Chapter 4

Construction of approximate solutions

In the preceding chapter, we have performed a quadratic normal form in order to reduce ourselves to an equation of the form (3.13). The right-hand side of this equation contains a source term and in Section 4.1 below, we construct an approximate solution solving the linear equation whose right-hand side is essentially this source term. We explained this part of the proof in Section 2.5, see equations (2.48)–(2.49). The construction of the approximate solution relies on Appendix C below.

On the other hand, because of the coupling between a dispersive equation and the evolution equation for the bound state, we have seen in Section 2.2 that we have also to study an ordinary differential equation (2.34), which is equivalent to the first equation in (2.9). We have explained at the end of Section 2.5 what is the form of that ODE, and how we can show that its solutions are global and decaying using Fermi's golden rule. Section 4.2 below is devoted to the asymptotic analysis of this ODE. Of course, the study is more technical than in the presentation in Chapter 2 since we have to fully take into account those terms on the right-hand side that come from the interaction between the bound state and the dispersive part of our problem.

4.1 Approximate solution to the dispersive equation

The proof of our main theorem being done by bootstrap, we shall assume that we know, on some interval $[1, T]$, an approximation of the function $t \mapsto a(t)$ that is present on the right-hand side of (3.13).

Let $\varepsilon_0 \in]0,1]$, $A, A' > 1$, $\theta' \in]0, \frac{1}{2}[$ (close to $\frac{1}{2}$) be given. Let $T \in [1, \varepsilon^{-4}]$. We shall denote for $t \geq 1$, $\varepsilon \in]0, \varepsilon_0[$,

$$t_\varepsilon = \varepsilon^{-2}\langle t\varepsilon^2\rangle \tag{4.1}$$

and assume given functions

$$\begin{aligned} g : [1,T] &\to \mathbb{C}, & \tilde{u}_\pm : [1,T]\times\mathbb{R} &\to \mathbb{C}, \\ t &\mapsto g(t), & (t,x) &\mapsto \tilde{u}_\pm(t,x) \end{aligned} \tag{4.2}$$

and $x \mapsto Z(x)$ in $\mathcal{S}(\mathbb{R})$, real valued, satisfying the following conditions:

$$|g(t)| \leq At_\varepsilon^{-\frac{1}{2}}, \quad |\partial_t g(t)| \leq A'\big(t_\varepsilon^{-\frac{3}{2}} + (\varepsilon^2\sqrt{t})^{\frac{3}{2}\theta'}t^{-\frac{3}{2}}\big), \quad t \in [1,T], \tag{4.3}$$

$$|\langle Z, \tilde{u}_\pm(t,\cdot)\rangle| \leq (\varepsilon^2\sqrt{t})^{\theta'}t^{-\frac{3}{4}}, \quad t \in [1,T]. \tag{4.4}$$

Moreover, we assume given $\widetilde{\mathcal{W}}$ a neighborhood of $\{-1,1\}$ in $\mathbb{R}$ and for any λ in $\mathbb{R} - \widetilde{\mathcal{W}}$, two functions

$$t \mapsto \varphi_\pm(\lambda,t), \quad t \mapsto \psi_\pm(\lambda,t) \tag{4.5}$$

satisfying for any $t \in [1, T]$, any $\lambda \in \mathbb{R} - \widetilde{\mathcal{W}}$,

$$|\varphi_\pm(\lambda, t)| \leq (\varepsilon^2\sqrt{t})^{\theta'} t^{-\frac{1}{2}}, \quad |\psi_\pm(\lambda, t)| \leq (\varepsilon^2\sqrt{t})^{\theta'} t^{-1} \tag{4.6}$$

and solving the equation

$$(D_t - \lambda)\varphi_\pm(\lambda, t) = \langle Z, \tilde{u}_\pm \rangle + \psi_\pm(\lambda, t). \tag{4.7}$$

We define from the above data

$$\begin{aligned} a_+^{\mathrm{app}}(t) &= e^{it\frac{\sqrt{3}}{2}} g(t) + \omega_2 g(t)^2 e^{it\sqrt{3}} + \omega_0 |g(t)|^2 + \omega_{-2} \overline{g(t)}^2 e^{-it\sqrt{3}} \\ &\quad + e^{it\frac{\sqrt{3}}{2}} \big(g(t)\varphi_+(0, t) - g(t)\varphi_-(0, t)\big) \\ &\quad + e^{-it\frac{\sqrt{3}}{2}} \big(\overline{g(t)}\varphi_+(\sqrt{3}, t) - \overline{g(t)}\varphi_-(\sqrt{3}, t)\big), \end{aligned} \tag{4.8}$$

where $\omega_0, \omega_2, \omega_{-2}$ are given complex constants. We set

$$a_-^{\mathrm{app}} = -\overline{a_+^{\mathrm{app}}}, \quad a^{\mathrm{app}}(t) = \frac{\sqrt{3}}{3}\big(a_+^{\mathrm{app}}(t) - a_-^{\mathrm{app}}(t)\big). \tag{4.9}$$

We assume given, as in the statement of Proposition 3.2.1, symbols $m'_{1,I}$ for $|I| = 1$ (i.e. $I = +$ or $-$) belonging to the class $\tilde{S}'_{1,0}(\langle \xi \rangle^{-1}, 1)$ satisfying (3.7). We want to construct an approximate solution u_+^{app} to the equation

$$\big(D_t - p(D_x)\big)u_+^{\mathrm{app}} = F_0^2[a^{\mathrm{app}}] + F_0^3[a^{\mathrm{app}}] + a^{\mathrm{app}}(t) \sum_{|I|=1} \mathrm{Op}(m'_{1,I})[u_I^{\mathrm{app}}] \tag{4.10}$$

that is deduced from (3.13) computing the source terms F_0^2, F_0^3 at a^{app}, and retaining from the other terms on the right-hand side only those that are linear both in a and $u_\pm$.

Before stating the main proposition, let us re-express the source term in (4.10).

Lemma 4.1.1. *Under the preceding assumptions on a^{app}, one may rewrite*

$$F_0^2[a^{\mathrm{app}}] + F_0^3[a^{\mathrm{app}}] = I_1 + I_2 + I_3 + R(t, x), \tag{4.11}$$

where

$$I_1(t, x) = \sum_{j \in \{-2, 0, 2\}} e^{ijt\frac{\sqrt{3}}{2}} M_j(t, x) \tag{4.12}$$

for smooth odd functions of x, $M_j(t, x)$, satisfying for any $\alpha, N \in \mathbb{N}$,

$$\begin{aligned} |\partial_\xi^\alpha \hat{M}_j(t, \xi)| &\leq C_{\alpha,N} t_\varepsilon^{-1} \langle \xi \rangle^{-N}, \\ |\partial_\xi^\alpha \partial_t \hat{M}_j(t, \xi)| &\leq C_{\alpha,N} \langle \xi \rangle^{-N} t_\varepsilon^{-\frac{1}{2}} \big(t_\varepsilon^{-\frac{3}{2}} + t^{-\frac{3}{2}} (\varepsilon^2\sqrt{t})^{\frac{3}{2}\theta'}\big) \end{aligned} \tag{4.13}$$

with constants $C_{\alpha,N}$ depending on A, A' in (4.3)–(4.4), where

$$I_2(t, x) = \sum_{j \in \{-3, -1, 1, 3\}} e^{ijt\frac{\sqrt{3}}{2}} M_j(t, x) \tag{4.14}$$

for smooth odd functions of x satisfying

$$
\begin{aligned}
|\partial_\xi^\alpha \hat{M}_j(t,\xi)| &\le C_{\alpha,N} t_\varepsilon^{-\frac{3}{2}} \langle\xi\rangle^{-N},\\
|\partial_\xi^\alpha \partial_t \hat{M}_j(t,\xi)| &\le C_{\alpha,N} \langle\xi\rangle^{-N} t_\varepsilon^{-1}\big(t_\varepsilon^{-\frac{3}{2}} + t^{-\frac{3}{2}}(\varepsilon^2\sqrt{t})^{\frac{3}{2}\theta'}\big),
\end{aligned}
\tag{4.15}
$$

and where I_3 is a sum of terms

$$
I_3(t,x) = \sum_{j=-1}^{1} e^{ijt\sqrt{3}} M_j^3(t,x), \tag{4.16}
$$

where M_j^3 are odd and satisfy the following conditions: First, for any j with $|j| \le 1$, any α, N,

$$
\begin{aligned}
|\partial_\xi^\alpha \hat{M}_j^3(t,\xi)| &\le C_{\alpha,N} t_\varepsilon^{-1} t^{-\frac{1}{2}} \langle\xi\rangle^{-N}\\
|\partial_\xi^\alpha \partial_t \hat{M}_j^3(t,\xi)| &\le C_{\alpha,N} t_\varepsilon^{-1} t^{-\frac{3}{4}} \langle\xi\rangle^{-N}.
\end{aligned}
\tag{4.17}
$$

Moreover, for $j = 1$, and when ξ is a point in a small neighborhood $\mathcal{W}$ of the set $\{\xi : \sqrt{1+\xi^2} = \sqrt{3}\}$, one may find functions $\tilde{\Phi}_1(t,\xi)$, $\tilde{\Psi}_1(t,\xi)$, satisfying

$$
|\tilde{\Phi}_1(t,\xi)| \le C t_\varepsilon^{-1} t^{-\frac{1}{2}}, \quad |\tilde{\Psi}_1(t,\xi)| \le C t_\varepsilon^{-1} t^{-1} \tag{4.18}
$$

such that for $\xi \in \mathcal{W}$,

$$
D_t \hat{M}_1^3(t,\xi) = \big(D_t + (\sqrt{3} - \sqrt{1+\xi^2})\big)\tilde{\Phi}_1(t,\xi) + \tilde{\Psi}_1(t,\xi). \tag{4.19}
$$

A similar decomposition holds for xM_1^3 instead of M_1^3.

Finally, the remainder R in (4.11) satisfies for any $\alpha, N \in \mathbb{N}$,

$$
|\partial_x^\alpha R(t,x)| \le C_{\alpha,N} t^{-1} t_\varepsilon^{-1} \langle x\rangle^{-N} \tag{4.20}
$$

and we have for $M_j(t,x)$ in (4.12) the following explicit expressions:

$$
\begin{aligned}
M_2(t,x) &= \frac{1}{3} g(t)^2 Y_2(x),\\
M_0(t,x) &= \frac{2}{3} |g(t)|^2 Y_2(x),\\
M_{-2}(t,x) &= \frac{1}{3} \overline{g(t)}^2 Y_2(x),
\end{aligned}
\tag{4.21}
$$

where Y_2 is given by

$$
Y_2(x) = b(x, D_x)^*(\kappa(x) Y(x)^2) \in \mathcal{S}(\mathbb{R}). \tag{4.22}
$$

Moreover, the constants in all above inequalities depend only on A, A' in (4.3)–(4.4).

Proof. Consider first the contribution $F_0^2[a^{\mathrm{app}}]$ that is given according to (2.28), (4.9) and (4.22) by

$$
\frac{1}{3}\big(a_+^{\mathrm{app}} + \overline{a_+^{\mathrm{app}}}\big)^2 Y_2(x).
$$

We replace a_+^{app} by its expansion (4.8). We get terms of the following form (up to irrelevant multiplicative constants):

$$e^{it\sqrt{3}}g(t)^2Y_2, \quad |g(t)|^2Y_2, \quad e^{-it\sqrt{3}}\overline{g(t)}^2Y_2, \tag{4.23}$$

$$e^{i(2\ell-3)t\frac{\sqrt{3}}{2}}g(t)^\ell\overline{g(t)}^{3-\ell}Y_2, \quad 0\le\ell\le 3, \tag{4.24}$$

and

$$\begin{aligned}
&e^{it\sqrt{3}}g_2(t)\big(\varphi_+(0,t)-\varphi_-(0,t)+\overline{\varphi_+(\sqrt{3},t)}-\overline{\varphi_-(\sqrt{3},t)}\big)Y_2,\\
&g_0(t)\,\mathrm{Re}\big(\varphi_+(0,t)-\varphi_-(0,t)+\varphi_+(\sqrt{3},t)-\varphi_-(\sqrt{3},t)\big)Y_2,\\
&e^{-it\sqrt{3}}g_{-2}(t)\big(\overline{\varphi_+(0,t)}-\overline{\varphi_-(0,t)}+\varphi_+(\sqrt{3},t)-\varphi_-(\sqrt{3},t)\big)Y_2
\end{aligned} \tag{4.25}$$

with g_{2j}, $j=-1,0,1$ satisfying, according to (4.3), the bounds

$$|g_{2j}(t)|\le C(A)t_\varepsilon^{-1}, \quad |\partial_t g_{2j}(t)|\le C(A,A')t_\varepsilon^{-\frac{1}{2}}\big(t_\varepsilon^{-\frac{3}{2}}+t^{-\frac{3}{2}}(\varepsilon^2\sqrt{t})^{\frac{3}{2}\theta'}\big), \tag{4.26}$$

and expressions that are, according to conditions (4.3) and (4.6), $O(t_\varepsilon^{-\frac{3}{2}}t^{-\frac{1}{2}}\langle x\rangle^{-N})$ or $O(t_\varepsilon^{-1}t^{-1}\langle x\rangle^{-N})$ for any N, as well as their ∂_x derivatives, so that they will satisfy (4.20). Terms (4.23) give I_1 with actually the explicit expression (4.21) for M_2, M_0, M_{-2}. Terms (4.24) provide contributions to I_2 in (4.14).

To study terms in (4.25) that will provide I_3, let us define

$$\tilde\varphi_\pm(\lambda,t)=e^{-i\lambda t}\varphi_\pm(\lambda,t). \tag{4.27}$$

By (4.7), we have

$$D_t\tilde\varphi_\pm(\lambda,t)=\langle Z,\tilde u_\pm\rangle e^{-i\lambda t}+\psi_\pm(\lambda,t)e^{-i\lambda t}. \tag{4.28}$$

Then all contributions in (4.25) may be written under the form $e^{ijt\sqrt{3}}M_j^\pm(t,x)$, $j=-1,0,1$, with $M_j^\pm$ given by linear combinations of expressions

$$\begin{aligned}
&e^{it\sqrt{3}}g_{2\ell}(t)\tilde\varphi_\pm(\delta\sqrt{3},t)Y_2,\ \ell+\delta=1, 0\le\delta,\ell\le 1,\ \text{if } j=1\\
&g_{-2\ell}(t)\tilde\varphi_\pm(\ell\sqrt{3},t)Y_2,\ g_{2\ell}(t)\overline{\tilde\varphi_\pm(\ell\sqrt{3},t)}Y_2,\ \ell=0,1,\ \text{if } j=0\\
&e^{-it\sqrt{3}}g_{-2\ell}(t)\overline{\tilde\varphi_\pm(\delta\sqrt{3},t)}Y_2,\ \ell+\delta=1, 0\le\delta,\ell\le 1,\ \text{if } j=-1.
\end{aligned} \tag{4.29}$$

Since by (4.28), (4.6), (4.7), (4.4),

$$|D_t\tilde\varphi_\pm(\delta\sqrt{3},t)|\le Ct^{-\frac{3}{4}}(\varepsilon^2\sqrt{t})^{\theta'}$$

we deduce from (4.3) and (4.6) that (4.17) holds for M_j^3 which is a combination of M_j^+ and M_j^-, $-1\le j\le 1$. In the case $j=1$, we have to obtain (4.19), i.e. to find functions $\tilde\Phi_{1,\ell}^\pm$, $\tilde\Psi_{1,\ell}^\pm$, $\ell=0,1$ satisfying (4.18), such that if we define according to the first line in (4.29)

$$M_{1,\ell}^\pm(t,x)=g_{2\ell}(t)\tilde\varphi_\pm\big((1-\ell)\sqrt{3},t\big)Y_2(x), \tag{4.30}$$

for ξ in the neighborhood $\mathcal{W}$ of $\{-\sqrt{2}, \sqrt{2}\}$, we have

$$D_t \hat{M}^{\pm}_{1,\ell}(t,\xi) = \Big(D_t + \big(\sqrt{3} - \sqrt{1+\xi^2}\big)\Big)\tilde{\Phi}^{\pm}_{1,\ell}(t,\xi) + \tilde{\Psi}^{\pm}_{1,\ell}(t,\xi). \tag{4.31}$$

Let us apply (4.7) with λ replaced by $\lambda(\xi) = \sqrt{1+\xi^2} - \ell\sqrt{3}$ and $\xi \in \mathcal{W}$, so that $\lambda(\xi)$ remains close to $\mathbb{Z}\sqrt{3}$, and thus outside a neighborhood of $\{-1, 1\}$. We may then find functions $\varphi_{\pm}(\lambda(\xi), t)$, $\psi_{\pm}(\lambda(\xi), t)$ such that

$$\big(D_t - \sqrt{1+\xi^2} + \ell\sqrt{3}\big)\varphi_{\pm}(\lambda(\xi), t) = \langle Z, \tilde{u}_{\pm}\rangle + \psi_{\pm}(\lambda(\xi), t) \tag{4.32}$$

with estimates of the form

$$|\varphi_{\pm}(\lambda(\xi), t)| \le (\varepsilon^2\sqrt{t})^{\theta'} t^{-\frac{1}{2}}, \quad |\psi_{\pm}(\lambda(\xi), t)| \le (\varepsilon^2\sqrt{t})^{\theta'} t^{-1} \tag{4.33}$$

uniformly for ξ in $\mathcal{W}$. Define

$$\tilde{\Phi}^{\pm}_{1,\ell}(t,\xi) = \varphi_{\pm}(\lambda(\xi), t) e^{-it(1-\ell)\sqrt{3}} g_{2\ell}(t) \hat{Y}_2(\xi).$$

Then (4.33) implies that

$$\begin{aligned}
&\Big(D_t - \big(\sqrt{1+\xi^2} - \sqrt{3}\big)\Big)\tilde{\Phi}^{\pm}_{1,\ell}(t,\xi) \\
&\quad = \langle Z, \tilde{u}_{\pm}\rangle e^{-it(1-\ell)\sqrt{3}} g_{2\ell}(t)\hat{Y}_2(\xi) \\
&\qquad + \psi_{\pm}(\lambda(\xi), t) e^{-it(1-\ell)\sqrt{3}} g_{2\ell}(t)\hat{Y}_2(\xi) \\
&\qquad + \varphi_{\pm}(\lambda(\xi), t) e^{-it(1-\ell)\sqrt{3}} D_t g_{2\ell}(t)\hat{Y}_2(\xi).
\end{aligned} \tag{4.34}$$

On the other hand, (4.30), (4.28), (4.6) and (4.26) imply that

$$D_t \hat{M}^{\pm}_{1,\ell}(t,\xi) = \langle Z, \tilde{u}_{\pm}\rangle e^{-it(1-\ell)\sqrt{3}} g_{2\ell}(t)\hat{Y}_2(\xi) + R^{\pm}_{1,\ell}(t,\xi) \tag{4.35}$$

with

$$|\partial^{\alpha}_{\xi} R^{\pm}_{1,\ell}(t,\xi)| \le C t^{-1} t_{\varepsilon}^{-1} (\varepsilon^2\sqrt{t})^{\theta'} \langle\xi\rangle^{-N} \tag{4.36}$$

for any N. Making the difference between (4.34) and (4.35), and using (4.3) and (4.6), we obtain that (4.31) holds, with functions $\Phi^{\pm}_{1,\ell}$, $\Psi^{\pm}_{1,\ell}$ satisfying (4.18) since the last two terms in (4.34) and (4.36) are

$$O(t^{-1}t_{\varepsilon}^{-1} + t_{\varepsilon}^{-\frac{1}{2}} t^{-1} (\varepsilon^2\sqrt{t})^{\frac{3}{2}\theta'}) = O(t_{\varepsilon}^{-1} t^{-1})$$

for $t \le \varepsilon^{-4}$.

As $xM^{\pm}_{1,\ell}(t,x)$ is also of the form (4.30), with Y_2 replaced by xY_2, the same reasoning applies to that function and shows that (4.19) holds as well for xM^3_1 (with different functions $\tilde{\Phi}_1$, $\tilde{\Psi}_1$ on the right-hand side).

We have thus obtained that the first term $F^2_0[a^{\mathrm{app}}]$ in (4.11) has the wanted structure.

To study $F_0^3[a^{\text{app}}]$, we notice that by (2.28), (4.9), (4.8), it may be written as a linear combination of expressions of the form (4.24) (with Y_2 replaced by another function in $\mathcal{S}(\mathbb{R})$), that have been already treated, and of products of an $\mathcal{S}(\mathbb{R})$ function by expressions that are, by (4.3) and (4.6), $O(t_\varepsilon^{-1}t^{-1})$, so that form part of the remainder term (4.20). ∎

We may now state the main proposition of this section.

Proposition 4.1.2. *Assume that properties (4.3)–(4.7) hold. One may construct a function $u_+^{\text{app}} : [1,T] \times \mathbb{R} \to \mathbb{C}$ (where $T < \varepsilon^{-4}$ is the length of the interval on which a_+^{app} is defined by (4.8)), solving the equation*

$$\begin{aligned}\big(D_t - p(D_x)\big)u_+^{\text{app}} &= F_0^2(a^{\text{app}}) + F_0^3(a^{\text{app}}) \\ &\quad + a^{\text{app}} \sum_{|I|=1} \operatorname{Op}(m'_{1,I})(u_I^{\text{app}}) + R(t,x), \\ u_+^{\text{app}}|_{t=1} &= 0,\end{aligned} \tag{4.37}$$

where $m'_{1,I}$ is the symbol in the last sum of (3.13), where the remainder R satisfies bounds

$$|\partial_x^\alpha R(t,x)| \leq C_{\alpha,N} t_\varepsilon^{-1} t^{-1} \log(1+t)\langle x\rangle^{-N} \tag{4.38}$$

for any α, N in $\mathbb{N}$, with constants $C_{\alpha,N}(A,A')$ depending on the constants A, A' in (4.3), and where u_+^{app} has the following structure: One may decompose

$$u_+^{\text{app}} = u'^{\text{app}}_+ + u''^{\text{app}}_+,$$

where u'^{app}_+ satisfies for any $r \in \mathbb{N}$,

$$\|u'^{\text{app}}_+(t,\cdot)\|_{H^r} \leq C(A,A')\varepsilon^2 t^{\frac14}, \tag{4.39}$$

$$\|u'^{\text{app}}_+(t,\cdot)\|_{W^{r,\infty}} \leq C(A,A')\varepsilon^2, \tag{4.40}$$

$$\|L_+ u'^{\text{app}}_+(t,\cdot)\|_{H^r} \leq C(A,A') t^{\frac14}\big((\varepsilon^2\sqrt{t}) + (\varepsilon^2\sqrt{t})^{\frac78}\varepsilon^{\frac18}\big), \tag{4.41}$$

where

$$L_+ = x + tp'(D_x), \tag{4.42}$$

and where u''^{app}_+ satisfies for any r,

$$\|u''^{\text{app}}_+(t,\cdot)\|_{H^r} \leq C(A,A')\varepsilon\bigg(\frac{t\varepsilon^2}{\langle t\varepsilon^2\rangle}\bigg)^{\frac12}, \tag{4.43}$$

$$\|u''^{\text{app}}_+(t,\cdot)\|_{W^{r,\infty}} \leq C(A,A')\varepsilon^2 \log(1+t)^2, \tag{4.44}$$

$$\|L_+ u''^{\text{app}}_+(t,\cdot)\|_{W^{r,\infty}} \leq C(A,A') \log(1+t)\log(1+\varepsilon^2 t). \tag{4.45}$$

For the action of the half-Klein–Gordon operator on u'^{app}_+, we have estimates

$$\|(D_t - p(D_x))u'^{\text{app}}_+(t,\cdot)\|_{H^r} \leq C(A,A')\varepsilon^2 t^{-\frac34} \tag{4.46}$$

and

$$\|L_+(D_t - p(D_x))u'^{\text{app}}_+(t,\cdot)\|_{H^r} \leq C(A,A') t^{-\frac34}\big((\varepsilon^2\sqrt{t}) + (\varepsilon^2\sqrt{t})^{\frac78}\varepsilon^{\frac18}\big). \tag{4.47}$$

Moreover, we may write also another decomposition of u_+^{app}*, of the form*

$$u_+^{\mathrm{app}}(t,x) = u_+^{\mathrm{app},1}(t,x) + \Sigma_+(t,x), \tag{4.48}$$

where $u_+^{\mathrm{app},1}$ *is a sum*

$$u_+^{\mathrm{app},1}(t,x) = \sum_{j\in\{-2,0,2\}} U_{j,+}(t,x), \tag{4.49}$$

where $U_{j,+}$ *solves the equation*

$$\begin{aligned}\big(D_t - p(D_x)\big)U_{j,+} &= e^{itj\frac{\sqrt{3}}{2}} M_j(t,x),\\ U_{j,+}|_{t=1} &= 0,\end{aligned} \tag{4.50}$$

with source term M_j *given by* (4.21)*. The second contribution* Σ_+ *on the right-hand side of* (4.48) *may be also written as a sum*

$$\sum_{j=-3}^{3} \underline{U}_j(t,x),$$

with $\underline{U}_j$ *solving an equation of the form* (4.50)*, with source terms* $e^{ijt\frac{\sqrt{3}}{2}}\underline{M}_j(t,x)$*, where* $\underline{M}_j$ *satisfies for any* α, N*,*

$$|\partial_\xi^\alpha \hat{\underline{M}}_j(t,\xi)| \le C_{\alpha,N}(A,A')t_\varepsilon^{-1}t^{-\frac{1}{2}}\langle\xi\rangle^{-N} \tag{4.51}$$

and for any symbol m' *in the class* $\tilde{S}'_{0,0}(\langle\xi\rangle^{-1},1)$ *of Definition* 3.1.1*, one has for any* $\alpha, N \in \mathbb{N}$ *estimates*

$$|x^N\partial_x^\alpha \mathrm{Op}(m')(\Sigma_+(t,x))| \le C(A,A')\big(t_\varepsilon^{-\frac{3}{2}} + t^{-1}t_\varepsilon^{-\frac{1}{2}} + t^{-1}\varepsilon^2\big)\log(1+t). \tag{4.52}$$

In addition, all constants $C(A,A')$ *in the above inequality depend only on* A *and* A' *in* (4.3) *and* (4.4)*.*

Moreover, $u_+^{\mathrm{app},1}$ *may be decomposed as* $u_+^{\mathrm{app},1} = u'^{\mathrm{app},1}_+ + u''^{\mathrm{app},1}_+$*, with* $u'^{\mathrm{app},1}_+$ *(resp.* $u''^{\mathrm{app},1}_+$*) satisfying* (4.39)–(4.41) *and* (4.46)*,* (4.47) *(resp.* (4.43)–(4.45)*).*

Finally, all functions above are odd.

Proof. The proof of the proposition will be divided in several steps, and use the results of Appendix C below.

First step. We have decomposed in equation (4.11) the source term of (4.37), i.e. $F_0^2[a^{\mathrm{app}}] + F_0^3[a^{\mathrm{app}}]$. In this first step, we construct a first contribution $u_+^{\mathrm{app},1}$ to the solution of (4.37) taking as forcing term the contribution I_1 given by (4.12) to (4.11), i.e. we solve, with the notation (4.12)

$$\begin{aligned}\big(D_t - p(D_x)\big)u_+^{\mathrm{app},1} &= \sum_{j\in\{-2,0,2\}} e^{itj\frac{\sqrt{3}}{2}} M_j(t,x),\\ u_+^{\mathrm{app},1}|_{t=1} &= 0.\end{aligned} \tag{4.53}$$

The functions M_j on the right-hand side are given by (4.21), satisfy (4.13), and one may thus write $u_+^{\mathrm{app},1}$ under the form (4.49), with $U_{j,+}$ given as the solution of (4.50). We apply Appendix C. The solution of (4.50) is given by (C.3) with $\lambda = j\sqrt{3}/2$ and may be decomposed according to (C.4) in $U'_{j,+} + U''_{j,+}$. We define

$$u'^{\mathrm{app},1}_+ = \sum_{j\in\{-2,0,2\}} U'_{j,+}, \quad u''^{\mathrm{app},1}_+ = \sum_{j\in\{-2,0,2\}} U''_{j,+} \tag{4.54}$$

and check that they give contributions to $u'^{\mathrm{app}}_+, u''^{\mathrm{app}}_+$ that satisfy (4.39)–(4.41) and (4.43)–(4.45). By (4.13), the functions M_j on the right-hand side of (4.53) satisfy (C.7) with $\omega = 1$, i.e. Assumption $(\mathrm{H1})_1$ holds. By (i) of Proposition C.1.1, we thus get bounds of the form (4.39)–(4.41), and by (i) of Proposition C.1.2, we have (4.43)–(4.45). We shall define the contribution $u_+^{\mathrm{app},1}$ in (4.48) by

$$u_+^{\mathrm{app},1} = u'^{\mathrm{app},1}_+ + u''^{\mathrm{app},1}_+, \tag{4.55}$$

i.e. by the right-hand side of (4.49). Moreover, as M_j is odd in x, so are $U_{j,+}$, $U'_{j,+}$ and $U''_{j,+}$.

Second step. We consider now the term involving $\mathrm{Op}(m'_{1,I})$ on the right-hand side of (4.37), where we replace $u_\pm^{\mathrm{app}}$ by $u_\pm^{\mathrm{app},1}$ given by (4.49) (with $u_-^{\mathrm{app},1} = -\overline{u_+^{\mathrm{app},1}}$), i.e.

$$a^{\mathrm{app}}(t) \sum_{|I|=1} \sum_{j\in\{-2,0,2\}} \mathrm{Op}(m'_{I,I})(U_{j,I}) \tag{4.56}$$

with $U_{j,-} = -\overline{U}_{j,+}$. Recall that we decomposed $U_{j,+} = U'_{j,+} + U''_{j,+}$ according to (C.4). Let us examine first the contribution coming from $\mathrm{Op}(m'_{1,I})(U''_{j,I})$ to (4.56). The symbol $m'_{1,I}$ lies in $\tilde{S}'_{1,0}(\langle\xi\rangle^{-1}M_0^{\nu}, 1)$, which is contained in $\tilde{S}'_{0,0}(1,1)$ (recall that $M_0 \equiv 1$ when there is only one ξ variable), and it satisfies (3.7). Since $U''_{j,+}$ is defined by (C.4) with $\lambda = j\sqrt{3}/2$ from some odd M_j, we may apply Proposition C.2.1, with M_j satisfying Assumption $(\mathrm{H1})_1$, i.e. (C.7) with $\omega = 1$ according to (4.13). We shall thus get from (C.89)

$$\mathrm{Op}(m'_{1,+})(U''_{j,+}) = e^{ijt\frac{\sqrt{3}}{2}} M^{(1)}_{j,+}(t,x) + r_+(t,x) \tag{4.57}$$

with for any α, N, by (C.91),

$$|\partial_x^\alpha r(t,x)| \le C_{\alpha,N}\varepsilon^2 t^{-1}\log(1+t)\langle x\rangle^{-N} \tag{4.58}$$

and where $M^{(1)}_{j,+}$ satisfies by (C.90)

$$\begin{aligned} |\partial_x^\alpha M^{(1)}_{j,+}(t,x)| &\le C_{\alpha,N} t_\varepsilon^{-1}\langle x\rangle^{-N}, \\ |\partial_x^\alpha \partial_t M^{(1)}_{j,+}(t,x)| &\le C_{\alpha,N} t_\varepsilon^{-\frac{1}{2}}\big(t_\varepsilon^{-\frac{3}{2}} + t^{-\frac{3}{2}}(\varepsilon^2\sqrt{t})^{\frac{3}{2}\theta'}\big)\langle x\rangle^{-N}. \end{aligned} \tag{4.59}$$

By conjugation, we shall have also

$$\mathrm{Op}(m'_{1,+})(U''_{j,-}) = e^{-ijt\frac{\sqrt{3}}{2}} M^{(1)}_{j,-}(t,x) + r_-(t,x) \tag{4.60}$$

with $M_{j,-}^{(1)}$ (resp. r_-) satisfying also (4.59) (resp. (4.58)). We plug (4.57) and (4.60) in (4.56) and use the expression (4.8)–(4.9) of a^{app}. We get that (4.56) is a sum of quantities of the following form:

- Terms of the form

$$e^{ij't\frac{\sqrt{3}}{2}} M_{j'}^{(1)}(t,x), \quad j' = -3,-1,1,3, \tag{4.61}$$

  coming from the product of the first term in (4.8) (or its conjugate) and of the $M_{j,\pm}^{(1)}$ terms in (4.57) and (4.60). One gets thus smooth odd functions of x, that satisfy by (4.59) and (4.3) estimates

$$\begin{aligned} |\partial_x^\alpha M_{j'}^{(1)}(t,x)| &\le C_{\alpha,N} t_\varepsilon^{-\frac{3}{2}} \langle x\rangle^{-N}, \\ |\partial_x^\alpha \partial_t M_{j'}^{(1)}(t,x)| &\le C_{\alpha,N} t_\varepsilon^{-1}\big(t_\varepsilon^{-\frac{3}{2}} + t^{-\frac{3}{2}}(\varepsilon^2\sqrt{t})^{\frac{3}{2}\theta'}\big)\langle x\rangle^{-N}. \end{aligned} \tag{4.62}$$

- Terms satisfying (4.38) and thus contributing to R in (4.37). These terms come from the product of (4.57) or (4.60) with all terms on the right-hand side of (4.8), except $e^{it\sqrt{3}/2} g(t)$ (and its conjugate), and from the product of a^{app} with $r_\pm$ in (4.57) and (4.60). As

$$\varepsilon^2 t^{-1} t_\varepsilon^{-\frac{1}{2}} \le C t^{-1} t_\varepsilon^{-1}$$

  if $t \le \varepsilon^{-4}$, we do get that these terms satisfy (4.38).

- Terms of the form

$$a^{\mathrm{app}}(t) \sum_{|I|=1} \sum_{j\in\{-2,0,2\}} \mathrm{Op}(m'_{1,I})(U'_{j,I}), \tag{4.63}$$

  where $U'_{j,I}$ is given by (C.4) in terms of M_j satisfying Assumption $(\mathrm{H1})_\omega$ with $\omega = 1$. We shall see in fifth step below that (4.63) satisfies also (4.38) and thus contributes to R.

It follows thus from (4.53) and the fact that (4.56) is given by (4.61) up to remainders, that

$$\big(D_t - p(D_x)\big)u_+^{\mathrm{app},1} - a^{\mathrm{app}}(t) \sum_{|I|=1} \mathrm{Op}(m'_{1,I})(u_I^{\mathrm{app},1}) = I_1 - I_2^{(1)} + R(t,x), \tag{4.64}$$

where I_1 is given by (4.12), $I_2^{(1)}$ is the sum of terms (4.61) and R satisfies (4.38). Making the difference between (4.37) and (4.64), we get, taking (4.11) into account

$$\begin{aligned} &\big(D_t - p(D_x)\big)(u_+^{\mathrm{app}} - u_+^{\mathrm{app},1}) \\ &\quad = I_2 + I_3 + I_2^{(1)} + a^{\mathrm{app}}(t) \sum_{|I|=1} \mathrm{Op}(m'_{1,I})(u_I^{\mathrm{app}} - u_I^{\mathrm{app},1}) + R(t,x), \end{aligned} \tag{4.65}$$

with R satisfying (4.38). Notice that by (4.62), $I_2^{(1)}$ has the same form as I_2 given by (4.14) and (4.15) so that we shall be able to treat both terms altogether.

Third step. We now construct an approximate solution in order to eliminate $I_2+I_2^{(1)}$ on the right-hand side of (4.65). Define $u_+^{\mathrm{app},2}$ as the solution to the linear equation

$$\begin{aligned}\big(D_t - p(D_x)\big)u_+^{\mathrm{app},2} &= I_2 + I_2^{(1)},\\ u_+^{\mathrm{app},2}|_{t=1} &= 0.\end{aligned} \tag{4.66}$$

As the right-hand side has structure (4.14) with M_j satisfying (4.15), we may express the solution as a sum $\sum_{j\in\{-3,-1,1,3\}} U_{j,+}(t,x)$, where $U_{j,+}$ is obtained from the j-th term in (4.14) and expressed under form (C.3) with $\lambda = j\sqrt{3}/2$. By (C.4),

$$U_{j,+} = U'_{j,+} + U''_{j,+}$$

and since (4.15) shows that (C.7) holds with $\omega = 3/2$, Assumption $(\mathrm{H1})_{3/2}$ holds. By Proposition C.1.1, bounds (C.18)–(C.20) with $\omega = 3/2$ hold for $U'_{j,+}$, and by Proposition C.1.2, (C.24), (C.25) and (C.27) are true. If we set

$$u'^{\,\mathrm{app},2}_+ = \sum_{j\in\{-3,-1,1,3\}} U'_{j,+},\quad u''^{\,\mathrm{app},2}_+ = \sum_{j\in\{-3,-1,1,3\}} U''_{j,+}, \tag{4.67}$$

this shows that these functions provide to $u'^{\mathrm{app}}_+, u''^{\mathrm{app}}_+$ contributions satisfying estimates (4.39)–(4.41) and (4.43)–(4.45).

Let us study

$$a^{\mathrm{app}}(t)\sum_{|I|=1}\mathrm{Op}(m'_{1,I})(u_I^{\mathrm{app},2}). \tag{4.68}$$

If we apply Proposition C.2.1, using that Assumption $(\mathrm{H1})_{3/2}$ holds, we get from (C.89), (C.90), (C.91) and the fact that $a^{\mathrm{app}}(t)$ is $O(t_\varepsilon^{-1/2})$, that the contribution of $u''^{\,\mathrm{app},2}_+$ to (4.68) is $O(t_\varepsilon^{-1}t^{-1}\langle x\rangle^{-N})$, i.e. may be included in R satisfying (4.38). On the other hand, if we replace in (4.68) $u_+^{\mathrm{app},2}$ by $u'^{\,\mathrm{app},2}_+$, we shall get terms of the form (4.63), with $U'_{j,I}$ given by (C.4) in terms of M_j satisfying Assumption $(\mathrm{H1})_\omega$ with $\omega = \frac{3}{2}$. These terms are thus better than those in (4.63) and the fact that they fulfill remainder estimates (4.38) will be seen in Step 5 below.

Consequently, we have shown that

$$\big(D_t - p(D_x)\big)u_+^{\mathrm{app},2} - a^{\mathrm{app}}(t)\sum_{|I|=1}\mathrm{Op}(m'_{1,I})(u_I^{\mathrm{app},2}) = I_2 + I_2^{(1)} + R(t,x) \tag{4.69}$$

with R satisfying (4.38). Making the difference between (4.65) and (4.69), we get

$$\begin{aligned}&\big(D_t - p(D_x)\big)\big(u_+^{\mathrm{app}} - u_+^{\mathrm{app},1} - u_+^{\mathrm{app},2}\big)\\ &\quad = I_3 + a^{\mathrm{app}}(t)\Big(\sum_{|I|=1}\mathrm{Op}(m'_{1,I})(u_I^{\mathrm{app}} - u_I^{\mathrm{app},1} - u_I^{\mathrm{app},2})\Big) + R(t,x).\end{aligned} \tag{4.70}$$

Fourth step. We construct an approximate solution in order to eliminate I_3 in (4.70), i.e. we solve

$$\begin{aligned}\big(D_t - p(D_x)\big)u_+^{\mathrm{app},3} &= I_3,\\ u_+^{\mathrm{app},3}|_{t=1} &= 0\end{aligned} \tag{4.71}$$

with I_3 given by equation (4.16). For each contribution $e^{ijt\sqrt{3}}M_j^3(t,x)$ to (4.16), with $-1 \leq j \leq 1$, we get an equation of the form (C.2) with $\lambda = j\sqrt{3}$. Moreover, by (4.17)–(4.19) assumptions (C.8)–(C.10) hold (the last two ones being empty if $\lambda = j\sqrt{3}$ with $j = 0$ or -1), i.e. Assumption (H2) of section (C.2) holds. We may thus apply (ii) of Proposition C.1.1 and Proposition C.1.2 that allow to write $u_+^{\mathrm{app},3}$ as a sum

$$u_+^{\mathrm{app},3} = \sum_{j=-1}^{1} U_{j,+}(t,x), \quad U_{j,+} = U'_{j,+} + U''_{j,+} \tag{4.72}$$

with $U'_{j,+}$ satisfying (C.21)–(C.23) and $U''_{j,+}$ satisfying (C.28)–(C.30). If we now set $u_+^{\mathrm{app},3} = u'^{\,\mathrm{app},3}_+ + u''^{\,\mathrm{app},3}_+$ with

$$u'^{\,\mathrm{app},3}_+ = \sum_{j=-1}^{1} U'_{j,+}(t,x), \quad u''^{\,\mathrm{app},3}_+ = \sum_{j=-1}^{1} U''_{j,+}(t,x), \tag{4.73}$$

it follows that (4.39)–(4.41) and (4.43)–(4.45) hold true. Let us check that

$$a^{\mathrm{app}}(t) \sum_{|I|=1} \mathrm{Op}(m'_{1,I})(u_+^{\mathrm{app},3}) \tag{4.74}$$

is a remainder satisfying (4.38). Since we are here under Assumption (H2), we shall apply Proposition C.2.4 splitting each $U_{j,+}$ in (4.72) as

$$U_{j,+} = U'_{j,+,1} + U''_{j,+,1} \tag{4.75}$$

according to (C.110). Then by (C.111), and the fact that $a^{\mathrm{app}} = O(t_\varepsilon^{-\frac{1}{2}})$, the contribution coming from $U''_{j,+,1}$ obeys remainder estimates (4.38), so that (4.74) may be written as a contribution to R in (4.37) and as

$$a^{\mathrm{app}}(t) \sum_{|I|=1} \mathrm{Op}(m'_{1,I})(u'^{\mathrm{app},3}_{+,1}) \tag{4.76}$$

with

$$u'^{\mathrm{app},3}_{+,1} = \sum_{j=-1}^{1} U'_{j,+,1}(t,x). \tag{4.77}$$

We shall see in Step 5 below that (4.76) provides also a contribution to R. Consequently, we have obtained that

$$\big(D_t - p(D_x)\big)u_+^{\mathrm{app},3} - a^{\mathrm{app}}(t) \sum_{|I|=1} \mathrm{Op}(m'_{1,I})(u_I^{\mathrm{app},3}) = I_3 + R(t,x).$$

Making the difference with (4.70), we conclude that u_+^{app} will solve (4.37) if and only if

$$\begin{aligned}\big(D_t - p(D_x)\big)&\Big(u_+^{\mathrm{app}} - \sum_{\ell=1}^{3} u_+^{\mathrm{app},\ell}\Big)\\ &- a^{\mathrm{app}}(t) \sum_{|I|=1} \mathrm{Op}(m'_{1,I})\Big(u_I^{\mathrm{app}} - \sum_{\ell=1}^{3} u_I^{\mathrm{app},\ell}\Big) = R(t,x).\end{aligned}$$

Consequently, we just have to take $u_+^{\mathrm{app}} = u_+^{\mathrm{app},1} + u_+^{\mathrm{app},2} + u_+^{\mathrm{app},3}$. We have checked that then estimates (4.39)–(4.41) and (4.43)–(4.45) hold. It remains to check that terms of the form (4.63) and (4.76) provide remainders, and that estimates (4.46)–(4.47) hold true, as well as the properties of the decomposition (4.48). This will be done in the following steps.

Fifth step. Let us show that (4.63) and (4.76) are remainders. Let us use the same notation $U'_{j,+}$ for either $U'_{j,+}$ in (4.63) or $U'_{j,+,1}$ in (4.77). Notice that since the functions M_j in (4.12), (4.14), (4.16) are odd in x, so are the $U'_{j,+}$ defined from them. Moreover, as $m'_{1,I}$ is in $\tilde{S}'_{1,0}(\langle\xi\rangle^{-1}, 1)$, we may write

$$\mathrm{Op}(m'_{1,\pm})(U'_{j,\pm}) = \mathrm{Op}(\tilde{m}_{1,\pm})(\langle D_x\rangle^{-1} U'_{j,\pm}) \tag{4.78}$$

with $\tilde{m}'_{1,I}$ in $\tilde{S}'_{1,0}(1,1)$. By oddness of $U'_{j,+}$

$$\begin{aligned} \langle D_x\rangle^{-1} U'_{j,+} &= \frac{ix}{2}\int_{-1}^{1}\Big(\frac{D_x}{\langle D_x\rangle}U'_{j,+}\Big)(t,\mu x)\,d\mu \\ &= \frac{ix}{2t}\int_{-1}^{1}\big((L_+U'_{j,+})(t,\mu x) - \mu x U'_{j,+}(t,\mu x)\big)\,d\mu. \end{aligned} \tag{4.79}$$

As $\tilde{m}_{1,I}$ has rapidly decaying coefficients in x, we rewrite (4.78) as a linear combination of expressions

$$\frac{1}{t}\mathrm{Op}(\hat{m}'_{1,I})\Big(\int_{-1}^{1}(L_\pm^k U'_{j,\pm})(t,\mu x)\mu^{1-k}\,d\mu\Big), \quad k = 0, 1, \tag{4.80}$$

for new symbols $\hat{m}'_{1,I}$ in the class $\tilde{S}'_{1,0}(1,1)$. Using (C.92) with $\omega = 1$ or (C.112), we bound any L^∞ norm of $x^\beta\partial_x^\alpha$ acting on (4.80) by $C\varepsilon^2 t^{-1}$. Taking into account that $a^{\mathrm{app}}(t)$ is $O(t_\varepsilon^{-1/2})$, we see that (4.63) and (4.76) satisfy (4.38) (using again $t \le \varepsilon^{-4}$).

Sixth step. We shall prove estimates (4.46) and (4.47). Recall that by definition

$$u'^{\mathrm{app}}_+ = u'^{\ \mathrm{app},1}_+ + u'^{\ \mathrm{app},2}_+ + u'^{\ \mathrm{app},3}_+$$

with $u'^{\ \mathrm{app},1}_+$ given by (4.54), $u'^{\ \mathrm{app},2}_+$ given by (4.67) and $u'^{\ \mathrm{app},3}_+$ given by (4.73). Consequently, the term $(D_t - p(D_x))u'^{\mathrm{app}}_+$ is a sum of expressions $(D_t - p(D_x))U'_{j,+}$, where $U'_{j,+}$ is given by an integral of the form (C.4) (resp. (C.110)) with M replaced by an M_j satisfying either (4.13) (for those coming from (4.54)) or (4.15) (for those coming from (4.67)) (resp. satisfying (4.17) for those coming from (4.73)). Consequently, for contributions of the form (C.4),

$$\big(D_t - p(D_x)\big)U'_{j,+} = -\frac{1}{2t}\int_1^{+\infty} e^{i(t-\tau)p(D_x)+i\lambda_j\tau}\tilde{\chi}\Big(\frac{\tau}{\sqrt{t}}\Big)M_j(\tau,\cdot)\,d\tau, \tag{4.81}$$

where $\tilde{\chi}(\tau) = \tau\chi'(\tau)$ and λ_j is some integer multiple of $\frac{\sqrt{3}}{2}$. In other words, we obtain still an expression of the form of the first line in (C.4), but with a gain of a factor t^{-1}. Estimates (4.39) and (4.41) that we have already obtained for u'^{app}_+ furnish thus (4.46) and (4.47) multiplying them by t^{-1} (the change of cut-off $\tilde{\chi}$ does not matter, as it has support contained in the one of χ). This shows also that (4.46) and (4.47) hold for $u'^{\mathrm{app},1} + u'^{\mathrm{app},2}$. The case of $u'^{\mathrm{app},3}$ is similar, using (C.110) to get an expression of the form (4.81), but with $\tilde{\chi}(\frac{\tau}{\sqrt{t}})$ replaced by $\tilde{\chi}(\frac{\tau}{t})$, i.e. again an integral of form (C.110) with the gain of a pre-factor t^{-1}.

Seventh step. We have to establish still (4.48). The contribution $u_+^{\mathrm{app},1}$ on the right-hand side is the one that has been defined in the first step by (4.53), with right-hand side given in terms of M_j defined in (4.21). The term Σ_+ in (4.48) is thus given by $u_+^{\mathrm{app},2} + u_+^{\mathrm{app},3}$ introduced in (4.67) and (4.72). These functions are constructed as sums of contributions $\underline{U}_j$ that satisfy equations of the form (4.50), where the source term satisfies (4.15) or (4.17) and thus (4.51). It remains to show (4.52). As m' has rapidly decaying coefficients in x, we may forget the x^N factor in (4.52), and are thus reduced to the study of $\partial_x^\alpha \mathrm{Op}(m')(u_+^{\mathrm{app},2})$ and $\partial_x^\alpha \mathrm{Op}(m')(u_+^{\mathrm{app},3})$.

Consider first $\partial_x^\alpha \mathrm{Op}(m')(u_+^{\mathrm{app},2})$. By (4.67), we express that from

$$\partial_x^\alpha \mathrm{Op}(m')(U'_{j,+}),\ \partial_x^\alpha \mathrm{Op}(m')(U''_{j,+}). \tag{4.82}$$

As Assumption (H1)$_\omega$ holds with $\omega = \frac{3}{2}$, according to (4.15), the second term above is given by (C.89) of Proposition C.2.1. It follows from (C.90) and (C.91) that its modulus is smaller than

$$t_\varepsilon^{-\frac{3}{2}} + \varepsilon^3 t^{-1}\log(1+t),$$

so than the right-hand side of (4.52). On the other hand, $\mathrm{Op}(m')(U'_{j,+})$ has been expressed in fifth step under the form (4.80). If we plug there estimates (C.92), we see that the modulus of the first term in (4.82) is $O(\varepsilon^3 t^{-1})$, so better than the right-hand side of (4.52).

Consider next $\partial_x^\alpha \mathrm{Op}(m')(u_+^{\mathrm{app},3})$. Solving (4.71), we have written $u_+^{\mathrm{app},3}$ under the form $\sum_{j=-1}^{1}(U'_{j,+,1} + U''_{j,+,1})$ according to (4.75). If we plug this decomposition in $\partial_x^\alpha \mathrm{Op}(m')(\cdot)$, we get on the one hand expressions of the form (C.111), that are bounded by the right-hand side of (4.52). For the contribution $\partial_x^\alpha \mathrm{Op}(m')(U'_{j,+,1})$, we use again that we can write an expression of the form (4.80) and bounds (C.112). We get an estimate in $O(\varepsilon^2 t^{-1})$ that is better than the right-hand side of (4.52). This concludes the proof. ∎

To conclude this section, let us compute some integrals that will be useful in the sequel.

Proposition 4.1.3. *Let Y_2 be the function defined in (4.22). The functions $U_{j,+}$, $j = -2, 0, 2$, on the right-hand side of (4.49) satisfy the following:*

$$\int U_{2,+}(t,x)p(D_x)^{-1}Y_2\,dx = (\alpha_2 + i\beta_2)e^{it\sqrt{3}}g(t)^2 + r(t), \tag{4.83}$$

where α_2 is real,

$$\beta_2 = -\frac{\sqrt{2}}{6}\hat{Y}_2(\sqrt{2})^2 \tag{4.84}$$

for the function Y_2 defined in (2.6), and where $r(t)$ satisfies

$$|r(t)| \leq C(A, A')\Big(\varepsilon^2 t^{-\frac{3}{2}} + t_\varepsilon^{-2} + \varepsilon t^{-\frac{3}{2}}(\varepsilon^2\sqrt{t})^{\frac{3}{2}\theta'}\Big) \leq C(A, A')t_\varepsilon^{-1}. \tag{4.85}$$

Moreover,

$$\int U_{0,+}(t,x)p(D_x)^{-1}Y_2\,dx = \alpha_0|g(t)|^2 + r(t), \tag{4.86}$$

$$\int U_{2,-}(t,x)p(D_x)^{-1}Y_2\,dx = \alpha_{-2}\overline{g(t)}^2 e^{-it\sqrt{3}} + r(t), \tag{4.87}$$

where α_0, α_{-2} are real constants, and where r satisfies (4.85). Finally, the function Σ_+ in (4.48) satisfies

$$\begin{aligned}&\Big|\int \Sigma_+(t,x)p(D_x)^{-1}Y_2\,dx\Big|\\ &\quad\leq C(A, A')\big(t_\varepsilon^{-\frac{3}{2}} + \varepsilon^2 t_\varepsilon^{-1} + t^{-1}t_\varepsilon^{-\frac{1}{2}}\big)\log(1+t).\end{aligned} \tag{4.88}$$

Proof. Let us establish (4.83). The function $U_{2,+}$ is defined as the solution of (4.50) with $j = 2$ and M_2 on the right-hand side given by (4.21). We write (4.83) as

$$\frac{1}{2\pi}\int \hat{U}_{2,+}(t,\xi)p(\xi)^{-1}\hat{Y}_2(-\xi)\,d\xi.$$

Since Y_2 is odd, we get from equation (C.124) applied with $\hat{Z}(\xi) = -p(\xi)^{-1}\hat{Y}_2(\xi)$, $\hat{M}(t,\xi) = \hat{M}_2(t,\xi)$, $\lambda = \sqrt{3}$, a contribution to r and two integral terms. By (4.21), the second one is

$$-\frac{e^{it\sqrt{3}}}{6\pi}\int \frac{(1-\chi_{\sqrt{3}})(\xi)}{\sqrt{3}-\sqrt{1+\xi^2}}\frac{\hat{Y}_2(\xi)^2}{\sqrt{1+\xi^2}}\,d\xi g(t)^2 \tag{4.89}$$

which may be written since Y_2 is real and odd, under the form $\alpha_2' e^{it\sqrt{3}}g(t)^2$ for some real α_2'.

Using the definition (C.123) of χ_λ, and the fact that $\hat{Y}_2(\xi)^2$ is even, the first term on the right-hand side of (C.124) brings the contribution

$$\begin{aligned}-\frac{i}{3\pi}e^{it\sqrt{3}}g(t)^2\lim_{\sigma\to 0+}\int_0^{+\infty}\int e^{i\tau(\sqrt{1+\xi^2}-\sqrt{3})-\sigma\tau}\chi(\xi-\sqrt{2})&\\ \times\frac{\hat{Y}_2(\xi)^2}{\sqrt{1+\xi^2}}\,d\xi\,d\tau.&\end{aligned} \tag{4.90}$$

Denote by $\xi(\zeta)$ the reciprocal of the change of variables $\xi \mapsto \zeta = \sqrt{3}-\sqrt{1+\xi^2}$ defined from a neighborhood of $\xi = \sqrt{2}$ to a neighborhood of $\zeta = 0$. We rewrite (4.90) as

$$-\frac{i}{3\pi}e^{it\sqrt{3}}g(t)^2\lim_{\sigma\to 0+}\int_0^{+\infty}\int e^{-i\tau\zeta-\sigma\tau}\chi(\xi(\zeta)-\sqrt{2})\hat{Y}_2(\xi(\zeta))^2\frac{d\zeta}{|\xi(\zeta)|}\,d\tau. \tag{4.91}$$

Notice that

$$\lim_{\sigma\to 0+}\int_0^{+\infty} e^{-i\tau\zeta-\sigma\tau}\,d\tau = -i(\zeta-i0)^{-1} = \pi\delta_0 - i\,\text{p.v.}\,\frac{1}{\zeta}.$$

Plugging in (4.91), we obtain an expression $\alpha_2' + i\beta_2$ with α_2' real and β_2 given by (4.84).

To obtain (4.86) and (4.87), we apply again Proposition C.3.1 but with $\lambda = 0$ or $\lambda = -\sqrt{3}$ so that $\chi_\lambda = 0$ and in (C.124) the first term on the right-hand side disappears. Only the second one and r remain, so that one gets no imaginary contribution to (4.86) and (4.87).

Finally, let us prove (4.88). As Y_2 is in $\mathcal{S}(\mathbb{R})$, the integral may be expressed as an integral of $\text{Op}(m')(\Sigma_+)$ for the symbol $m' = Y_2(x)p(\xi)^{-1}$, so that (4.52) brings the conclusion. ■

4.2 Asymptotic analysis of the ODE

In this section, we shall prove that solutions of the ordinary differential equation (2.34) have a certain asymptotic expansion by a bootstrap argument.

We make some a priori assumptions on the functions Φ_j and Γ_j on the right-hand side of (2.34).

Assumption ($\mathbf{H}_1'$). Assume that u_+ is a solution to equation (2.27) defined on the set $[1,T]\times\mathbb{R}$ for some $T \le \varepsilon^{-4}$ such that the functions Φ_2 and Γ_j, $j = 1,2,3$, defined on (2.36) satisfy the inequality

$$\begin{aligned}|\Phi_2(u_+(t,\cdot),u_-(t,\cdot))| + \sum_{j=1}^{3} t_\varepsilon^{-\frac{3}{2}+\frac{j}{2}} |\Gamma_j(u_+(t,\cdot),u_-(t,\cdot))| \\ \le B' t^{-\frac{3}{2}}(\varepsilon^2\sqrt{t})^{2\theta'}\end{aligned} \tag{4.92}$$

for some constant B', some $\theta' \in]0,\frac{1}{2}[$ (close to $\frac{1}{2}$), all $t \in [1,T]$, and assume that the function Φ_1 given by (2.36) satisfies for any $t \in [1,T]$,

$$\begin{aligned}\Big|\Phi_1(u_+(t,\cdot),u_-(t,\cdot)) - \frac{\sqrt{3}}{3}\langle Y, Y\kappa(x)b(x,D_x)p(D_x)^{-1}\big(u_+^{\text{app}} - u_-^{\text{app}}\big)\rangle \\ - \big(\langle Z,\tilde{u}_+\rangle - \langle Z,\tilde{u}_-\rangle\big)\Big| \le B' t^{-\frac{3}{2}}(\varepsilon^2\sqrt{t})^{2\theta'},\end{aligned} \tag{4.93}$$

where u_+^{app} is the approximate solution constructed in Section 4.1, Z is a function in $\mathcal{S}(\mathbb{R})$, $\tilde{u}_\pm$ are functions verifying inequality (4.4) such that for any λ in $\mathbb{R}-\{-1,1\}$, one may find functions $\varphi_\pm(\lambda,t)$ and $\psi_\pm(\lambda,t)$ as in (4.5), solving equation (4.7) and such that estimates (4.6) hold true, for λ outside a given neighborhood $\widetilde{\mathcal{W}}$ of $\{-1,1\}$ in $\mathbb{R}$.

We consider on the interval $[1,T]$ the solution a_+ of equation (2.34), namely

$$\Big(D_t-\frac{\sqrt{3}}{2}\Big)a_+=\sum_{j=0}^{2}(a_+-a_-)^{2-j}\Phi_j[u_+,u_-]+\sum_{j=0}^{3}(a_+-a_-)^{3-j}\Gamma_j[u_+,u_-] \tag{4.94}$$

with an initial condition at $t=1$ satisfying

$$|a_+(1)|\leq A_0\varepsilon \tag{4.95}$$

for some constant A_0. We introduce as a second assumption an estimate on a_+, that we give in terms of upper bounds (4.99) below:

Assumption ($\mathbf{H}_2'$). The solution of equation (4.94) with initial condition (4.95) exists on some interval $[1,T]$ with $T\leq\varepsilon^{-4}$ and satisfies on that interval the following requirements: One may write

$$a_+(t)=a_+^{\mathrm{app}}(t)+S(t), \tag{4.96}$$

where $a_+^{\mathrm{app}}(t)$ has the structure

$$\begin{aligned}a_+^{\mathrm{app}}(t)&=e^{it\frac{\sqrt{3}}{2}}g(t)+\omega_2g(t)^2e^{it\sqrt{3}}+\omega_0|g(t)|^2+\omega_{-2}\overline{g(t)}^2e^{-it\sqrt{3}}\\&\quad+e^{it\frac{\sqrt{3}}{2}}g(t)\big(\varphi_+(0,t)-\varphi_-(0,t)\big)\\&\quad+e^{-it\frac{\sqrt{3}}{2}}\overline{g(t)}\big(\varphi_+(\sqrt{3},t)-\varphi_-(\sqrt{3},t)\big)\end{aligned} \tag{4.97}$$

and where

$$S(t)=\omega_3g(t)^3e^{3it\frac{\sqrt{3}}{2}}+\omega_{-1}|g(t)|^2\overline{g(t)}e^{-it\frac{\sqrt{3}}{2}}+\omega_{-3}\overline{g(t)}^3e^{-3it\frac{\sqrt{3}}{2}} \tag{4.98}$$

with the following notation:

- The coefficients ω_j in (4.97) (resp. (4.98)) are real (resp. complex) constants that will be chosen below.
- The function g satisfies, for some constants A, A' and $t\in[1,T]$,

  $$|g(t)|\leq At_\varepsilon^{-\frac{1}{2}},\ |\partial_tg(t)|\leq A'(t_\varepsilon^{-\frac{3}{2}}+t^{-\frac{3}{2}}(\varepsilon^2\sqrt{t})^{\frac{3}{2}\theta'}), \tag{4.99}$$

  where $\theta'\in\,]0,\frac{1}{2}[$ is close to $\frac{1}{2}$ and has been introduced in (H_1').
- The functions $\varphi_\pm(0,t)$, $\varphi_\pm(\sqrt{3},t)$ satisfy conditions (4.5)–(4.7) with Z and $\tilde{u}_\pm$ introduced in (4.93), i.e. one has estimates

  $$\begin{aligned}|\varphi_\pm(\lambda,t)|&\leq(\varepsilon^2\sqrt{t})^{\theta'}t^{-\frac{1}{2}},\quad|\psi_\pm(\lambda,t)|\leq(\varepsilon^2\sqrt{t})^{\theta'}t^{-1}\\|\langle Z,\tilde{u}_\pm(t,\cdot)\rangle|&\leq(\varepsilon^2\sqrt{t})^{\theta'}t^{-\frac{3}{4}}\end{aligned} \tag{4.100}$$

  (when ε is small enough) and one has the equation

  $$(D_t-\lambda)\varphi_\pm(\lambda,t)=\langle Z,\tilde{u}_\pm(t,\cdot)\rangle+\psi_\pm(\lambda,t) \tag{4.101}$$

  for $\lambda=0$ or $\sqrt{3}$.

We shall bootstrap Assumption (H$'_2$), i.e. estimates (4.99) assuming that (H$'_1$) holds:

Proposition 4.2.1. *Let $c \in]0,1[$ and $\theta' \in]0, \frac{1}{2}[$, θ' close to $\frac{1}{2}$. There are constants $A, A', \varepsilon_0 > 0$ such that if Assumption (H$'_1$) holds and if the solution a_+ of (4.94) exists on $[1,T]$ and has structure (4.96) with g satisfying (4.99) on $[1,T]$, then if $\varepsilon \in]0, \varepsilon_0[$, $T \leq \varepsilon^{-4+c}$, one has actually, for any $t \in [1,T]$,*

$$|g(t)| \leq \frac{1}{2} A t_\varepsilon^{-\frac{1}{2}}, \quad |\partial_t g(t)| \leq \frac{1}{2} A' \Big(t_\varepsilon^{-\frac{3}{2}} + t^{-\frac{3}{2}} (\varepsilon^2 \sqrt{t})^{\frac{3}{2}\theta'} \Big). \tag{4.102}$$

As a first step towards the proof of the proposition, let us rewrite equation (4.94).

Lemma 4.2.2. *There are a real constant γ_1 and complex constants $\gamma_3, \gamma_{-1}, \gamma_{-3}$ such that, under the assumptions of the proposition,*

$$\begin{aligned}\Big(D_t - \frac{\sqrt{3}}{2}\Big) a_+ = {}& e^{it\frac{\sqrt{3}}{2}} |g(t)|^2 g(t) \Big(\gamma_1 - i \frac{\sqrt{6}}{18} \hat{Y}_2 (\sqrt{2})^2 \Big) \\ & + e^{3it\frac{\sqrt{3}}{2}} g(t)^3 \gamma_3 + e^{-it\frac{\sqrt{3}}{2}} |g(t)|^2 \overline{g(t)} \gamma_{-1} \\ & + e^{-3it\frac{\sqrt{3}}{2}} \overline{g(t)}^3 \gamma_{-3} \\ & + (a_+ - a_-)^2 \Phi_0 + (a_+ - a_-)^3 \Gamma_0 \\ & + (a_+ - a_-)(\langle Z, \tilde{u}_+ \rangle - \langle Z, \tilde{u}_- \rangle) + r(t),\end{aligned} \tag{4.103}$$

where $r(t)$ satisfies

$$|r(t)| \leq C(A, A', B') t^{-\frac{3}{2}} (\varepsilon^2 \sqrt{t})^{2\theta'} \tag{4.104}$$

for a constant depending only on the constants A, A', B' of (4.99), (4.92), (4.93).

Proof. Consider the right-hand side of equation (4.94). By (4.92), the Φ_2 contribution is bounded by $B' t^{-\frac{3}{2}} (\varepsilon^2 \sqrt{t})^{2\theta'}$, so satisfies (4.104). By (4.96), (4.97), (4.99), (4.100)

$$|a_+(t)| + |a_-(t)| \leq C(A) t_\varepsilon^{-\frac{1}{2}} \tag{4.105}$$

so that (4.92) implies that the contributions $(a_+ - a_-)^{3-j} \Gamma_j$, $j = 1,2,3$, to (4.94) satisfy (4.104). We are thus left with studying

$$\Phi_0 (a_+ - a_-)^2 + \Phi_1 [u_+, u_-](a_+ - a_-) + \Gamma_0 (a_+ - a_-)^3. \tag{4.106}$$

The first and last terms in (4.106) are present on the right-hand side of (4.103). Consider $(a_+ - a_-)\Phi_1$. By (4.93), up to another contribution to r, we get on the one hand the last but one term on the right-hand side of (4.103) and the quantity

$$\frac{\sqrt{3}}{3} (a_+ - a_-) \langle Y, Y\kappa(x) b(x, D_x) p(D_x)^{-1} (u_+^{\text{app}} - u_-^{\text{app}}) \rangle$$

that, according to the definition (4.22) of Y_2, may be written

$$\frac{\sqrt{3}}{3} (a_+ - a_-) \langle Y_2, p(D_x)^{-1} (u_+^{\text{app}} - u_-^{\text{app}}) \rangle. \tag{4.107}$$

We replace above u_+^{app} by expansion (4.48). According to (4.88),

$$|\langle Y_2, p(D_x)^{-1}\Sigma_+\rangle| \le C(A,A')\big(t_\varepsilon^{-\frac{3}{2}} + t^{-1}\varepsilon^2 + t^{-1}t_\varepsilon^{-\frac{1}{2}}\big)\log(1+t).$$

If we use also (4.105) and (4.1), we conclude, since

$$t_\varepsilon^{-2} \le Ct^{-\frac{3}{2}}(\varepsilon^2\sqrt{t}), \quad t_\varepsilon^{-\frac{1}{2}}t^{-1}\varepsilon^2 \le Ct^{-\frac{3}{2}}(\varepsilon^2\sqrt{t}), \quad t^{-1}t_\varepsilon^{-1} \le Ct^{-\frac{3}{2}}(\varepsilon^2\sqrt{t}),$$

that (4.107) satisfies inequality (4.104) (if we absorb the logarithm using that we assume $\varepsilon^2\sqrt{t} \le \varepsilon^{\frac{c}{2}}$, $\theta' < \frac{1}{2}$, and that we take ε small). We are thus left with the contribution to (4.107) of

$$\frac{\sqrt{3}}{3}(a_+ - a_-)\langle Y_2, p(D_x)^{-1}(u_+^{\text{app},1} - u_-^{\text{app},1})\rangle \tag{4.108}$$

with $u_+^{\text{app},1}$ given by (4.49). The bracket above has been computed in (4.83), (4.86) and (4.87). It is in particular $O(C(A,A')t_\varepsilon^{-1})$. By equations (4.96)–(4.100) the difference $a_+ - e^{it\sqrt{3}/2}g$ is bounded by $C(A)(t_\varepsilon^{-1} + t_\varepsilon^{-1/2}t^{-1/2}(\varepsilon^2\sqrt{t})^{\theta'})$, so that if we replace in (4.108) a_+ by $e^{it\sqrt{3}/2}g$, we get an error bounded by

$$C(A,A')\Big(t_\varepsilon^{-2} + t_\varepsilon^{-\frac{3}{2}}t^{-\frac{1}{2}}(\varepsilon^2\sqrt{t})^{\theta'}\Big) \le C(A,A')t^{-\frac{3}{2}}(\varepsilon^2\sqrt{t})^{2\theta'}, \tag{4.109}$$

so that we get a remainder. Consequently, using again (4.49), we have reduced (4.108) to

$$\frac{\sqrt{3}}{3}\Big(g(t)e^{it\frac{\sqrt{3}}{2}} + \overline{g(t)}e^{-it\frac{\sqrt{3}}{2}}\Big)\Big[\sum_{j\in\{-2,0,2\}}\langle Y_2, p(D_x)^{-1}(U_{j,+} + \overline{U}_{j,+})\rangle\Big] \tag{4.110}$$

up to remainders. We have computed the bracket above in (4.83), (4.86) and (4.87). Up to terms bounded by the product of (4.85) with $t_\varepsilon^{-1/2}$, which still provides remainders satisfying (4.104), we get that (4.110) is given by

$$e^{3it\frac{\sqrt{3}}{2}}\gamma_3 g(t)^3 + e^{it\frac{\sqrt{3}}{2}}\tilde{\gamma}_1|g(t)|^2 g(t) + e^{-it\frac{\sqrt{3}}{2}}\gamma_{-1}|g(t)|^2\overline{g(t)} + e^{-3it\frac{\sqrt{3}}{2}}\gamma_{-3}\overline{g(t)}^3,$$

where γ_j are complex constants, with $\tilde{\gamma}_1 = \frac{\sqrt{3}}{3}(2\alpha_0 + \alpha_2 + \alpha_{-2} + i\beta_2)$, where α_0, α_2, α_{-2} are real and β_2 is given by (4.84). We obtain thus the first four terms on the right-hand side of (4.103). This concludes the proof. ∎

We shall next compute from expression (4.96) of a_+ and from (4.103) an equation satisfied by g.

Lemma 4.2.3. *One may choose the coefficients ω_j, $-3 \le j \le 3$, $j \ne 1$, in (4.97) and (4.98) such that if a_+ is given by (4.96) and satisfies (4.103), then g solves*

$$D_t g(t) = \Big(\alpha - i\frac{\sqrt{6}}{18}\hat{Y}_2(\sqrt{2})^2\Big)|g(t)|^2 g(t) + r_1(t), \tag{4.111}$$

where α is real, $\hat{Y}_2(\sqrt{2})^2$ is negative and $r_1(t)$ satisfies

$$\begin{aligned}|r_1(t)| \le C(A)t_\varepsilon^{-\frac{1}{2}}t^{-1}(\varepsilon^2\sqrt{t})^{\theta'} + C(A,A',B')\Big(t_\varepsilon^{-2} + t_\varepsilon^{-1}t^{-1}(\varepsilon^2\sqrt{t})^{\theta'} \\ + t^{-\frac{3}{2}}(\varepsilon^2\sqrt{t})^{2\theta'} + t_\varepsilon^{-\frac{1}{2}}t^{-\frac{3}{2}}(\varepsilon^2\sqrt{t})^{\frac{3}{2}\theta'} + t^{-2}(\varepsilon^2\sqrt{t})^{\frac{5}{2}\theta'}\Big),\end{aligned} \tag{4.112}$$

where $C(\cdot)$ are constants depending only on the indicated quantities.

Proof. Let us express in a more explicit way the right-hand side of (4.103). By equations (4.96)–(4.100),

$$\begin{aligned}\Big|a_+(t) - \Big(e^{it\frac{\sqrt{3}}{2}}g(t) + \omega_2 g(t)^2 e^{it\sqrt{3}} + \omega_0|g(t)|^2 + \omega_{-2}\overline{g(t)}^2 e^{-it\sqrt{3}}\Big)\Big| \\ \le C(A)t_\varepsilon^{-\frac{1}{2}}t^{-\frac{1}{2}}(\varepsilon^2\sqrt{t})^{\theta'} + C(A)t_\varepsilon^{-\frac{3}{2}}\end{aligned} \tag{4.113}$$

for constants $C(A)$ depending only on A.

It follows that

$$\begin{aligned}(a_+(t) - a_-(t))^2 = {}& e^{it\sqrt{3}}g(t)^2 + 2|g(t)|^2 + e^{-it\sqrt{3}}\overline{g(t)}^2 \\ & + 2e^{3it\frac{\sqrt{3}}{2}}g(t)^3(\omega_2 + \omega_{-2}) \\ & + 2e^{it\frac{\sqrt{3}}{2}}|g(t)|^2 g(t)(2\omega_0 + \omega_2 + \omega_{-2}) \\ & + 2e^{-it\frac{\sqrt{3}}{2}}|g(t)|^2\overline{g(t)}(2\omega_0 + \omega_2 + \omega_{-2}) \\ & + 2e^{-3it\frac{\sqrt{3}}{2}}\overline{g(t)}^3(\omega_2 + \omega_{-2}) + r(t),\end{aligned} \tag{4.114}$$

where r satisfies (4.112).

In the same way

$$\begin{aligned}(a_+(t) - a_-(t))^3 = {}& e^{3it\frac{\sqrt{3}}{2}}g(t)^3 + 3e^{it\frac{\sqrt{3}}{2}}|g(t)|^2 g(t) \\ & + 3e^{-it\frac{\sqrt{3}}{2}}|g(t)|^2\overline{g(t)} + e^{-3it\frac{\sqrt{3}}{2}}\overline{g(t)}^3 + r(t)\end{aligned} \tag{4.115}$$

where r satisfies (4.112). We plug (4.114)–(4.115) in the right-hand side of (4.103). We get, as Φ_0, Γ_0 given by (2.35) are real constants, the expression

$$\begin{aligned}& e^{it\sqrt{3}}\Phi_0 g(t)^2 + 2|g(t)|^2\Phi_0 + e^{-it\sqrt{3}}\Phi_0\overline{g(t)}^2 \\ & \quad + e^{it\frac{\sqrt{3}}{2}}|g(t)|^2 g(t)\Big(\underline{\gamma}_1 - i\frac{\sqrt{6}}{18}\hat{Y}_2(\sqrt{2})^2\Big) \\ & \quad + e^{3it\frac{\sqrt{3}}{2}}g(t)^3\underline{\gamma}_3 + e^{-it\frac{\sqrt{3}}{2}}|g(t)|^2\overline{g(t)}\underline{\gamma}_{-1} + e^{-3it\frac{\sqrt{3}}{2}}\overline{g(t)}^3\underline{\gamma}_{-3} \\ & \quad + e^{it\frac{\sqrt{3}}{2}}g(t)(\langle Z,\tilde{u}_+\rangle - \langle Z,\tilde{u}_-\rangle) \\ & \quad + e^{-it\frac{\sqrt{3}}{2}}\overline{g(t)}(\langle Z,\tilde{u}_+\rangle - \langle Z,\tilde{u}_-\rangle) + r(t),\end{aligned} \tag{4.116}$$

where $\underline{\gamma}_j$, $j = -3,-1,1,3$, are new constants with $\underline{\gamma}_1$ real, $\underline{\gamma}_{-3}, \underline{\gamma}_{-1}, \underline{\gamma}_3$ depending on $\omega_{-2}, \omega_0, \omega_2$ but not on $\omega_{-3}, \omega_{-1}, \omega_3$, and where $r(t)$ satisfies (4.112), and contains in particular the product of $\langle Z, \tilde{u}_\pm \rangle$ with $a_+(t) - e^{it\sqrt{3}/2}g(t)$, $a_-(t) + e^{it\sqrt{3}/2}\overline{g(t)}$, according to estimates (4.113) and (4.100).

On the other hand, we may compute the left-hand side of (4.103) replacing a_+ by its expression (4.96). We get, using (4.101) with $\lambda = 0$ or $\sqrt{3}$,

$$\begin{aligned}\Big(D_t - \frac{\sqrt{3}}{2}\Big)a_+ = e^{it\frac{\sqrt{3}}{2}} D_t g &+ \frac{\sqrt{3}}{2} e^{it\sqrt{3}} \omega_2 g(t)^2 - \frac{\sqrt{3}}{2}\omega_0 |g(t)|^2 \\ &- 3\frac{\sqrt{3}}{2}\omega_{-2} e^{-it\sqrt{3}}\overline{g(t)}^2 + \sqrt{3}\omega_3 e^{3it\frac{\sqrt{3}}{2}} g(t)^3 \\ &- \sqrt{3}\omega_{-1} e^{-it\frac{\sqrt{3}}{2}} |g(t)|^2 \overline{g(t)} \\ &- 2\sqrt{3}\omega_{-3} e^{-3it\frac{\sqrt{3}}{2}} \overline{g(t)}^3 \\ &+ e^{it\frac{\sqrt{3}}{2}} g(t)(\langle Z, \tilde{u}_+\rangle - \langle Z, \tilde{u}_-\rangle) \\ &+ e^{-it\frac{\sqrt{3}}{2}} \overline{g(t)}(\langle Z, \tilde{u}_+\rangle - \langle Z, \tilde{u}_-\rangle) + r_1(t),\end{aligned} \tag{4.117}$$

where $r_1(t)$ is made of terms of the form

$$\begin{aligned}&O(|gD_t g|), &&O(|D_t g \varphi_\pm(0,t)|), &&O(|D_t g \varphi_\pm(\sqrt{3},t)|), \\ &O(|g\psi_\pm(0,t)|), &&O(|g\psi_\pm(\sqrt{3},t)|), &&O(|g^2 D_t g|).\end{aligned} \tag{4.118}$$

By a priori estimate (4.99) and (4.100), these terms are bounded by

$$\begin{aligned}&C(A, A')\Big(t_\varepsilon^{-2} + t_\varepsilon^{-\frac{1}{2}} t^{-\frac{3}{2}} (\varepsilon^2\sqrt{t})^{\frac{3}{2}\theta'} + t^{-\frac{1}{2}} t_\varepsilon^{-\frac{3}{2}} (\varepsilon^2\sqrt{t})^{\theta'} + t^{-2}(\varepsilon^2\sqrt{t})^{\frac{5}{2}\theta'}\Big) \\ &\quad + C(A) t_\varepsilon^{-\frac{1}{2}} t^{-1} (\varepsilon^2\sqrt{t})^{\theta'},\end{aligned} \tag{4.119}$$

the last contribution coming from the first two terms in the second line of (4.118). We choose now the free parameters ω_j, $j \in \{-3, \dots, 3\} - \{1\}$ setting

$$\begin{aligned}\omega_3 &= \frac{\sqrt{3}}{3}\underline{\gamma}_3, &\omega_2 &= \frac{2\sqrt{3}}{3}\Phi_0, &\omega_0 &= -\frac{4\sqrt{3}}{3}\Phi_0, \\ \omega_{-1} &= -\frac{\sqrt{3}}{3}\underline{\gamma}_{-1}, &\omega_{-2} &= -\frac{2\sqrt{3}}{9}\Phi_0, &\omega_{-3} &= -\frac{\sqrt{3}}{6}\underline{\gamma}_{-3}\end{aligned}$$

(which is possible as $\underline{\gamma}_{-3}, \underline{\gamma}_{-1}, \underline{\gamma}_3$ do not depend on $\omega_{-3}, \omega_{-1}, \omega_3$). In that way, when we make the difference between the two expressions (4.116) and (4.117) of $(D_t - \frac{\sqrt{3}}{2})$, we obtain equation (4.111) with a remainder satisfying (4.119). This concludes the proof, as $\hat{Y}_2(\sqrt{2})$ being purely imaginary (since Y_2 is real and odd), $\hat{Y}_2(\sqrt{2})^2 \leq 0$ and moreover, by Proposition G.1.2, $\hat{Y}_2(\sqrt{2}) \neq 0$. ■

Proof of Proposition 4.2.1. Let us show first that under the assumptions of the proposition, the first inequality of (4.102) holds if A has been chosen large enough, ε small enough and $t \le \varepsilon^{-4+c}$. In a first step, consider the case when $\varepsilon^2 t$ is small, i.e. let us show that there is $\tau_0 \in]0,1]$ such that if $1 \le t \le \frac{\tau_0}{\varepsilon^2}$, and ε is small enough,

$$|g(t)| \le \frac{A}{4} t_\varepsilon^{-\frac{1}{2}}. \tag{4.120}$$

Since for these t one has $\frac{\varepsilon^2}{2} \le t_\varepsilon^{-1} \le \varepsilon^2$, the a priori bound (4.99), equation (4.111) and estimates (4.112) imply that, for any such t,

$$|g(t)| \le |g(1)| + KA^3\varepsilon^3 t + C(A, A', B')(\varepsilon^{1+\theta'} + \varepsilon^{4\theta'}),$$

where $K = |\alpha - i\frac{\sqrt{6}}{18}\hat{Y}_2(\sqrt{2})^2|$ and $C(\cdot)$ is a new constant depending on A, A', B' (and τ_0). If A is taken such that

$$|g(1)| \le \frac{A}{8}\frac{\varepsilon}{\sqrt{2}},$$

and τ_0 small enough so that

$$KA^2\tau_0 < \frac{1}{16\sqrt{2}},$$

and if we take ε small enough, we get, using that θ' is close to $\frac{1}{2}$, that

$$|g(t)| \le \frac{A}{4\sqrt{2}}\varepsilon \le \frac{A}{4} t_\varepsilon^{-\frac{1}{2}},$$

i.e. (4.120).

We shall thus study from now on equation (4.111) for $t \ge \frac{\tau_0}{\varepsilon^2}$ and initial condition at $\frac{\tau_0}{\varepsilon^2}$ bounded by $\frac{A}{4\sqrt{2}}\varepsilon$. In this regime, for some new constant $C(A, A', B')$, (4.112) implies

$$|r_1(t)| \le C(A, A', B')\big(t^{-\frac{3}{2}}(\varepsilon^2\sqrt{t})^{\theta'} + t^{-2}\big), \tag{4.121}$$

remembering that t stays in $[\tau_0\varepsilon^{-2}, \varepsilon^{-4+c}]$. For t in $[\tau_0, \varepsilon^{-2+c}]$, set

$$e(t) = \varepsilon^{-1}(1+t)^{\frac{1}{2}} g\Big(\frac{t}{\varepsilon^2}\Big). \tag{4.122}$$

We deduce from (4.111) and (4.121) that if $\beta = -\frac{\sqrt{6}}{18}\hat{Y}_2(\sqrt{2})^2 > 0$,

$$\partial_t e(t) = \frac{1}{2}\frac{e(t)}{1+t} + \frac{-\beta + i\alpha}{1+t}|e(t)|^2 e(t) + R(t), \tag{4.123}$$

where

$$\begin{aligned} |R(t)| &\le C(A, A', B')\Big(\frac{(1+t)^{\frac{1}{2}}}{t^{\frac{3}{2}}}(\varepsilon\sqrt{t})^{\theta'} + \varepsilon\frac{(1+t)^{\frac{1}{2}}}{t^2}\Big) \\ &\le \frac{C(A, A', B')}{1+t}(1+\tau_0^{-1})^{\frac{3}{2}}\big(\varepsilon^{\frac{\theta'}{2}c} + \varepsilon\tau_0^{-\frac{1}{2}}\big). \end{aligned} \tag{4.124}$$

Denote $w(t) = |e(t)|^2$. Then

$$\partial_t w(t) = \frac{1}{1+t}(w(t) - 2\beta w(t)^2 + Q(t)), \tag{4.125}$$

where according to (4.124), for $t \in [\tau_0, \varepsilon^{-2+c}]$,

$$|Q(t)| \leq C\left(\varepsilon^{\frac{\theta'}{2}c} + \varepsilon\tau_0^{-\frac{1}{2}}\right)|w(t)|^{\frac{1}{2}} \tag{4.126}$$

for some constant depending on A, A', B', τ_0. Moreover, we have

$$w(\tau_0) \leq \left(\frac{A}{4}\right)^2. \tag{4.127}$$

We fix A large enough so that $(\frac{A}{2})^2 - 2\beta(\frac{A}{2})^4 \leq -\frac{A}{2}$ and then take $\varepsilon < \varepsilon_0$ small enough (in function of A, A', B', τ_0) such that (4.126) implies $|Q(t)| \leq \frac{1}{2}|w(t)|^{1/2}$. Then it follows that if, at some time t_*, $w(t_*)$ reaches $(\frac{A}{2})^2$, the right-hand side of (4.125) is strictly negative. Consequently, taking (4.127) into account, we get $w(t) \leq (\frac{A}{2})^2$ for any t in $[\tau_0, \varepsilon^{-2+c}]$. Using (4.122), we conclude that

$$|g(t)| \leq \frac{A}{2} t_\varepsilon^{-\frac{1}{2}}$$

for t in $[\frac{\tau_0}{\varepsilon^2}, \varepsilon^{-4+c}]$. This gives the first inequality of (4.102).

To get the second one, we notice that we may bound the right-hand side of (4.112) by

$$\begin{aligned} &C(A)\left(t_\varepsilon^{-\frac{3}{2}} + t^{-\frac{3}{2}}(\varepsilon^2\sqrt{t})^{\frac{3}{2}\theta'}\right) \\ &\qquad + C(A, A', B')\left(\varepsilon + (\varepsilon^2\sqrt{t})^{\frac{\theta'}{2}}\right)\left(t_\varepsilon^{-\frac{3}{2}} + t^{-\frac{3}{2}}(\varepsilon^2\sqrt{t})^{\frac{3}{2}\theta'}\right) \end{aligned}$$

for new constants $C(A), C(A, A', B')$, depending only on the indicated arguments. Plugging this in (4.111), we get

$$|\partial_t g(t)| \leq K|g(t)|^3 + (C(A) + C(A, A', B')e(t, \varepsilon))\left(t_\varepsilon^{-\frac{3}{2}} + t^{-\frac{3}{2}}(\varepsilon^2\sqrt{t})^{\frac{3}{2}\theta'}\right)$$

with

$$\lim_{\varepsilon \to 0+} \sup_{t \in [1, \varepsilon^{-4+c}]} e(t, \varepsilon) = 0.$$

If we plug there the first inequality of (4.102), choose A' large enough relatively to A, so that

$$K\left(\frac{A}{2}\right)^3 + C(A) \leq \frac{A'}{4}$$

and then take ε small enough relatively to A, A', B', we get the second inequality of (4.102). This concludes the proof. ■