Chapter 3

First quadratic normal form

In Section 2.2 of the preceding chapter, we have introduced an evolution equation (2.27) for a function u_+. This equation is of the type of (1.58) in the introduction, except that its nonlinearity is non-local (see (2.31) and (2.32)). In this chapter, we shall express these nonlinearities in terms of multilinear operators, that are a special case of classes introduced in Appendix B. This will give us a general framework that will be stable under the reductions we shall have to perform.

The nonlinearity in our equation contains quadratic terms. We have already explained in Section 1.6 of the introduction that such terms have to be eliminated by normal form. This is the goal of Section 3.2 of this chapter, following the guidelines explained in Section 2.4 of Chapter 2.

3.1 Expression of the equation from multilinear operators

Let us define the classes of multilinear operators we shall use. They are special cases of the operators introduced in Appendix B, that will be useful in the rest of the paper. We introduce in this section only the subclasses we need in Chapter 3.

In this chapter, an order function on $\mathbb{R}^p$ is a function from $\mathbb{R}^p$ to $\mathbb{R}_+$ such that there is some $N_0 \in \mathbb{N}$ so that, for any $(\xi_1, \dots, \xi_p), (\xi_1', \dots, \xi_p') \in \mathbb{R}^p$,

$$M(\xi_1', \dots, \xi_p') \le C \prod_{j=1}^{p} \langle \xi_j - \xi_j' \rangle^{N_0} M(\xi_1, \dots, \xi_p). \tag{3.1}$$

(In Appendix B, we shall allow order functions depending also on a space variable x.)

Definition 3.1.1. Let M be an order function on $\mathbb{R}^p$, with $p \in \mathbb{N}^*$, $\kappa \in \mathbb{N}$. We denote by $\tilde{S}_{\kappa,0}(M, p)$ the space of smooth functions

$$\begin{gathered}(y, \xi_1, \dots, \xi_p) \mapsto a(y, \xi_1, \dots, \xi_p),\\ \mathbb{R} \times \mathbb{R}^p \to \mathbb{C}\end{gathered} \tag{3.2}$$

satisfying for any $\alpha \in \mathbb{N}^p$,

$$|\partial_\xi^\alpha a(y, \xi)| \le C M(\xi) M_0(\xi)^{\kappa|\alpha|} \tag{3.3}$$

and for any $\alpha \in \mathbb{N}^p$, any $\alpha_0' \in \mathbb{N}^*$, any $N \in \mathbb{N}$,

$$|\partial_\xi^\alpha \partial_y^{\alpha_0'} a(y, \xi)| \le C M(\xi) M_0(\xi)^{\kappa|\alpha|} \big(1 + M_0(\xi)^{-\kappa} |y|\big)^{-N}, \tag{3.4}$$

where $M_0(\xi)$ denotes

$$M_0(\xi_1, \dots, \xi_p) = \Bigg(\sum_{1 \le i < j \le p} \langle \xi_i \rangle^2 \langle \xi_j \rangle^2 \Bigg) \Bigg(\sum_{i=1}^{p} \langle \xi_i \rangle^2 \Bigg)^{-\frac{1}{2}} \tag{3.5}$$

and is equivalent to $1 + \max_2(|\xi_1|, \dots, |\xi_p|)$, $\max_2$ standing for the second largest of the arguments.

We denote by $\tilde{S}'_{\kappa,0}(M, p)$ the subspace of $\tilde{S}_{\kappa,0}(M, p)$ of those a for which (3.4) holds including for $\alpha'_0 = 0$.

The symbols of Definition 3.1.1 are the special case of those defined in Definition B.1.2 of Appendix B when there is no x dependence in (B.11). We associate to them operators through the quantization rule

$$\begin{aligned}\mathrm{Op}(a)(v_1, \dots, v_p) = \frac{1}{(2\pi)^p} \int e^{ix(\xi_1 + \cdots + \xi_p)} a(x, \xi_1, \dots, \xi_p) \\ \times \prod_{j=1}^{p} \hat{v}_j(\xi_j)\, d\xi_1 \cdots d\xi_p\end{aligned} \tag{3.6}$$

for any $a \in \tilde{S}_{\kappa,0}(M, p)$, any test functions $v_1, \dots, v_p$. This is the rule defined in (B.17) of the appendix in the case of general symbols $a(y, x, \xi)$, specialized to the subclass of symbols that do not depend on x, as in Definition 3.1.1. We shall also impose on our symbols the extra condition

$$a(-y, -\xi_1, \dots, -\xi_p) = (-1)^{p-1} a(y, \xi_1, \dots, \xi_p). \tag{3.7}$$

Under this condition, the operator $\mathrm{Op}(a)$ sends a p-tuple of odd functions to an odd function.

Let us state the symbolic calculus result that is proved in Appendix B (see Corollary B.2.6, (B.42), (B.43)) and that we shall use below.

Proposition 3.1.2. *The following statements hold.*

(i) *Let $n', n'' \in \mathbb{N}^*$, $n = n' + n'' - 1$, let $M'(\xi_1, \dots, \xi_{n'})$, $M''(\xi_{n'}, \dots, \xi_n)$ be two order functions. Let a (resp. b) be in $\tilde{S}_{\kappa,0}(M', n')$ (resp. $\tilde{S}_{\kappa,0}(M'', n'')$). Define*

$$M(\xi_1, \dots, \xi_n) = M'(\xi_1, \dots, \xi_{n'-1}, \xi_{n'} + \cdots + \xi_n) M''(\xi_{n'}, \dots, \xi_n). \tag{3.8}$$

There are $\nu \in \mathbb{N}$, depending only on the order functions M' and M'', and a symbol c'_1 in $\tilde{S}'_{\kappa,0}(M M_0^{\nu\kappa}, n)$ such that if

$$\begin{aligned}c(y, \xi_1, \dots, \xi_n) = a(y, \xi_1, \dots, \xi_{n'-1}, \xi_{n'} + \cdots + \xi_n) b(y, \xi_{n'}, \dots, \xi_n) \\ + c'_1(y, \xi_1, \dots, \xi_n),\end{aligned} \tag{3.9}$$

then for all test functions $v_1, \dots, v_n$,

$$\mathrm{Op}(a)[v_1, \dots, v_{n'-1}, \mathrm{Op}(b)(v_{n'}, \dots, v_n)] = \mathrm{Op}(c)[v_1, \dots, v_n]. \tag{3.10}$$

Moreover, if a and b satisfy (3.7), so do c and c'_1.

(ii) *If a is in $\tilde{S}_{0,0}(M, 1)$, there is a symbol a^* in $\tilde{S}_{0,0}(M, 1)$ such that* $\mathrm{Op}(a^*) = \mathrm{Op}(a)^*$. *Moreover, if a satisfies (3.7), so does a^*.*

We shall use the above class of symbols to re-express equation (2.27).

Proposition 3.1.3. *For any multiindex $I = (i_1, \dots, i_p) \in \{-,+\}^p$ with $2 \le |I| = p \le 3$, one may find symbols $m_{0,I}$ in $\tilde{S}_{0,0}(\prod_{j=1}^{p} \langle \xi_j \rangle^{-1}, p)$ satisfying condition (3.7), and for any multiindex $I = (i_1, \dots, i_p) \in \{-,+\}^p$ with $1 \le |I| = p \le 2$, one may find symbols $m'_{1,I}$ in $\tilde{S}'_{0,0}(\prod_{j=1}^{p} \langle \xi_j \rangle^{-1}, p)$ satisfying condition (3.7), such that equation (2.27) may be written*

$$
\begin{aligned}
\big(D_t - p(D_x)\big)u_+ = F_0^2[a] + F_0^3[a] + \sum_{2 \le |I| \le 3} \mathrm{Op}(m_{0,I})[u_I] \\
+ a(t) \sum_{1 \le |I| \le 2} \mathrm{Op}(m'_{1,I})[u_I] \\
+ a(t)^2 \sum_{|I|=1} \mathrm{Op}(m'_{2,I})[u_I],
\end{aligned}
\tag{3.11}
$$

where u_I is defined in (2.25) and (2.26).

Proof. Consider first the terms on the right-hand side of equation (2.27) that do not depend on a, i.e. with notation (2.29) $\sum_{|I|=2} F_{2,I}^2[u_I]$ and $\sum_{|I|=3} F_{3,I}^3[u_I]$. These terms are given by the first equality in (2.31) and (2.32). A symbol of the form $\kappa(y) \prod_{\ell=1}^{2} b(y, \xi_j) p(\xi_j)^{-1}$ or $\prod_{\ell=1}^{3} b(y, \xi_j) p(\xi_j)^{-1}$ belongs respectively to $\tilde{S}_{0,0}(\prod_{\ell=1}^{2} \langle \xi_j \rangle^{-1}, 2)$ and $\tilde{S}_{0,0}(\prod_{\ell=1}^{3} \langle \xi_j \rangle^{-1}, 3)$ and because of property (A.9) satisfied by b and the oddness of κ, condition (3.7) holds. If we apply the results of Proposition 3.1.2, we conclude that the contributions to (2.27) that do not depend on a have the structure of the first sum on the right-hand side of (3.11).

Consider next terms of the form $a(t) F_{1,I}^2[u_I]$, $|I| = 1$ or $a(t) F_{2,I}^3[u_I]$, $|I| = 2$ in equation (2.29). They may be expressed from the first line in (2.30) and the second line in (2.31). Since Y is rapidly decaying, the symbols $Y(y)\kappa(y)b(y, \xi)p(\xi)^{-1}$ and $Y(y) \prod_{\ell=1}^{2} b(y, \xi_j) p(\xi_j)^{-1}$ are in $\tilde{S}'_{0,0}(\langle \xi \rangle^{-1}, 1)$ and $\tilde{S}'_{0,0}(\prod_{j=1}^{2} \langle \xi_j \rangle^{-1}, 2)$. Because of the oddness of Y, κ and (A.9), they satisfy (3.7). Using again the composition result of Proposition 3.1.2, and noticing that as soon as at least one of the symbols a and b in (3.9) is in the $\tilde{S}'$ class, so is the composed symbol c, we conclude that the linear term in $a(t)$ on the right-hand side of (2.27) is given by the second sum in (3.11).

In the same way, the contributions $a(t)^2 F_{1,I}^3[u_I]$ coming from the second line (2.29) with $j = 1$, with $F_{1,I}^3$ given by (2.30), provide the last sum in (3.11). This concludes the proof. ∎

On the right-hand side of equation (3.11), terms with higher degree of homogeneity in $(a, u_\pm)$ will have better decay estimates. Moreover, an expression of the form $\mathrm{Op}(m')[u_I]$ with $|I| = p$ and a symbol m' in $\tilde{S}'_{0,0}(M, p)$, i.e. with rapid decay in y, will have better time decay than a term $\mathrm{Op}(m)[u_I]$ with $|I| = p$ and a symbol m in $\tilde{S}_{0,0}(M, p)$. Consequently, we expect that the terms in $\sum_{|I|=2} \mathrm{Op}(m_{0,I})[u_I]$ will be, among all $u_\pm$-dependent terms on the right-hand side of (3.11), those having the worst time decay. In next section, we shall get rid of these terms by normal form.

3.2 First quadratic normal form

Proposition 3.2.1. *Define from the symbols $m_{0,I}$, $|I| = 2$ of Proposition 3.1.3 new functions*

$$\tilde{m}_{0,I}(y,\xi_1,\xi_2) = m_{0,I}(y,\xi_1,\xi_2)\big(-p(\xi_1+\xi_2) + i_1 p(\xi_1) + i_2 p(\xi_2)\big)^{-1} \tag{3.12}$$

if $I = (i_1, i_2)$. Then $\tilde{m}_{0,I}$ belongs to $\tilde{S}_{1,0}(\prod_{j=1}^{2}\langle\xi_j\rangle^{-1}M_0(\xi_1,\xi_2), 2)$. Moreover, there are new symbols

- $(m'_{0,I})_{|I|=2}$ *belonging to* $\tilde{S}'_{1,0}(\prod_{j=1}^{2}\langle\xi_j\rangle^{-1}M_0(\xi), 2)$,
- $(m'_{j,I})_{1\leq|I|\leq 4-j}$, $1 \leq j \leq 3$, *in* $\tilde{S}'_{1,0}(\prod_{j=1}^{|I|}\langle\xi_j\rangle^{-1}M_0(\xi)^{\nu}, |I|)$ *for some* ν,
- $(m_{0,I})_{3\leq|I|\leq 4}$ *belonging to* $\tilde{S}_{1,0}(\prod_{j=1}^{|I|}\langle\xi_j\rangle^{-1}M_0(\xi), |I|)$

such that

$$\begin{aligned}&\big(D_t - p(D_x)\big)\Big(u_+ - \sum_{|I|=2}\mathrm{Op}(\tilde{m}_{0,I})[u_I]\Big)\\ &\quad = F_0^2[a] + F_0^3[a] + \sum_{3\leq|I|\leq 4}\mathrm{Op}(m_{0,I})[u_I] + \sum_{|I|=2}\mathrm{Op}(m'_{0,I})[u_I]\\ &\qquad + \sum_{j=1}^{3} a(t)^j \sum_{1\leq|I|\leq 4-j}\mathrm{Op}(m'_{j,I})[u_I].\end{aligned} \tag{3.13}$$

Finally, all above symbols satisfy (3.7).

Proof. We notice first that

$$\begin{aligned}\langle\xi_1\rangle + \langle\xi_2\rangle - \langle\xi_1+\xi_2\rangle &= \frac{1 + 2(\langle\xi_1\rangle\langle\xi_2\rangle - \xi_1\xi_2)}{\langle\xi_1\rangle + \langle\xi_2\rangle + \langle\xi_1+\xi_2\rangle}\\ &\geq c\big(1 + \max_2(|\xi_1|,|\xi_2|)\big)^{-1}\\ &\geq c M_0(\xi_1,\xi_2)^{-1}.\end{aligned} \tag{3.14}$$

This implies that

$$\langle\xi_1+\xi_2\rangle + \langle\xi_2\rangle - \langle\xi_1\rangle \geq c\big(1 + \max_2(|\xi_1+\xi_2|,|\xi_2|)\big)^{-1}$$

which is larger than the right-hand side of (3.14), except when $|\xi_2| \gg |\xi_1|$. But then the left-hand side is larger than one. Consequently, we deduce from these inequalities that, for any sign i_1, i_2, we have for any $\alpha \in \mathbb{N}^2$,

$$\Big|\partial_\xi^\alpha\big(\langle\xi_1+\xi_2\rangle + i_1\langle\xi_1\rangle + i_2\langle\xi_2\rangle\big)^{-1}\Big| \leq C_\alpha M_0(\xi_1,\xi_2)^{1+|\alpha|}. \tag{3.15}$$

This implies that $\tilde{m}_{0,I}$ belongs to the wanted class of symbols. It obeys trivially (3.7) since $m_{0,I}$ does.

Denoting for $|I| = 2$, $u_I = (u_{i_1}, u_{i_2})$ as in (2.25), we compute

$$\begin{aligned}
&\big(D_t - p(D_x)\big)\big[\mathrm{Op}(\tilde{m}_{0,I})[u_I]\big] \\
&\quad = -\mathrm{Op}(p(\xi)) \circ \mathrm{Op}(\tilde{m}_{0,I})[u_I] + \mathrm{Op}(\tilde{m}_{0,I})[i_1 \mathrm{Op}(p(\xi))u_{i_1}, u_{i_2}] \\
&\qquad + \mathrm{Op}(\tilde{m}_{0,I})[u_{i_1}, i_2 \mathrm{Op}(p(\xi))u_{i_2}] \\
&\qquad + \mathrm{Op}(\tilde{m}_{0,I})[(D_t - i_1 p(D_x))u_{i_1}, u_{i_2}] \\
&\qquad + \mathrm{Op}(\tilde{m}_{0,I})[u_{i_1}, (D_t - i_2 p(D_x))u_{i_2}].
\end{aligned} \tag{3.16}$$

By Corollary B.2.7, the sum of the first three terms on the right-hand side may be written as a contribution to $\sum_{|I|=2} \mathrm{Op}(m'_{0,I})[u_I]$ in (3.13) plus the expression

$$\mathrm{Op}\big((-p(\xi_1 + \xi_2) + i_1 p(\xi_1) + i_2 p(\xi_2))\tilde{m}_{0,I}\big)[u_I]. \tag{3.17}$$

By (3.12), (3.17) will cancel the term $\sum_{|I|=2} \mathrm{Op}(m_{0,I})[u_I]$ in (3.11). Since the other terms on the right-hand side of (3.11) are still present in (3.13), we see that to conclude the proof, we just need to show that the last two terms in (3.16) provide as well contributions to the three sums on the right-hand side of (3.13). We express $(D_t \mp p(D_x))u_\pm$ from (3.11) (or its conjugate). To fix ideas, consider for instance

$$\mathrm{Op}(\tilde{m}_{0,(+,i_2)})[(D_t - p(D_x))u_+, u_{i_2}]. \tag{3.18}$$

If we replace $(D_t - p(D_x))u_+$ by the contribution $F_0^2[a] + F_0^3[a]$, which by (2.28) may be written $a(t)^2 Y_2 + a(t)^3 Y_3$, with odd functions Y_2, Y_3 in $\mathcal{S}(\mathbb{R})$, we see applying Corollary B.2.8 of Appendix B that expression (3.18) will provide contributions to the $\sum_{j=2}^3 a(t)^j \sum_{|I|=1} \mathrm{Op}(m'_{j,I})[u_I]$ term in (3.13).

We replace next $(D_t - p(D_x))u_+$ in (3.18) by the $a(t)$ or $a(t)^2$ terms in (3.11). We use (i) of Proposition 3.1.2, noticing that if in (3.9), either a is in $\tilde{S}'_{\kappa,0}(M', n')$ or b is in $\tilde{S}'_{\kappa,0}(M'', n'')$, then c is in $\tilde{S}'_{\kappa,0}(M, n)$. Consequently, we get contributions to $a(t) \sum_{2 \le |I| \le 3} \mathrm{Op}(m'_{1,I})[u_I]$ and $a(t)^2 \sum_{|I|=2} \mathrm{Op}(m'_{1,I})[u_I]$ in (3.13). Finally, if we replace in (3.18) $(D_t - p(D_x))u_+$ by the first sum on the right-hand side of (3.11), we obtain contributions to $\sum_{3 \le |I| \le 4} \mathrm{Op}(m_{0,I}[u_I])$ in (3.13) using again (i) of Proposition 3.1.2. This concludes the proof as property (3.7) of the symbols is preserved under composition. ∎