathbb{R})$-function Z_2 (resp. Z_3). The other terms on the right-hand side of (2.42) are expressed in terms of multilinear operators $\mathrm{Op}(m)(u_1,\dots,u_p)$, defined if $m(x,\xi_1,\dots,\xi_p)$ is a smooth function satisfying convenient estimates, as

$$\begin{aligned}\mathrm{Op}(m)(u_1,\dots,u_p) = \frac{1}{(2\pi)^p}\int e^{ix(\xi_1+\dots+\xi_p)} m(x,\xi_1,\dots,\xi_p) \\ \times \prod_{j=1}^{p} \hat{u}_j(\xi_j)\, d\xi_1\cdots d\xi_p.\end{aligned} \tag{2.43}$$

On the right-hand side of (2.42), we denote by I p-tuples $I = (i_1,\dots,i_p)$ where $i_\ell = \pm$ and set $|I| = p$. Then u_I stands for a p-tuple $u_I = (u_{i_1},\dots,u_{i_p})$ whose components are equal to u_+ or u_- defined in (2.24). The symbols $m_{0,I}$, $m'_{1,I}$, $m'_{2,I}$ are functions of $(x,\xi_1,\dots,\xi_p)$ with $p = |I|$. We do not write explicitly in this presentation of the proof the estimates that are assumed on these functions and their derivatives: we refer to Definition 3.1.1 below and to Appendix B for the precise description of the classes of symbols we consider. Let us just say that symbols $m_{0,I}$ are bounded in x, while their ∂_x-derivatives are rapidly decaying in x. This comes from the fact that the symbol $b(x,\xi)$ and the functions κ, Y in (2.20) satisfy such properties. On the other hand, symbols $m'_{1,I}, m'_{2,I}$ (and more generally any symbol that we shall denote as m' in what follows) decay rapidly in x even without taking derivatives. It turns out that operators with decaying symbol in x acting on functions we shall introduce below will give quantities with a better time decay than operators associated to non-decaying symbols.

2.4 Step 2: First quadratic normal form

The goal of the whole paper is to obtain energy estimates for the solution u_+ to (2.27) and a_+ to (2.34).

As we have seen in Section 1.6 of the Introduction, the first thing to do in order to get Sobolev estimates for an equation like (2.27) is to eliminate the quadratic contributions $\sum_{|I|=2} \mathrm{Op}(m_{0,I})[u_I]$. We do that through a "time normal form" à la Shatah [76] and Simon and Taflin [77] (see also for one-dimensional Klein–Gordon equations Moriyama, Tonegawa and Tsutsumi [71], Moriyama [70], Hayashi and Naumkin [39] and the very recent works of Germain and Pusateri [33], of Lindblad, Lührmann and Soffer [60] and of Lindblad, Lührmann, Schlag and Soffer [59]). Actually, we construct new symbols $(\tilde{m}_{0,I})_{|I|=2}$ such that

$$\begin{aligned}
&\big(D_t - p(D_x)\big)\Big(u_+ - \sum_{|I|=2} \mathrm{Op}(\tilde{m}_{0,I})[u_I]\Big) \\
&\quad = F_0^2[a] + F_0^3[a] + \sum_{3\le|I|\le4} \mathrm{Op}(m_{0,I})[u_I] + \sum_{|I|=2} \mathrm{Op}(m'_{0,I})[u_I] \\
&\qquad + \sum_{j=1}^{3} a(t)^j \sum_{1\le|I|\le4-j} \mathrm{Op}(m'_{j,I})[u_I],
\end{aligned} \tag{2.44}$$

where on the right-hand side, we eliminated the quadratic contributions $\mathrm{Op}(m_{0,I})[u_I]$, but made appear new quadratic terms $\mathrm{Op}(m'_{0,I})[u_I]$ given in terms of new symbols $m'_{0,I}$ that *decay rapidly* when x goes to infinity. These corrections come from the fact that, at the difference with a usual normal form method where one eliminates quadratic expressions like (2.43) with $p = 2$ and a symbol $m(\xi_1, \xi_2)$ independent of x, we have here to cope with symbols $m(x, \xi_1, \xi_2)$. This x dependence makes appear some commutator, given essentially in terms of $\mathrm{Op}(\frac{\partial m}{\partial x}(x, \xi_1, \xi_2))$, with a symbol *rapidly decaying* in x. These commutators are the new quadratic terms $\mathrm{Op}(m'_{0,I})[u_I]$ on the right-hand side of (2.44). As already mentioned, such expressions will have better time decay estimates than the quadratic expressions given by non-space decaying symbols that we have eliminated, and are actually better than most remaining terms on the right-hand side of (2.44). They are not completely negligible, but will be treated only at the end of the reasoning.

2.5 Step 3: Approximate solution

Our general strategy is to define from the solution u_+ of (2.44) a new unknown $\tilde{u}_+$ that would satisfy similar estimates as those of the bootstrap (1.39) of the introduction. More precisely, we aim at constructing a new unknown $\tilde{u}_+$ for which we could get, for $t \in [1, \varepsilon^{-4+c}]$ with $c > 0$ given, bounds of the following form:

$$\|\tilde{u}_+(t,\cdot)\|_{H^s} = O(\varepsilon t^{\delta}), \tag{2.45}$$

$$\|L_+\tilde{u}_+(t,\cdot)\|_{L^2} = O\big((\varepsilon^2\sqrt{t})^{\theta} t^{\frac{1}{4}}\big), \tag{2.46}$$

$$\|\tilde{u}_+(t,\cdot)\|_{W^{\rho,\infty}} = O\Big(\frac{(\varepsilon^2\sqrt{t})^{\theta'}}{\sqrt{t}}\Big), \tag{2.47}$$

where $\delta > 0$ is small, $\theta' < \theta < \frac{1}{2}$ with θ' close to $\frac{1}{2}$, $s \gg \rho \gg 1$, and where we denoted $\|w\|_{W^{\rho,\infty}} = \|\langle D_x\rangle^\rho w\|_{L^\infty}$. The first estimate (2.45) is the one that would follow by energy inequality for the solution of (1.32), assuming that (2.47) holds (since, for $t \le \varepsilon^{-4+c}$, (2.47) implies a bound in $c(\varepsilon)t^{-\frac{1}{2}}$, with $c(\varepsilon)$ going to zero when ε goes to zero). In the same way, assuming (2.47) and assuming that $\tilde{u}_+$ solves an equation of the form (1.26) with $p = 1$, one could bootstrap a bound of the form (2.46). Finally, an estimate of the form (2.47) will have to be deduced from (2.46) constructing from the PDE solved by $\tilde{u}_+$ an ODE with remainder term controlled from (2.46).

Of course, the right-hand side of (2.44) is far from having the nice structure of the one of (1.32), and this is why we shall have to modify the unknown u_+ in order to eliminate all bad terms on the right-hand side of (2.44). In Chapter 4 of the paper we shall get rid of the contributions $F_0^2[a]$, $F_0^3[a]$. These functions are bounded as well as their space derivatives by $t^{-1}\langle x\rangle^{-N}$ for any N. Clearly, if we make act L_+ on them and compute the L^2 norm, we shall get an $O(1)$ quantity. If we were integrating such a bound, we would deduce that $\|L_+u_+(t,\cdot)\|_{L^2} = O(t)$, a much worse estimate than the one (2.46) we want. We shall thus remove from u_+ the solution of the linear equation with force terms $F_0^2[a] + F_0^3[a]$, i.e. we shall solve

$$\begin{aligned} \big(D_t - p(D_x)\big)U &= F_0^2[a] + F_0^3[a], \\ U|_{t=1} &= 0 \end{aligned} \tag{2.48}$$

and then make the difference between (2.44) and (2.48) in order to eliminate $F_0^2[a]$ and $F_0^3[a]$ from the right-hand side of the new equation obtained in that way. Actually, one needs to take also into account at this stage bilinear terms in (a,u) in (2.44). We thus construct in Proposition 4.1.2 an approximate solution u_+^{app} of

$$\begin{aligned} \big(D_t - p(D_x)\big)u_+^{\mathrm{app}} &= F_0^2(a^{\mathrm{app}}) + F_0^3(a^{\mathrm{app}}) \\ &\quad + a^{\mathrm{app}} \sum_{|I|=1} \mathrm{Op}(m'_{1,I})(u_I^{\mathrm{app}}) + \text{remainder}, \\ u_+^{\mathrm{app}}|_{t=1} &= 0, \end{aligned} \tag{2.49}$$

where a^{app} is some approximation of the function $a(t)$ solving the first equation (2.9).

Let us explain what are the bounds satisfied by the approximate solution u_+^{app} of equation (2.49) that we obtain in Proposition 4.1.2 using the results of Appendix C. We decompose $u_+^{\mathrm{app}} = u'^{\mathrm{app}}_+ + u''^{\mathrm{app}}_+$. The term u'^{app}_+ satisfies the kind of estimates we aim at proving, namely (2.45)–(2.47) (and actually slightly better ones) for times $t = O(\varepsilon^{-4+c})$. On the other hand, inequalities (2.45) and (2.47) hold for u''^{app}_+ (and even actually slightly better ones), but $L_+u''^{\mathrm{app}}_+$ does not verify (2.46). On the other hand, $L_+u''^{\mathrm{app}}_+$ obeys good estimates in L^∞ norms, of the form

$$\|L_+u''^{\mathrm{app}}_+\|_{W^{r,\infty}} = O\big(\log(1+t)\log(1+\varepsilon^2 t)\big) \tag{2.50}$$

that will allow us to estimate conveniently nonlinear terms containing u''^{app}_+. Let us stress that the limitation of our main result to times $O(\varepsilon^{-4})$ comes from the degeneracy of bound (2.46) for $L_+ u'^{\text{app}}_+$ when t becomes larger than ε^{-4}. We do not claim that, in such a regime, an estimate of the form (2.46) would be optimal. But we remark that in the construction of u'^{app}_+ made from the results of Appendix C, the main contribution comes from quantities that have pretty explicit bounds: see Proposition C.1.4 and in particular bound (C.40) with $\omega = 1$ (that gives the main contribution to u'^{app}_+) and (C.42) with $\omega = 1$ (that gives the main contribution to $L_+ u'^{\text{app}}_+$). If we extrapolate estimate (C.40) for $t \gg \varepsilon^{-4}$ (which is of course not legitimate, as we prove it only for times $O(\varepsilon^{-4})$), we see that outside a conical neighborhood of the two lines $x = \pm t\sqrt{2/3}$, an estimate of $|u'^{\text{app}}_+(t,x)|$ in $O(\varepsilon^2 t^{-\frac{1}{2}})$ would hold. On the other hand, along these two lines, a degeneracy happens, and we do not expect to be able to prove that, for $t \gg \varepsilon^{-4}$, $|u'^{\text{app}}_+(t, \pm t\sqrt{2/3})|\sqrt{t}$ remains small (or even bounded). Because of that, we do not hope to push estimates of the form (2.45)–(2.47) for such times, without taking into account first some extra corrections. In particular, going back to (1.105), we do not expect an $O(t^{-\frac{1}{2}})$ bound for $|P_{\text{ac}}\varphi(t,x)|$ along these lines.

Notice that such a phenomenon cannot be detected using weighted space estimates an in [56]: actually, along the lines $x = \pm t\sqrt{2/3}$, a space decaying weight is also time decaying and kills bad bounds of u'^{app}_+ along these lines. We shall comment more extensively on that issue in Section 2.10 below.

In addition to the proof of estimates of the form (2.45)–(2.47), we need, in order to obtain (1.105), to study the solution of the first equation (2.9). We do that in Section 4.2 of Chapter 4. Setting

$$a_+(t) = \Big(D_t + \frac{\sqrt{3}}{2}\Big)a, \quad a_-(t) = \Big(D_t - \frac{\sqrt{3}}{2}\Big)a = -\bar{a}_+,$$

the first equation (2.9) may be rewritten as

$$\begin{aligned}\Big(D_t - \frac{\sqrt{3}}{2}\Big)a_+ = \sum_{j=0}^{2}(a_+ - a_-)^{2-j}\,\Phi_j[u_+, u_-]& \\ + \sum_{j=0}^{3}(a_+ - a_-)^{3-j}\,\Gamma_j[u_+, u_-],&\end{aligned} \tag{2.51}$$

where Φ_j, Γ_j are expressions in the solution u_+ to (2.42) or (2.44). The goal of Section 4.2 is to uncover the structure of a_+. We write

$$a_+(t) = a^{\text{app}}_+(t) + O\big(\varepsilon^3(1 + t\varepsilon^2)^{-\frac{3}{2}}\big),$$

where $a^{\text{app}}_+(t)$ has structure (4.97), that implies in particular

$$a^{\text{app}}_+(t) = e^{it\frac{\sqrt{3}}{2}}g(t) + \text{more decaying terms.} \tag{2.52}$$

The main goal of Section 4.2 is to prove by bootstrap that $g(t)$ satisfies bounds

$$|g(t)| = O(\varepsilon(1 + t\varepsilon^2)^{-\frac{1}{2}}), \quad |\partial_t g(t)| = O(t^{-\frac{3}{2}}). \tag{2.53}$$

(Actually, we get more precise bounds for $\partial_t g$: see (4.99)). These bounds are obtained showing that (2.51) implies that g satisfies an ODE

$$D_t g(t) = \left(\alpha - i\frac{\sqrt{6}}{18}\hat{Y}_2(\sqrt{2})^2\right)|g(t)|^2 g(t) + \text{remainder}, \tag{2.54}$$

where Y_2 is some explicit function in $\mathcal{S}(\mathbb{R})$ and α is real. The coefficient of the cubic term on the right-hand side comes from some of the terms on the right-hand side of (2.51) where we replace $u_\pm$ by the approximate solution u_+^{app} determined in Section 4.1. The main contribution to u_+^{app}, integrated against an $\mathcal{S}(\mathbb{R})$ function, may be computed explicitly in terms of g (see Proposition 4.1.3), and brings the right-hand side of (2.54). The key point in that equation is that $\hat{Y}_2(\sqrt{2})^2 < 0$. This implies that g satisfies bounds (2.53) for $t \geq 1$ if $g(1) = O(\varepsilon)$. The inequality $\hat{Y}_2(\sqrt{2})^2 < 0$ is nothing but Fermi's golden rule. Actually, $\hat{Y}_2(\sqrt{2})^2 \leq 0$ holds trivially and the key point is to check that $\hat{Y}_2(\sqrt{2}) \neq 0$. This reduces to showing that some explicit integral is non-zero. Kowalczyk, Martel and Muñoz checked that numerically in [56]. In Appendix G, we compute explicitly this integral by residues.

2.6 Step 4: Reduced form of dispersive equation

The goal of this step is to rewrite equation (2.44) in terms of a new unknown $\tilde{u}_+$ that will satisfy estimates (2.45)–(2.47). We define

$$\tilde{u}_+ = u_+ - \sum_{|I|=2} \text{Op}(\tilde{m}_{0,I})(u_I) - u'^{\text{app}}_+ - u''^{\text{app}}_+, \tag{2.55}$$

and set $\tilde{u}_- = -\overline{\tilde{u}_+}$. Making the difference between (2.44) and (2.49), we show in Section 5.2 (see Proposition 5.2.1) that $\tilde{u}_+$ satisfies

$$\begin{aligned}\big(D_t - p(D_x)\big)\tilde{u}_+ = &\sum_{3\leq|I|\leq 4,\, I=(I',I'')} \text{Op}(\tilde{m}_I)(\tilde{u}_{I'}, u^{\text{app}}_{I''})\\ &+ \sum_{|I|=2,\, I=(I',I'')} \text{Op}(m'_{0,I})(\tilde{u}_{I'}, u^{\text{app}}_{I''})\\ &+ \underline{a}^{\text{app}}(t)\sum_{|I|=1}\text{Op}(m'_{1,I})(\tilde{u}_I)\\ &+ \frac{1}{3}\Big(e^{it\frac{\sqrt{3}}{2}}g(t) + e^{-it\frac{\sqrt{3}}{2}}\overline{g(t)}\Big)^2\sum_{|I|=1}\text{Op}(m'_{0,I})(\tilde{u}_I)\\ &+ \text{remainder},\end{aligned} \tag{2.56}$$

where:

- For $3 \leq |I| \leq 4$, $\tilde{m}_I$ are symbols $\tilde{m}_I(x, \xi_1, \dots, \xi_p)$, $p = |I| = |I'| + |I''|$ which are $O(1)$ as functions of x, but $O(\langle x\rangle^{-\infty})$ if one takes at least one ∂_x-derivative.
- For $1 \leq |I| \leq 2$, $m'_{0,I}, m'_{1,I}$ are symbols that are $O(\langle x\rangle^{-\infty})$, even without taking any derivative.

- Function of time g has been introduced in (2.52) and gives the principal term in the expansion of $a_+^{\mathrm{app}}(t)$ or $a_+(t)$.
- Function $\underline{a}^{\mathrm{app}}(t) = \frac{\sqrt{3}}{3}(\underline{a}_+^{\mathrm{app}}(t) - \underline{a}_-^{\mathrm{app}}(t))$, where

$$\underline{a}_+^{\mathrm{app}}(t) = e^{it\frac{\sqrt{3}}{2}} g(t) + \omega_2 e^{it\sqrt{3}} g(t)^2 + \omega_0 |g(t)|^2 + \omega_{-2} e^{-it\sqrt{3}} \overline{g(t)}^2 \quad (2.57)$$

with convenient constants $\omega_2, \omega_0, \omega_{-2}$ and $\underline{a}_-^{\mathrm{app}}(t) = -\overline{\underline{a}_+^{\mathrm{app}}(t)}$.

We cannot derive directly from equation (2.56) estimate (2.46) for $\tilde{u}_+$, as the right-hand side of (2.56) has not the nice structure (1.32). Before applying an energy method, we shall have to use several normal forms in order to reduce ourselves to such a nice nonlinearity. As a preparation to that step, we show in Corollary 5.2.3 that (2.56) may be rewritten under the following equivalent form:

$$\begin{aligned}
\big(D_t - p(D_x)\big)\tilde{u}_+ &- \sum_{j=-2}^{2} e^{itj\frac{\sqrt{3}}{2}} \mathrm{Op}(b'_{j,+})\tilde{u}_+ - \sum_{j=-2}^{2} e^{itj\frac{\sqrt{3}}{2}} \mathrm{Op}(b'_{j,-})\tilde{u}_- \\
&= \sum_{3\le|I|\le 4,\, I=(I',I'')} \mathrm{Op}(\tilde{m}_I)(\tilde{u}_{I'}, u^{\mathrm{app}}_{I''}) + \sum_{|I|=2} \mathrm{Op}(m'_{0,I})(\tilde{u}_I) \\
&\quad + \sum_{I=(I',I''),\, |I'|=|I''|=1} \mathrm{Op}(m'_{0,I})(\tilde{u}_{I'}, u'^{\mathrm{app},1}_{I''}) \\
&\quad + \sum_{|I|=2} \mathrm{Op}(m'_{0,I})(u'^{\mathrm{app},1}_{I}) + \text{remainder},
\end{aligned} \quad (2.58)$$

where, in comparison with (2.56), all linear terms in $\tilde{u}_+, \tilde{u}_-$ have been sent to the left-hand side, and are expressed from symbols $b'_{j,\pm}(t,x,\xi)$ that are rapidly decaying in x at infinity. Moreover, on the right-hand side, we still use the convention of denoting by $m'_{0,I}$ symbols rapidly decaying in x, while $\tilde{m}_I$ are $O(1)$ in x, with ∂_x-derivatives rapidly decaying in x. Furthermore, in the last two sums in (2.58), we replaced u'^{app} by $u'^{\mathrm{app},1}$, which is actually the main contribution (in terms of time decay) to u'^{app}. If we set $\tilde{u} = [\begin{smallmatrix}\tilde{u}_+ \\ \tilde{u}_-\end{smallmatrix}]$, we may rewrite (2.58) as a system of the form

$$\begin{aligned}
\big(D_t - P_0 - \mathcal{V}\big)\tilde{u} &= \mathcal{M}_3(\tilde{u}, u^{\mathrm{app}}) + \mathcal{M}_4(\tilde{u}, u^{\mathrm{app}}) \\
&\quad + \mathcal{M}'_2(\tilde{u}, u'^{\mathrm{app},1}) + \text{remainder},
\end{aligned} \quad (2.59)$$

where

$$P_0 = \begin{bmatrix} p(D_x) & 0 \\ 0 & -p(D_x) \end{bmatrix},$$

$\mathcal{V}$ is a 2×2 matrix of operators of the form

$$\mathcal{V} = \sum_{j=-2}^{2} e^{ijt\frac{\sqrt{3}}{2}} \mathrm{Op}(M'_j(t,x,\xi)) \quad (2.60)$$

with M'_j 2×2 matrix of symbols whose entries are given in terms of the $b'_{j,\pm}$ in (2.58), and where the 2-vectors $\mathcal{M}_3$ (resp. $\mathcal{M}_4$, resp. $\mathcal{M}'_2$) come from the cubic (resp. quartic, resp. quadratic) terms on the right-hand side of (2.58).

To obtain the wanted estimates (2.45) and (2.46) for $\tilde{u}_+$, we have next to reduce (2.59) to an equation essentially of the form (1.32). This is the object of Step 5 of the proof.

2.7 Step 5: Normal forms

Equation (2.59) has not structure of the form (1.32), in that sense that if we make act

$$L = \begin{bmatrix} L_+ & 0 \\ 0 & L_- \end{bmatrix},$$

with $L_- = x - tp'(D_x)$, first L does not commute to the potential term $\mathcal{V}$, and second the action of L on the nonlinearities on the right-hand side does not give quantities whose L^2 norm is $O(\|\tilde{u}\|_{L^\infty}^2 \|L\tilde{u}\|_{L^2})$ (which is essentially necessary if we want to get (2.46) by energy estimates). To cope with the lack of commutation of L with $\mathcal{V}$, we shall construct a wave operator and use it to eliminate $\mathcal{V}$ by conjugation of the equation. This is similar to what has been done to pass from the second equation (2.9), that was involving the potential $2V(x)$ to equation (2.17), where there was no longer any potential. The difference here is that $\mathcal{V}$ given by (2.60) is time dependent (with $O(t^{-\frac{1}{2}})$ decay). We thus cannot rely on existing references, and have to construct by hand operators $B(t), C(t)$ (depending on time) such that

$$C(t)\big(D_t - P_0 - \mathcal{V}\big) = (D_t - P_0)C(t). \tag{2.61}$$

In that way, if $\tilde{u}$ solves (2.59), then $C(t)\tilde{u}$ solves the new equation without potential

$$\begin{aligned}(D_t - P_0)C(t)\tilde{u} &= C(t)\mathcal{M}_3(\tilde{u}, u^{\mathrm{app}}) + C(t)\mathcal{M}_4(\tilde{u}, u^{\mathrm{app}}) \\ &\quad + C(t)\mathcal{M}_2'(\tilde{u}, u'^{\mathrm{app},1}) + \text{remainder}\end{aligned} \tag{2.62}$$

(see Proposition 6.1.2). Moreover, since we want to pass from an L^2 bound on $L\tilde{u}$ to an L^2 bound on $LC(t)\tilde{u}$ and conversely, we need to relate $L \circ C(t)$ and L, proving that

$$L \circ C(t) = \tilde{C}(t) \circ L + \tilde{C}_1(t), \tag{2.63}$$

where $\tilde{C}(t)$ is bounded on L^2 uniformly in t and $\tilde{C}_1(t)$ is bounded with a small time growth when t goes to infinity. The construction of operator $C(t)$ is made in Appendix E by a pretty standard series expansion. We notice however that we need to use in that construction the fact that we are dealing with *odd* functions $\tilde{u}$.

Once reduced to (2.62), we still have to handle those nonlinear terms on the right-hand side that do not have a structure of the form (1.32), i.e. we have to cope with nonlinearities that have the same structure as in the model (1.68) of Section 1.6 of the introduction. We have seen there that this problem may be solved using "space-time normal forms". We shall follow essentially the approach of [20], already described in Section 1.6 of the introduction, that we have to adapt to the more general operators $\mathcal{M}_3, \mathcal{M}_4$ on the right-hand side of (2.62). Remark that the components of the vectors

$\mathcal{M}_3, \mathcal{M}_4$ are, according to (2.58), given by expressions $\mathrm{Op}(\tilde{m})(\tilde{u}_\pm, \dots, u_\pm^{\mathrm{app}})$, where $\tilde{m}(x, \xi_1, \dots, \xi_p)$ is a symbol that is $O(1)$ when $|x|$ goes to infinity, but $O(\langle x\rangle^{-\infty})$ if one takes at least one ∂_x-derivative. We have to distinguish between to type of terms, the characteristic and the non-characteristic ones. The former correspond to the case when, among the p arguments of $\mathrm{Op}(\tilde{m})(\tilde{u}_\pm, \dots, u_\pm^{\mathrm{app}})$, $\frac{p+1}{2}$ are equal to $\tilde{u}_+$ or u_+^{app} and $\frac{p-1}{2}$ are equal to $\tilde{u}_-$ or u_-^{app}.

In the case of simple monomial nonlinearities, example of characteristic terms are given by the right-hand side $|u_+|^2 u_+$ of (1.32), which, when making act L_+ on it, may be estimated in L^2 by $\|u_+(t,\cdot)\|_{L^\infty}^2 \|L_+ u_+(t,\cdot)\|_{L^2}$. If $\tilde{m}$ were independent of x, the same would hold for the action of the operator L_+ on any characteristic term like $\mathrm{Op}(\tilde{m})(\tilde{u}_\pm, \dots, \tilde{u}_\pm)$, as $L_+\mathrm{Op}(\tilde{m})(\tilde{u}_\pm, \dots, \tilde{u}_\pm)$ could be expressed from $\mathrm{Op}(\tilde{m})(L_\pm \tilde{u}_\pm, \dots, \tilde{u}_\pm), \dots, \mathrm{Op}(\tilde{m})(\tilde{u}_\pm, \dots, L_\pm \tilde{u}_\pm)$. Using the boundedness properties of $\mathrm{Op}(\tilde{m})$, one would then estimate the L^2 norm of these quantities by $\|\tilde{u}\|_{L^\infty}^{p-1} \|L\tilde{u}\|_{L^2}$. As $p \geq 3$, one could then obtain estimate (2.46) by energy inequality, as in (1.26). Since here $\tilde{m}$ does depend on x, there is no exact commutation relation in the characteristic case between $\mathrm{Op}(\tilde{m})$ and L_+, as some commutators of the form $t\mathrm{Op}(\partial_x \tilde{m})$ have to be taken into account. It turns out that, because $\partial_x \tilde{m}$ is rapidly decaying in x, and because $\tilde{u}_\pm$ is odd, $\|t\mathrm{Op}(\tilde{m})(\tilde{u}_\pm, \dots, \tilde{u}_\pm)\|_{L^2}$ may be also estimate by the right-hand side of (1.26). Actually, the kind of expressions one has to cope with is morally of the form

$$tZ(x)\big(\langle D_x\rangle^{-1}\tilde{u}_\pm\big)^3, \tag{2.64}$$

where Z is in $\mathcal{S}(\mathbb{R})$ (This reflects the fact that $\partial_x \tilde{m}$ is rapidly decaying in x). Since $\tilde{u}_+$ is odd, we may write using the definition of $L_+ = x + t\frac{D_x}{\langle D_x\rangle}$

$$\begin{aligned} \langle D_x\rangle^{-1}\tilde{u}_+ &= ix\int_{-1}^{1}\Big(\frac{D_x}{\langle D_x\rangle}\tilde{u}_+\Big)(\mu x)\,d\mu \\ &= i\frac{x}{t}\int_{-1}^{1}\big((L_+\tilde{u}_+)(\mu x) - \mu x\tilde{u}_+(\mu x)\big)\,d\mu. \end{aligned} \tag{2.65}$$

The rapid decay of $Z(x)$ allows one to absorb the powers of x on the right-hand side of (2.65), and to estimate the L^2 norm of (2.64) by

$$C\big(\|L_+\tilde{u}_+\|_{L^2} + \|\tilde{u}_+\|_{L^2}\big)\|\tilde{u}_+\|_{L^\infty}^2,$$

i.e. by the right-hand side of (1.26) with $p = 1$. Similar arguments apply when the factors $\tilde{u}_\pm$ are replaced by $u_\pm^{\mathrm{app}}$.

The above reasoning disposes of the characteristic components in $\mathcal{M}_j(\tilde{u}, u^{\mathrm{app}})$ in (2.62). The non-characteristic ones are for instance of the form $\mathrm{Op}(\tilde{m})(\tilde{u}_+, \dots, \tilde{u}_+)$ and we no longer have an approximate commutation property of L_+ with such operators. These terms have thus to be eliminated by a space-time normal form. We construct in Proposition 6.2.1, using the results of Appendix F, operators $\hat{\mathcal{M}}_j$, $j = 3, 4$, such that

$$(D_t - P_0)\hat{\mathcal{M}}_j(\tilde{u}, u^{\mathrm{app}}) = \mathcal{M}_j(\tilde{u}, u^{\mathrm{app}})_{\mathrm{nch}} + \text{remainder}, \tag{2.66}$$

where $\mathcal{M}_j(\tilde{u}, u^{\text{app}})_{\text{nch}}$ denotes the non-characteristic contributions to $\mathcal{M}_j(\tilde{u}, u^{\text{app}})$ on the right-hand side of (2.62). Actually, $\mathcal{M}_4(\tilde{u}, u^{\text{app}})_{\text{nch}} = \mathcal{M}_4(\tilde{u}, u^{\text{app}})$ as only $\mathcal{M}_3$ contains characteristic components. In that way, we deduce from (2.62) that

$$\begin{aligned}(D_t - P_0)\big(C(t)\big(\tilde{u} - \hat{\mathcal{M}}_3(\tilde{u}, u^{\text{app}}) - \hat{\mathcal{M}}_4(\tilde{u}, u^{\text{app}})\big)\big)\\ = C(t)\mathcal{M}_2'(\tilde{u}, u'^{\text{app},1}) + \mathcal{R},\end{aligned} \tag{2.67}$$

where the remainder $\mathcal{R}$ satisfies bounds of the form

$$\|L_+\mathcal{R}\|_{L^2} = O\big(\|\tilde{u}_+\|_{L^\infty}^2 \|L_+\tilde{u}_+\|_{L^2}\big)$$

as on the right-hand side of (1.26) with $p = 1$. Notice that to deduce (2.67) from (2.66), we have to compare $(D_t - P_0)C(t)\hat{\mathcal{M}}_j$ and $C(t)(D_t - P_0)\hat{\mathcal{M}}_j$ which by (2.61) makes appear a term $C(t)\mathcal{V}\hat{\mathcal{M}}_j$, but the time and space decay of operator $\mathcal{V}$ allows one to show that such errors form part of the remainder $\mathcal{R}$ in (2.67).

One has still on the right-hand side of (2.67) term $C(t)\mathcal{M}_2'(\tilde{u}, u'^{\text{app},1})$. Again $\mathcal{M}_2'$ may be expressed in terms of quantities $\text{Op}(m')(\tilde{u}_\pm, \tilde{u}_\pm)$ (and similar ones with $\tilde{u}_\pm$ replaced by $u'^{\text{app},1}_\pm$), so that one may gain some time decay using expressions of the form (2.65), but as this term is just quadratic, this gain is not sufficient to include $C(t)\mathcal{M}_2'$ into $\mathcal{R}$ in (2.67). As $C(t) - \text{Id}$ has some time decay, one may prove though that $(C(t) - \text{Id})\mathcal{M}_2'$ is a remainder, but the expression $\mathcal{M}_2'(\tilde{u}, u'^{\text{app},1})$ still needs to be eliminated from the right-hand side of (2.67). We do that in Proposition 6.2.4 of Chapter 6, using results of Appendix F. Actually, a quantity like $\text{Op}(m')(\tilde{u}_\pm, \tilde{u}_\pm)$ may be expressed, using the x-rapid decay of m' and the oddness of $\tilde{u}_\pm$, as a sum of expressions of the form

$$t^{-2}K\big(L_\pm^{\ell_1}\tilde{u}_\pm, L_\pm^{\ell_2}\tilde{u}_\pm\big), \quad 0 \le \ell_1, \ell_2 \le 1, \tag{2.68}$$

where K is an operator of form

$$\widehat{K(f_1, f_2)}(\xi_0) = \int k(\xi_0, \xi_1, \xi_2)\hat{f}_1(\xi_1)\hat{f}(\xi_2)\, d\xi_1\, d\xi_2, \tag{2.69}$$

where the kernel k has rapid decay in $\langle \xi_0 - \xi_1 - \xi_2\rangle$. An operator of form (2.68) slightly misses bounds in $O(t^{-1}\|L_+\tilde{u}_+\|_{L^2})$ when we make act on it $L_\pm$ and take the L^2 norm. But it does satisfy such estimates if we cut-off k in (2.69) on a domain $|\pm\langle\xi_0\rangle \pm \langle\xi_1\rangle \pm \langle\xi_2\rangle| \le ct^{-\frac{1}{2}}$. Consequently, one may assume that in (2.69), k is supported for $|\pm\langle\xi_0\rangle \pm \langle\xi_1\rangle \pm \langle\xi_2\rangle| \ge ct^{-\frac{1}{2}}$. This extra cut-off allows to construct by normal forms a quadratic term $\hat{\mathcal{M}}_2'(\tilde{u}, u'^{\text{app},1})$ such that

$$(D_t - P_0)\hat{\mathcal{M}}_2'(\tilde{u}, u'^{\text{app},1}) = \mathcal{M}_2'(\tilde{u}, u'^{\text{app},1}) + \text{remainder}.$$

Subtracting this equation from (2.67), one gets finally

$$(D_t - P_0)\mathring{u} = \hat{\mathcal{R}} \tag{2.70}$$

where

$$\mathring{u} = C(t)\Big(\tilde{u} - \sum_{j=3}^{4} \hat{\mathcal{M}}_j(\tilde{u}, u^{\mathrm{app}})\Big) - \hat{\mathcal{M}}'_2(\tilde{u}, u'^{\mathrm{app},1}). \tag{2.71}$$

and where $\hat{\mathcal{R}}$ will satisfy among other things essentially

$$\|L\hat{\mathcal{R}}(t,\cdot)\|_{L^2} = O(t^{-1}\|L_+\tilde{u}_+\|_{L^2}). \tag{2.72}$$

2.8 Step 6: Bootstrap of L^2 estimates

As seen above, the conclusion of the main theorem follows from the bootstrap of estimates (2.45)–(2.47). In Chapter 7, we perform the bootstrap of (2.45) and (2.46), assuming that (2.45)–(2.47) hold on some interval $[1, T]$ with $T \leq \varepsilon^{-4+c}$ and showing that (2.45)–(2.46) then actually hold with the implicit constant on the right-hand side divided by 2 for instance. As we have seen, estimate (2.46) cannot be obtained making act L directly on (2.59), as the action of L on the right-hand side of this equation has bad upper bounds in L^2. On the other hand, making act L on (2.70), commuting it to $D_t - P_0$ and using (2.72), one may obtain a bound of the form (2.46) for $\|L_+\mathring{u}_+(t,\cdot)\|_{L^2}$. Actually, to do so with an improved implicit constant, one has to show that the right-hand side of (2.72) is $o(t^{-1}\|L_+\tilde{u}_+\|_{L^2})$ instead of just $O(t^{-1}\|L_+\tilde{u}_+\|_{L^2})$, but this follows from the estimates we get if $t \leq \varepsilon^{-4+c}$ and $\varepsilon \ll 1$. The remaining thing to do is then to relate estimates for $L_+\mathring{u}_+$ in L^2 and estimates for $L_+\tilde{u}_+$, i.e. to show that the action of L_+ on the $\hat{\mathcal{M}}_j$, $\hat{\mathcal{M}}'_2$ terms in (2.71) do not perturb significantly the a priori bound of the left-hand side. We do that in Section 7.1 for $\hat{\mathcal{M}}_j$, $j = 3, 4$ and in Section 7.2 for $\hat{\mathcal{M}}'_2$. In this Chapter 7, we also check that the remainder $\hat{\mathcal{R}}$ in (2.70) satisfies (2.72). These estimates heavily rely on the boundedness properties of the different multilinear operators we use, that are discussed in Appendix D. Putting all of that together, we conclude the bootstrap for estimates (2.45)–(2.46) in Proposition 7.3.7.

2.9 Step 7: Bootstrap of L^∞ estimates and end of proof

The only remaining step in order to conclude the proof of the main theorem is to bootstrap bound (2.47). We do that in Chapter 8. We deduce from equation (2.56) satisfied by $\tilde{u}_+$ an ordinary differential equation. We proceed as in [1] for water waves, with simplifications inspired by Ifrim and Tataru [45] (see also [20, 82]). If we write equation (2.56) as $(D_t - p(D_x))\tilde{u}_+ = f_+$ and if we define $\underline{\tilde{u}}_+, \underline{f}_+$ by

$$\tilde{u}_+(t,x) = \frac{1}{\sqrt{t}}\underline{\tilde{u}}_+\Big(t, \frac{x}{t}\Big), \quad f_+(t,x) = \frac{1}{\sqrt{t}}\underline{f}_+\Big(t, \frac{x}{t}\Big), \tag{2.73}$$

we obtain

$$\Big(D_t - \mathrm{Op}_h^{\mathrm{W}}\big(x\xi + \sqrt{1+\xi^2}\big)\Big)\underline{\tilde{u}}_+ = \underline{f}_+, \tag{2.74}$$

where we used a Weyl semiclassical quantization, depending on the parameter $h = \frac{1}{t}$, defined in general by

$$\mathrm{Op}_h^{\mathrm{W}}(a(x,\xi)) = \frac{1}{2\pi h}\int e^{i(x-y)\frac{\xi}{h}} a\Big(\frac{x+y}{2},\xi\Big)u(y)\,dy\,d\xi. \tag{2.75}$$

We decompose then $\underline{\tilde{u}}_+ = \underline{\tilde{u}}_\Lambda + \underline{\tilde{u}}_{\Lambda^c}$, where

$$\underline{\tilde{u}}_\Lambda = \mathrm{Op}_h^{\mathrm{W}}\Big(\gamma\Big(\frac{x+p'(\xi)}{\sqrt{h}}\Big)\Big)\underline{\tilde{u}}_+ \tag{2.76}$$

with γ in $C_0^\infty(\mathbb{R})$, equal to one close to zero and with small enough support. Then $\underline{\tilde{u}}_\Lambda$ is localized close to the set $\Lambda = \{(x,\xi) : x = -p'(\xi)\}$, i.e. close to $\{\xi = d\varphi(x)\}$ if $\varphi(x) = \sqrt{1-x^2}$ is the phase of oscillations of solutions to linear Klein–Gordon equations (after rescaling (2.73)). One sees that the L^2 estimates (2.45)–(2.46) allow one to get wanted bounds for the component $\underline{\tilde{u}}_{\Lambda^c}$ (see Proposition 8.1.1). On the other hand, since $\underline{\tilde{u}}_\Lambda$ is microlocalized close to Λ, in the term $\mathrm{Op}_h^{\mathrm{W}}(x\xi + \sqrt{1+\xi^2})\underline{\tilde{u}}_\Lambda$ one may replace the symbol by its restriction to Λ, up to remainders that are well controlled thanks to the L^2 estimates (2.45)–(2.46). This brings an ODE for $\underline{\tilde{u}}_\Lambda$ that implies by integration the wanted bound (2.47). The end of Chapter 8 (Section 8.2) puts together these estimates and those obtained in Section 4.2 for $a(t)$ in order to close the bootstrap argument and prove the main conclusions (2.13) and (2.14).

2.10 Further comments

In the last section of the present chapter, we shall explain what is the difficulty in order to go beyond the time limit ε^{-4}. Since this is much related to a phenomenon extensively discussed in the two papers of Lindblad, Lührmann and Soffer [60] and Lindblad, Lührmann, Schlag and Soffer [59], as well as in the work of Germain and Pusateri [33], let us first recall some of the results of [60].

The authors of that paper consider an equation of the form

$$\big(D_t - \sqrt{1+D_x^2}\big)u = -\frac{1}{2}\langle D_x\rangle^{-1}\big(\alpha(\cdot)(u+\bar{u})^2\big) \tag{2.77}$$

on $\mathbb{R}\times\mathbb{R}$, where α is a smooth decaying function (say $\alpha \in \mathcal{S}(\mathbb{R})$, even if their assumptions are weaker), satisfying $\hat{\alpha}(\sqrt{3}) \neq 0$ or $\hat{\alpha}(-\sqrt{3}) \neq 0$. They prove that if (2.77) is supplemented by an initial data u_0 satisfying $\varepsilon = \|\langle x\rangle^2 u_0\|_{H^4} \ll 1$, then the solution to (2.77) may be decomposed as a sum

$$u(t,\cdot) = u_{\mathrm{free}}(t,x) + u_{\mathrm{mod}}(t,x), \tag{2.78}$$

where u_{free} satisfies the same dispersive estimates as a solution a linear Klein–Gordon equation, namely $\|u_{\mathrm{free}}(t,\cdot)\|_{L^\infty} = O(\varepsilon t^{-\frac{1}{2}})$ when t goes to $+\infty$, and where u_{mod} obeys only the weaker dispersive estimate

$$\|u_{\mathrm{mod}}(t,\cdot)\|_{L^\infty} = O\Big(\varepsilon^2\frac{\log t}{\sqrt{t}}\Big) \tag{2.79}$$

(see [60, Theorem 1.1] and in particular formulas (1.12) and (1.15)). Moreover, the logarithmic loss that appears on the right-hand side of (2.79), in comparison with the decay of linear solution, in unavoidable. Actually, Lindblad, Lührmann and Soffer show that along the rays $x = \pm\sqrt{3}t/2$, $u_{\text{mod}}(t, \pm\sqrt{3}t/2)$ behaves when t goes to $+\infty$ as

$$\frac{a_0^2}{\sqrt{8}} e^{i\frac{\pi}{4}} e^{i\frac{t}{2}} \hat{\alpha}(\mp\sqrt{3}) \frac{\log t}{\sqrt{t}} \tag{2.80}$$

for some complex coefficient $a_0 = O(\varepsilon)$. (See [60, (1.15)] and (1.16) of the same paper for an explicit expression of a_0 in terms of the solution u to (2.77)). On the other hand, outside a conical neighborhood of these two rays, u_{mod} has an $\varepsilon^2 t^{-\frac{1}{2}}$ bound, without any logarithmic loss. In order to relate this with the obstacle that prevents us from going above time ε^{-4} in our own result, let us recall the argument of the introduction of [60] that explains heuristically the appearance of the logarithmic factor in (2.80). The idea is that, since $\alpha(x)$ on the right-hand side of (2.77) is decaying when x goes to infinity, one may replace there $u(t,x)$ by $u(t,0)$, up to terms that are expected to have a stronger time decay. In that way, an approximation of (2.77) is

$$\big(D_t - \sqrt{1 + D_x^2}\big)u = -\frac{1}{2}\langle D_x\rangle^{-1}\big(\alpha(x)\big(u(t,0) + \bar{u}(t,0)\big)^2\big). \tag{2.81}$$

A second approximation (that is justified a posteriori) is to assume that $u(t,0)$ will have the same asymptotic behavior as a solution to a linear Klein–Gordon equation restricted to $x = 0$ when t goes to infinity. This allows one to replace in (2.81) $u(t,0)$ by $\varepsilon\frac{e^{it}}{\sqrt{t}}$, so that u_{mod} will be essentially the solution to

$$\big(D_t - \sqrt{1 + D_x^2}\big)u_{\text{mod}} = -\frac{\varepsilon^2}{2t}\big(\langle D_x\rangle^{-1}\alpha\big)\big(e^{2it} + 2 + e^{-2it}\big). \tag{2.82}$$

If more generally one considers an equation of the form

$$\big(D_t - \sqrt{1 + D_x^2}\big)u = \frac{1}{t}Y(x)e^{i\lambda t} \tag{2.83}$$

with Y in $\mathcal{S}(\mathbb{R})$ (or at least smooth enough and decaying enough at infinity), one may rewrite (2.83) as an equation for $u_\lambda(t,x) = e^{-i\lambda t}u(t,x)$ of the form

$$\big(D_t + \lambda - \sqrt{1 + D_x^2}\big)u_\lambda = \frac{1}{t}Y(x). \tag{2.84}$$

If $\lambda < 1$, the operator $\sqrt{1 + D_x^2} - \lambda$ is elliptic and the solution to (2.84) will be $O(t^{-\frac{1}{2}})$ in L^∞ when t goes to infinity: This may be seen using Duhamel formula and integrating by parts, or equivalently defining

$$w_\lambda = u_\lambda + \big(\sqrt{1 + D_x^2} - \lambda\big)^{-1}(t^{-1}Y(x)) \tag{2.85}$$

that satisfies a new equation

$$\big(D_t + \lambda - \sqrt{1 + D_x^2}\big)w_\lambda = \frac{1}{t^2}\tilde{Y}(x), \tag{2.86}$$

where $\tilde{Y}$ is some new $\mathcal{S}(\mathbb{R})$ function and the new right-hand side is time integrable. Because of that, the solution to (2.86) will have the same dispersive time decay rate as a solution to a linear Klein–Gordon equation, i.e. will be $O(t^{-\frac{1}{2}})$ in L^∞. This is what happens for the last two terms on the right-hand side of (2.82). On the other hand, for the first one, one gets an equation of the form (2.83), (2.84) with $\lambda = 2$, so that the symbol $\sqrt{1+\xi^2} - 2$ vanishes at $\xi = \pm\sqrt{3}$. In this case, the analysis of the solution to (2.86) expressed from Duhamel formula and Fourier transform shows that an asymptotic behavior of the form (2.80) holds along the two rays $x = \pm t\frac{\sqrt{3}}{2}$.

The logarithmic loss displayed in (2.80) seems incompatible with the known methods used to study global existence and asymptotic behavior for Klein–Gordon equations of the form (1.21) or (2.77) if we no longer assume that $\alpha(\cdot)$ is decaying at infinity. Actually, [60, Theorem 1.1] as well as [59, Theorem 1.1], uses in an essential way the fact that the space decay of this coefficient will provide, along the rays over which (2.80) holds, a time decay that will compensate the logarithmic loss.

Another situation when asymptotic behavior may be obtained for the solution of a problem of the form (2.77), including with nonlinearities involving terms like $(u+\bar{u})^2$, $(u+\bar{u})^3$ (without space decaying pre-factors), appears if the bad term (2.80) vanishes. This happens for the non-resonant case $\hat{\alpha}(\sqrt{3}) = \hat{\alpha}(-\sqrt{3}) = 0$ treated in [60, Theorem 1.6] and [59, Theorem 1.1], when one recovers the same asymptotics as those holding true for equations of the form (2.77) with the function α replaced by a constant.

The second case when (2.80) vanishes is when $a_0 = 0$. This happens for instance when α is an odd function and the initial condition in (2.77) is also odd (see (2.81) where the right-hand side vanishes for odd functions u, so that the contributions coming from (2.82) that were responsible of the bad term (2.80) disappear). Such a situation is studied by Germain and Pusateri [33], in a more general framework. They consider equations of the form

$$\left(\partial_t^2 - \partial_x^2 + V(x) + m^2\right)u = a(x)u^2, \tag{2.87}$$

where $a(x)$ is a smooth function that has different limits at $+\infty$ and $-\infty$ and $V(x)$ an $\mathcal{S}(\mathbb{R})$ potential that has no bound state. They prove a decay estimate for the solution in $O(t^{-\frac{1}{2}})$ when time goes to infinity, under some orthogonality assumption on the solution. This assumption always holds for generic potentials, and in the case of exceptional ones (like the zero potential), it holds under evenness or oddness conditions on V, a and the initial data. One of the key ingredients in the proof of [33, Theorem 1.1] is again related to the fact that a bad frequency $\pm\sqrt{3}$ appears. Actually, it shows up when one tries to perform a variable coefficients normal form. In order to overcome this difficulty, the authors introduce functional spaces, involving dyadic Fourier cut-offs close to the bad frequencies, and measuring the (distorted) Fourier transform of the solution in such spaces.

Let us go back to the problem we study in this book, and in particular to the limitation of our result to times $O(\varepsilon^{-4})$. We already discussed this issue in Section 2.5 after the introduction of the approximate solution in (2.49). Here, we want to explain how the problem we encounter to go beyond time ε^{-4} might be related to some of the works we just described, namely the possible appearance of some extra logarithm in pointwise estimates of the solution along two rays, as in (2.80). Remark first that we are dealing only with odd solutions. As already noticed, this implies that the coefficient a_0 in (2.80) vanishes, so that a solution of a problem of the form (2.77) has $O(t^{-\frac{1}{2}})\ L^\infty$ estimates. The point is that, in our problem, we do not study an equation of the form (2.77) or (2.87), but a *coupling* between a PDE and an ODE, namely system (2.11) or equivalently, a coupling between the PDE (2.27) and the ODE (2.34). Because of that, our PDE contains a source term given by (2.28), involving expressions of the form

$$a(t)^2 Y_2(x), a(t)^3 Y_3(x), \tag{2.88}$$

where Y_2, Y_3 are $\mathcal{S}(\mathbb{R})$ functions and $a(t)$, solution of the ODE, has an oscillatory behavior of the form

$$\frac{\varepsilon}{\sqrt{1+t\varepsilon^2}} e^{\pm it\frac{\sqrt{3}}{2}}. \tag{2.89}$$

When plugged in (2.88), this shows that our PDE will contain a source term that has a similar structure as the right-hand side of (2.82), with oscillating terms $e^{\pm it\sqrt{3}}$ instead of $e^{\pm 2it}$ and pre-factor $\frac{\varepsilon^2}{1+t\varepsilon^2}$ instead of $\frac{\varepsilon^2}{t}$ (for the quadratic contribution coming from (2.88)). Because of that, and by analogy with the study of [60], we may expect that the solution to our PDE contains contributions that might grow as $\frac{\log t}{\sqrt{t}}$ when t goes to infinity.

In this book, we prove that such a possible growth does not happen before at least time ε^{-4+0}. Let us return to the discussion on that issue that we started in Section 2.5. We introduced in (2.49) a solution u_+^{app} of a linear equation with source terms that are essentially of the form (2.88) (forgetting the second line of the first equation in (2.49)). If we retain only the quadratic term $a(t)^2 Y_2$ in (2.88), and use (2.89), this means that we have to solve essentially an equation of the form

$$\left(D_t - \sqrt{1+D_x^2}\right)U = \frac{\varepsilon^2}{1+t\varepsilon^2} e^{\pm it\sqrt{3}} M(x) \tag{2.90}$$

for some function M in $\mathcal{S}(\mathbb{R})$ and zero initial data at $t=1$. This is an equation of the form (2.83), and as we have seen after (2.84), the delicate case is the one corresponding to the phase $t\sqrt{3}$ in the exponential, so that in the sequel we discuss only (2.90) with sign $+$. Then U is one of the contribution to the approximate solution u_+^{app} of (2.49), and we decompose it as $U = U' + U''$ with essentially

$$U'(t,x) = i\int_1^{\sqrt{t}} e^{i(t-\tau)\sqrt{1+D_x^2}+it\sqrt{3}} M(\cdot)\,\frac{\varepsilon^2\,d\tau}{1+\tau\varepsilon^2}, \tag{2.91}$$

$$U''(t,x) = i\int_{\sqrt{t}}^{t} e^{i(t-\tau)\sqrt{1+D_x^2}+it\sqrt{3}} M(\cdot)\,\frac{\varepsilon^2\,d\tau}{1+\tau\varepsilon^2}. \tag{2.92}$$

This decomposition corresponds to $u_+^{\text{app}} = u_+'^{\text{app}} + u_+''^{\text{app}}$ introduced before (2.50) in Section 2.5, and we may prove some good L^∞ estimate for L_+U'' (see (2.50)) and some good L^2 estimate for L_+U' (of the form (2.46)) for times $t = O(\varepsilon^{-4+0})$. This last L^2 bound degenerates when t goes to ε^{-4}, and actually so does the pointwise estimate of U' that is obtained in Appendix C (see (C.40) with $\omega = 1$). We obtain there for U' a pointwise bound in

$$\frac{(\varepsilon^2\sqrt{t})}{\sqrt{t}}\Big\langle t^{\frac{1}{2}}\Big(\frac{x}{t} \pm \sqrt{\frac{2}{3}}\Big)\Big\rangle^{-1}. \tag{2.93}$$

Outside a conical neighborhood of the rays $x = \mp t\sqrt{2/3}$, (2.93) reduces to an $\varepsilon^2 t^{-\frac{1}{2}}$ decay (whatever the value of t). On the other hand, along the lines $x = \mp t\sqrt{2/3}$, we just get a bound in $(\varepsilon^2\sqrt{t})/\sqrt{t}$, that provides an $O(t^{-\frac{1}{2}})$ decay only for $t = O(\varepsilon^{-4})$. Past such a time, estimate (2.93) will no longer remain valid and, at the light of the results of [60] concerning (2.77) and [59], one may not exclude that some $\log t/\sqrt{t}$ behavior might hold along the two preceding rays. Since, unlike in (2.77), we no not have just nonlinearities involving rapidly space decaying coefficients, we do not know how such contributions might be handled in the nonlinear problem.