Chapter 5

Reduced form of dispersive equation

In Section 3.2, we performed a quadratic normal form on equation (3.11) satisfied by u_+ in order to get equation (3.13). On the other hand, in Section 4.1, we constructed some approximate solution solving equation (4.37). Making the difference between (3.13) and (4.37), we shall get an equation for the action of $D_t - p(D_x)$ on

$$\tilde{u}_+ = u_+ - \sum_{|I|=2} \mathrm{Op}(\tilde{m}_{0,I})(u_I) - u_+^{\mathrm{app}}.$$

The goal of this chapter is to invert in convenient spaces the map $u_+ \mapsto \tilde{u}_+$, to obtain an expression for u_+ in terms of $\tilde{u}_+$ and to write down the equation satisfied by $\tilde{u}_+$ in closed form.

5.1 A fixed point theorem

We establish first some abstract theorem. We consider E, F two Banach spaces with norms $\|\cdot\|_E, \|\cdot\|_F$. We consider also two other normed spaces $\tilde{E}, \tilde{F}$ such that $E \cap \tilde{E}$ (resp. $F \cap \tilde{F}$) is also a Banach space. We set $B_F(r), B_E(r)$ for the closed ball of center zero, radius r in F, E. We assume given a function

$$\begin{aligned} \Phi : (E \cap F) \times (E \cap F) &\to E \cap F, \\ (u'', f) &\mapsto \Phi(u'', f) \end{aligned} \tag{5.1}$$

satisfying the following estimates: There are $C > 0, \sigma > 0$ such that for any parameter $\lambda \geq 1$, any u'', f, f_1, f_2 in $E \cap F$, one has

$$\|\Phi(u'', f)\|_E \leq C\big(\|u''\|_F + \|f\|_F\big)\big(\|u''\|_E + \|f\|_E\big), \tag{5.2}$$

$$\begin{aligned} \|\Phi(u'', f)\|_F \leq{}& C\lambda^\sigma\big(\|u''\|_F + \|f\|_F\big)^2 \\ &+ C\lambda^{-1}\big(\|u''\|_F + \|f\|_F\big)\big(\|u''\|_E + \|f\|_E\big), \end{aligned} \tag{5.3}$$

$$\begin{aligned} &\|\Phi(u'', f_1) - \Phi(u'', f_2)\|_E \\ &\quad \leq C\big(\|u''\|_F + \|f_1\|_F + \|f_2\|_F\big)\|f_1 - f_2\|_E \\ &\qquad + C\big(\|u''\|_E + \|f_1\|_E + \|f_2\|_E\big)\|f_1 - f_2\|_F, \end{aligned} \tag{5.4}$$

$$\begin{aligned} &\|\Phi(u'', f_1) - \Phi(u'', f_2)\|_F \\ &\quad \leq C\big(\lambda^\sigma\big(\|u''\|_F + \|f_1\|_F + \|f_2\|_F\big) \\ &\qquad + \lambda^{-1}\big(\|u''\|_E + \|f_1\|_E + \|f_2\|_E\big)\big)\|f_1 - f_2\|_F \\ &\qquad + C\lambda^{-1}\big(\|u''\|_F + \|f_1\|_F + \|f_2\|_F\big)\big)\|f_1 - f_2\|_E. \end{aligned} \tag{5.5}$$

We assume also that if, in addition to preceding assumptions, u'' is in $\tilde{F}$ and f is in $\tilde{E}$, then $\Phi(u'', f)$ is in $\tilde{E}$, with estimate

$$\|\Phi(u'', f)\|_{\tilde{E}} \leq C(\|u''\|_{\tilde{F}}\|u''\|_E + \big(\|u''\|_F + \|f\|_F\big)\|f\|_{\tilde{E}}\big) \tag{5.6}$$

and if f_1, f_2 are in $\tilde{E}$,

$$\|\Phi(u'', f_1) - \Phi(u'', f_2)\|_{\tilde{E}} \leq C\big(\|u''\|_F + \|f_1\|_F + \|f_2\|_F\big)\|f_1 - f_2\|_{\tilde{E}}. \tag{5.7}$$

Lemma 5.1.1. *There is $r_0 > 0$ such that for any r in $]0, r_0[$, any $\lambda \geq 1$, any $u', u'', \tilde{u}$ in $B_E(r\lambda) \cap B_F(r\lambda^{-\sigma})$, the fixed point problem*

$$f = u' + \tilde{u} + \Phi(u'', f) \tag{5.8}$$

has a unique solution f in $B_E(3r\lambda) \cap B_F(3r\lambda^{-\sigma})$. Moreover, if one defines inductively

$$\begin{aligned} \Phi^1(u'', a, g) &= a + \Phi(u'', g), \\ \Phi^{n+1}(u'', a, g) &= \Phi^n(u'', a, \Phi^1(u'', a, g)) = \Phi^1(u'', a, \Phi^n(u'', a, g)), \end{aligned} \tag{5.9}$$

and if one sets

$$\mathcal{E}_\lambda = \lambda^\sigma\big(\|u''\|_F + \|u'\|_F + \|\tilde{u}\|_F\big) + \lambda^{-1}\big(\|u''\|_E + \|u'\|_E + \|\tilde{u}\|_E\big),$$

one has for any $N \geq 1$ and a new constant $C > 0$,

$$\begin{aligned} &\|f - \Phi^N(u'', u' + \tilde{u}, u')\|_E \\ &\quad\leq C^{N+1}\mathcal{E}_\lambda^N\|f - u'\|_E \\ &\qquad\quad + C^{N+1}\mathcal{E}_\lambda^{N-1}\big(\|u''\|_E + \|u'\|_E + \|\tilde{u}\|_E\big)\|f - u'\|_F, \\ &\|f - \Phi^N(u'', u' + \tilde{u}, u')\|_F \\ &\quad\leq C^{N+1}\mathcal{E}_\lambda^N\|f - u'\|_F + C^{N+1}\mathcal{E}_\lambda^N\lambda^{-1}\|f - u'\|_E. \end{aligned} \tag{5.10}$$

Furthermore, if one assumes that $u', \tilde{u}$ are also in $\tilde{E}$ and u'' is also in $\tilde{F}$, then f is in $\tilde{E}$ and one has for any $N \geq 1$,

$$\|f - \Phi^N(u'', u' + \tilde{u}, u')\|_{\tilde{E}} \leq C^N\big(\|u'\|_F + \|\tilde{u}\|_F + \|u''\|_F\big)^N\|f - u'\|_{\tilde{E}}. \tag{5.11}$$

Proof. We define the usual sequence of approximations

$$\begin{aligned} f_{N+1} &= \Phi^{N+1}(u'', u' + \tilde{u}, u') = u' + \tilde{u} + \Phi(u'', f_N), \\ f_0 &= 0 \end{aligned}$$

using notation (5.9). By (5.2) and (5.3), we have

$$\|f_{N+1}\|_E \leq \|u'\|_E + \|\tilde{u}\|_E + C\big(\|u''\|_F + \|f_N\|_F\big)\big(\|u''\|_E + \|f_N\|_E\big)$$

and

$$\|f_{N+1}\|_F \le \|u'\|_F + \|\tilde{u}\|_F + C\big(\lambda^\sigma\big(\|u''\|_F + \|f_N\|_F\big) + \lambda^{-1}\big(\|u''\|_E + \|f_N\|_E\big)\big)\big(\|u''\|_F + \|f_N\|_F\big).$$

It follows that if $u', u'', \tilde{u}$ are in $B_F(r\lambda^{-\sigma}) \cap B_E(\lambda r)$ with r small enough, one has for any N,

$$\|f_{N+1}\|_E \le \frac{4}{3}\big(\|u'\|_E + \|\tilde{u}\|_E\big) + \frac{1}{3}\|u''\|_E,$$
$$\|f_{N+1}\|_F \le \frac{4}{3}\big(\|u'\|_F + \|\tilde{u}\|_F\big) + \frac{1}{3}\|u''\|_F.$$

In particular, $(f_N)_N$ remains bounded in $B_F(3r\lambda^{-\sigma}) \cap B_E(3\lambda r)$. Moreover, by (5.4) and (5.5) and the above bounds, for r small enough, $(f_N)_N$ converges in $E \cap F$ to a limit f satisfying

$$f = u' + \tilde{u} + \Phi(u'', f) = \Phi^1(u'', u' + \tilde{u}, f).$$

Then (5.10) with $N = 1$ follows from (5.4) and (5.5). One obtains the general case by induction, using (5.4) and (5.5). In the same way, (5.11) follows from (5.7). ■

We shall apply the preceding lemma with $E = H^s(\mathbb{R})$, $F = W^{\rho,\infty}(\mathbb{R})$, $s > 0$, $\lambda = t \ge 1$, $\rho \in \mathbb{N}$. We define the spaces $\tilde{E}$, $\tilde{F}$ by

$$\tilde{E} = \{f \in L^2(\mathbb{R}) : xf \in L^2(\mathbb{R})\}, \quad \tilde{F} = \{f \in W^{\rho,\infty}(\mathbb{R}) : xf \in W^{\rho,\infty}(\mathbb{R})\} \tag{5.12}$$

and we endow them with norms depending on the parameter t:

$$\|f\|_{\tilde{E}} = t\|f\|_{L^2} + \|xf\|_{L^2}, \quad \|f\|_{\tilde{F}} = t\|f\|_{W^{\rho,\infty}} + \|xf\|_{W^{\rho,\infty}}.$$

The functions u', u'' of (5.8) will be the functions $u'^{\text{app}}_+, u''^{\text{app}}_+$ of Proposition 4.1.2. By (4.39)–(4.41) applied with a large enough r, and using (4.42), we get

$$\begin{aligned} &\|u'^{\text{app}}_+(t,\cdot)\|_E \le C(A, A')\varepsilon^2 t^{\frac{1}{4}},\\ &\|u'^{\text{app}}_+(t,\cdot)\|_F \le C(A, A')\varepsilon^2,\\ &\|u'^{\text{app}}_+(t,\cdot)\|_{\tilde{E}} \le C(A, A')(\varepsilon^2 t^{\frac{5}{4}} + t^{\frac{1}{4}}(\varepsilon^2\sqrt{t})^{\frac{7}{8}}\varepsilon^{\frac{1}{8}}). \end{aligned} \tag{5.13}$$

In particular, for ε small, $t^\sigma\|u'^{\text{app}}_+(t,\cdot)\|_F + t^{-1}\|u'^{\text{app}}_+(t,\cdot)\|_E$ may be made as small as we want (uniformly in $t \le \varepsilon^{-4}$) if $\varepsilon > 0$ is small enough. In the same way, by (4.43)–(4.45)

$$\begin{aligned} &\|u''^{\text{app}}_+(t,\cdot)\|_E \le C(A, A')\varepsilon,\\ &\|u''^{\text{app}}_+(t,\cdot)\|_F \le C(A, A')\varepsilon^2(\log(1+t))^2,\\ &\|u''^{\text{app}}_+(t,\cdot)\|_{\tilde{F}} \le C(A, A')t\varepsilon^2(\log(1+t))^2. \end{aligned} \tag{5.14}$$

Again, for $t \le \varepsilon^{-4}$, we see that $t^\sigma\|u''^{\text{app}}_+(t,\cdot)\|_F + t^{-1}\|u''^{\text{app}}_+(t,\cdot)\|_E$ may be made as small as we want for $\varepsilon > 0$ small.

We shall take some function $\tilde{u}_+$ in $B_E(\lambda r) \cap B_F(\lambda^{-\sigma} r) \cap \tilde{E}$, and shall solve in u_+ the equation

$$\tilde{u}_+ = u_+ - \sum_{|I|=2} \mathrm{Op}(\tilde{m}_{0,I})(u_I) - u'^{\mathrm{app}}_+ - u''^{\mathrm{app}}_+, \tag{5.15}$$

where $\tilde{m}_{0,I}$ are symbols in $\tilde{S}_{1,0}(\prod_{j=1}^{2} \langle \xi_j \rangle^{-1} M_0, 2)$ defined in Proposition 3.2.1. Setting $f_+ = u_+ - u''^{\mathrm{app}}_+$, we rewrite (5.15) as

$$f_+ = u'^{\mathrm{app}}_+ + \tilde{u}_+ + \Phi(u''^{\mathrm{app}}_+, f_+), \tag{5.16}$$

where

$$\Phi(u''^{\mathrm{app}}_+, f_+) = \sum_{|I|=2} \mathrm{Op}(\tilde{m}_{0,I})\big((u''^{\mathrm{app}} + f)_I\big). \tag{5.17}$$

Let us check that the assumptions of Lemma 5.1.1 are satisfied by the preceding map.

Lemma 5.1.2. *If we take $E = H^s(\mathbb{R})$, $F = W^{\rho,\infty}(\mathbb{R})$, with s, ρ large enough and $\tilde{E}, \tilde{F}$ defined by (5.12), then inequalities (5.2) to (5.7) are satisfied by the function Φ defined by (5.17).*

Proof. To prove (5.2) we have to check that, for any I with $|I| = 2$,

$$\|\mathrm{Op}(\tilde{m}_{0,I})\big((u'' + f)_I\big)\|_{H^s} \le C\big(\|u''\|_{W^{\rho,\infty}} + \|f\|_{W^{\rho,\infty}}\big)\big(\|u''\|_{H^s} + \|f\|_{H^s}\big)$$

which follows from (D.32) if ρ is large enough, since Proposition D.1.6 applies in particular to symbols that are independent of x, which is the case of elements of $\tilde{S}_{1,0}(\prod_{j=1}^{2} \langle \xi_j \rangle^{-1} M_0, 2)$ according to Definition 3.1.1. In the same way, (5.3) may be written

$$\begin{aligned}\|\mathrm{Op}(\tilde{m}_{0,I})&\big((u'' + f)_I\big)\|_{W^{\rho,\infty}}\\ &\le C\Big(t^{\sigma}\big(\|u''\|_{W^{\rho,\infty}} + \|f\|_{W^{\rho,\infty}}\big)\\ &\qquad + t^{-1}\big(\|u''\|_{H^s} + \|f\|_{H^s}\big)\Big)\big(\|u''\|_{W^{\rho,\infty}} + \|f\|_{W^{\rho,\infty}}\big)\end{aligned}$$

which follows from (D.39) with $r = 1$ if $(s - \rho)\sigma$ is large enough. Inequalities (5.4) and (5.5) are proved in the same way using the bilinearity of $\mathrm{Op}(\tilde{m}_{0,I})$.

Let us prove (5.6) and (5.7). To simplify notation, consider for instance the case $I = (2, 0)$. It is enough to prove the estimates

$$\|\mathrm{Op}(\tilde{m}_{0,I})(f_1, f_2)\|_{L^2} \le C\|f_1\|_{W^{\rho,\infty}}\|f_2\|_{L^2}, \tag{5.18}$$

$$\|x\mathrm{Op}(\tilde{m}_{0,I})(f_1, f_2)\|_{L^2} \le C\big(t\|f_1\|_{W^{\rho,\infty}} + \|xf_1\|_{W^{\rho,\infty}}\big)\|f_2\|_{L^2}, \tag{5.19}$$

$$\|x\mathrm{Op}(\tilde{m}_{0,I})(f_1, f_2)\|_{L^2} \le C\|f_1\|_{W^{\rho,\infty}}\big(t\|f_2\|_{L^2} + \|xf_2\|_{L^2}\big) \tag{5.20}$$

(and the symmetric ones) in order to get (5.6) and (5.7). But (5.18) (resp. (5.19)) follows from (D.33) (resp. (D.37)) if on the right-hand side of the latter inequality we estimate

$$\|L_\pm v_j\|_{W^{\rho_0,\infty}} \le C\big(\|xv_j\|_{W^{\rho_0,\infty}} + t\|v_j\|_{W^{\rho_0+1,\infty}}\big).$$

To get (5.20), one applies instead (D.33) after commuting x to $\mathrm{Op}(\tilde{m}_{0,I})$ in order to put it against the f_2 argument.

This concludes the proof of the lemma. ∎

We may now state the main result of this section, that will show that the implicit equation (5.16) may be solved in f_+, and that we get an expansion for f_+ in terms of $u'^{\mathrm{app}}_+, u''^{\mathrm{app}}_+$ and $\tilde{u}_+$.

Proposition 5.1.3. *Let $u'^{\mathrm{app}}_+, u''^{\mathrm{app}}_+$ be function satisfying (5.13)–(5.14). Let also $\tilde{u}_+$ be a function of $(t,x) \in [1,T] \times \mathbb{R}$, with $T \le \varepsilon^{-4+c}$ satisfying for some $0 < \theta' < \theta < \frac{1}{2}$ (θ' and θ being close to $\frac{1}{2}$), some $\delta > 0$, some constant D the following estimates*

$$\begin{aligned} \|\tilde{u}_+(t,\cdot)\|_E &\le D\varepsilon t^\delta, \\ \|\tilde{u}_+(t,\cdot)\|_F &\le D\frac{(\varepsilon^2\sqrt{t})^{\theta'}}{\sqrt{t}}, \\ \|\tilde{u}_+(t,\cdot)\|_{\tilde{E}} &\le D t^{\frac{5}{4}}(\varepsilon^2\sqrt{t})^\theta. \end{aligned} \tag{5.21}$$

Then, if ε is small enough, there is a unique function f_+ in $E \cap F$ with

$$\begin{aligned} \|f_+\|_F &\le 3\max(C(A,A'),D)\max\left(\varepsilon^2(\log(1+t))^2, \frac{(\varepsilon^2\sqrt{t})^{\theta'}}{\sqrt{t}}\right), \\ \|f_+\|_E &\le 3\max(C(A,A'),D)\varepsilon t^\delta \end{aligned} \tag{5.22}$$

such that, setting $f_- = -\bar{f}_+$,

$$f_+ = u'^{\mathrm{app}}_+ + \tilde{u}_+ + \sum_{|I|=2} \mathrm{Op}(\tilde{m}_{0,I})\big((u''^{\mathrm{app}} + f)_I\big). \tag{5.23}$$

Moreover, one may find symbols $(m_I)_{2\le|I|\le4}$ in the class $\tilde{S}_{1,0}(\prod_{j=1}^{|I|}\langle\xi_j\rangle^{-1}M_0^\nu, |I|)$ for some ν, such that one may write the solution f_+ to (5.23) under the form

$$f_+ = u'^{\mathrm{app}}_+ + \tilde{u}_+ + \sum_{\substack{2\le|I|\le4 \\ I=(I',I'')}} \mathrm{Op}(m_I)\big(\tilde{u}_{I'}, u^{\mathrm{app}}_{I''}\big) + R, \tag{5.24}$$

where R satisfies

$$\|R(t,\cdot)\|_{H^s} \le C'(A,A',D)\left(\frac{(\varepsilon^2\sqrt{t})^{\theta'}t^\sigma}{\sqrt{t}}\right)^4 \varepsilon t^\delta, \tag{5.25}$$

$$\|xR(t,\cdot)\|_{L^2} \le C'(A,A',D)\left(\frac{(\varepsilon^2\sqrt{t})^{\theta'}t^\sigma}{\sqrt{t}}\right)^4 t^{\frac{5}{4}}(\varepsilon^2\sqrt{t})^\theta \tag{5.26}$$

for some new constants $C'(A,A',D)$, $\sigma > 0$ as small as we want.

Proof. Equation (5.23) may be written under the form (5.16) with Φ given by (5.17). We have seen in Lemma 5.1.2 that inequalities (5.2) to (5.7) hold true, with the spaces $E, F, \tilde{E}, \tilde{F}$ defined in that lemma. By (5.13), (5.14) and (5.21), if $t \le \varepsilon^{-4}$ and ε is small enough, we can make $t^{\sigma}\|u'^{\text{app}}_{+}(t,\cdot)\|_{F}$, $t^{\sigma}\|u''^{\text{app}}_{+}(t,\cdot)\|_{F}$, $t^{\sigma}\|\tilde{u}'_{+}(t,\cdot)\|_{F}$ and $t^{-1}\|u'^{\text{app}}_{+}(t,\cdot)\|_{E}$, $t^{-1}\|u''^{\text{app}}_{+}(t,\cdot)\|_{E}$, $t^{-1}\|\tilde{u}'_{+}(t,\cdot)\|_{E}$ as small as we want. We may thus apply Lemma 5.1.1, that gives the solution f_{+} to (5.23) and its uniqueness. This lemma gives as well the first inequality of (5.22). To get the second one, we deduce from (5.8) and (5.2) that

$$\|f_{+}\|_{E} \leq \|u'^{\text{app}}_{+}\|_{E} + \|\tilde{u}_{+}\|_{E} + \sigma(\varepsilon)\big(\|f_{+}\|_{E} + \|u''^{\text{app}}_{+}\|_{E}\big), \tag{5.27}$$

where $\sigma(\varepsilon)$ is controlled by $\|f_{+}\|_{F}$ and $\|u''^{\text{app}}_{+}\|_{F}$, so goes to zero if ε goes to zero by the first inequality of (5.22) and (5.14). Using (5.13), (5.14), (5.21), it follows that, for ε small enough,

$$\|f_{+}\|_{E} \leq 3\max(C(A,A'),D)\varepsilon t^{\delta}. \tag{5.28}$$

In the same way, we get from (5.8) and (5.6),

$$\|f_{+}\|_{\tilde{E}} \leq \|u'^{\text{app}}_{+}\|_{\tilde{E}} + \|\tilde{u}_{+}\|_{\tilde{E}} + C\|u''^{\text{app}}_{+}\|_{\tilde{F}}\|u''^{\text{app}}_{+}\|_{E} + \sigma(\varepsilon)\|f_{+}\|_{\tilde{E}},$$

where $\sigma(\varepsilon)$ is controlled by $\|u''^{\text{app}}_{+}\|_{F} + \|f_{+}\|_{F}$, so goes to zero with ε. Plugging (5.13), (5.14), (5.21) in this inequality, we get for ε small enough, and some new constant $\tilde{C}(A,A',D)$,

$$\|f_{+}\|_{\tilde{E}} \leq \tilde{C}(A,A',D)t^{\frac{5}{4}}(\varepsilon^{2}\sqrt{t})^{\theta}. \tag{5.29}$$

We apply next (5.10) with $N = 4$. We obtain, using (5.13), (5.14), (5.21), (5.22) that

$$\left\|f_{+} - \Phi^{4}(u''^{\text{app}}_{+}, u'^{\text{app}}_{+} + \tilde{u}_{+}, u'^{\text{app}}_{+})\right\|_{E} \leq C'(A,A',D)\left(\frac{(\varepsilon^{2}\sqrt{t})^{\theta'}t^{\sigma}}{\sqrt{t}}\right)^{4}\varepsilon t^{\delta} \tag{5.30}$$

since we assume $t \leq \varepsilon^{-4+c}$ with some $c > 0$. In the same way, by (5.11)

$$\begin{aligned}&\left\|f_{+} - \Phi^{4}(u''^{\text{app}}_{+}, u'^{\text{app}}_{+} + \tilde{u}_{+}, u'^{\text{app}}_{+})\right\|_{\tilde{E}}\\ &\quad\leq C'(A,A',D)\left(\frac{(\varepsilon^{2}\sqrt{t})^{\theta'}t^{\sigma}}{\sqrt{t}}\right)^{4} t^{\frac{5}{4}}(\varepsilon^{2}\sqrt{t})^{\theta}.\end{aligned} \tag{5.31}$$

The right-hand side of (5.30) (resp. (5.31)) is controlled by (5.25) (resp. (5.26)).

To finish the proof, we have to rewrite $\Phi^{4}(u''^{\text{app}}_{+}, u'^{\text{app}}_{+} + \tilde{u}_{+}, u'^{\text{app}}_{+})$ as the main term on the right-hand side of (5.24), up to remainders. Let us show by induction that one may write

$$\Phi^{N}(u''^{\text{app}}_{+}, u'^{\text{app}}_{+} + \tilde{u}_{+}, u'^{\text{app}}_{+}) = u'^{\text{app}}_{+} + \tilde{u}_{+} + \sum_{\substack{2\leq |I|\leq N+1\\ I=(I',I'')}} \operatorname{Op}(m_{I}^{N})(\tilde{u}_{I'}, u^{\text{app}}_{I''}) \tag{5.32}$$

for some new symbols m_{I}^{N} in $\tilde{S}_{1,0}(\prod_{j=1}^{|I|}\langle\xi_{j}\rangle^{-1}M_{0}^{\nu}, |I|)$ for some ν. For $N = 1$ this follows from the definition (5.9) of Φ^{1} and of (5.17). The general case follows using (5.9) and Corollary B.2.6, i.e. the stability of operators of the form $\operatorname{Op}(m_{I}^{N})$ by composition.

We apply (5.32) with $N = 4$, and according to (5.30) and (5.31), equality (5.24) will be proved if we show that the contribution to the right-hand side of (5.32) given by I with $|I| = 5$ forms part of R in (5.24). Using (D.33), we estimate the H^s norm of such a term by

$$C\big(\|\tilde{u}_+\|_{W^{\rho_0,\infty}} + \|u'^{\text{app}}_+\|_{W^{\rho_0,\infty}} + \|u''^{\text{app}}_+\|_{W^{\rho_0,\infty}}\big)^4 \\ \times \big(\|\tilde{u}_+\|_{H^s} + \|u'^{\text{app}}_+\|_{H^s} + \|u''^{\text{app}}_+\|_{H^s}\big),$$

so by the right-hand side of (5.25), using (5.13), (5.14), (5.21).

To study the L^2 norm of the product of x and of the terms in the sum (5.32) with $|I| = 5$, we rewrite the latter, decomposing $u^{\text{app}} = u'^{\text{app}} + u''^{\text{app}}$ under the form

$$\sum_{\substack{|I|=5 \\ I=(I',I'',I''')}} \text{Op}(\tilde{m}_I^5)(\tilde{u}_{I'}, u'^{\text{app}}_{I''}, u''^{\text{app}}_{I'''}) \tag{5.33}$$

with symbols $\tilde{m}_I^5$ in $\tilde{S}_{1,0}(\prod_{j=1}^5 \langle \xi_j \rangle^{-1} M_0^\nu, 5)$.

In (5.33), we distinguish the cases $|I'''| < 5$ and $|I'''| = 5$. In the first one, we use (D.36), making play the special role to one argument different from $u''^{\text{app}}_\pm$. We obtain a bound in

$$\big(\|\tilde{u}_+\|_{W^{\rho_0,\infty}} + \|u'^{\text{app}}_+\|_{W^{\rho_0,\infty}} + \|u''^{\text{app}}_+\|_{W^{\rho_0,\infty}}\big)^4 \big(\|u'^{\text{app}}_+\|_{\tilde{E}} + \|\tilde{u}_+\|_{\tilde{E}}\big)$$

which is controlled by the right-hand side of (5.26). When $|I'''| = 5$, we use (D.37), to obtain a bound in

$$\|u''^{\text{app}}_+\|^3_{W^{\rho_0,\infty}} \|u''^{\text{app}}_+\|_{L^2} \|u''^{\text{app}}_+\|_{\tilde{F}} \le C(A, A')t\big(\log(1+t)\big)^8 \varepsilon^9$$

by (5.14). Since $t \le \varepsilon^{-4+c}$, the last bound is smaller, for ε small enough, than

$$C'(A, A', D)\bigg(\frac{(\varepsilon^2\sqrt{t})^{\theta'}}{\sqrt{t}}\bigg)^4 t^{\frac{5}{4}} (\varepsilon^2 \sqrt{t})^\theta,$$

so than the right-hand side of (5.26). This concludes the proof. ∎

5.2 Reduction of the dispersive equation

The goal of this section is to deduce from equation (3.13) satisfied by u_+ an equation satisfied by the function $\tilde{u}_+$ defined in (5.15). More precisely, we shall prove:

Proposition 5.2.1. *We fix $c > 0$, $0 < \theta' < \theta < \frac{1}{2}$, with θ' close to $\frac{1}{2}$ and $\delta > 0$ small. We take numbers satisfying $s \gg \rho \gg 1$ (that may depend on the preceding parameters c, θ, θ'). Let $\varepsilon \in]0, 1]$ and $T \in [1, \varepsilon^{-4+c}]$. Assume we are given on interval $[1, T]$ a solution $u^{\text{app}}_+ = u'^{\text{app}}_+ + u''^{\text{app}}_+$ of (4.37) satisfying bounds (4.39)–(4.41) and (4.43)–(4.45). Assume also given a function u_+ in $C([1, T], H^s(\mathbb{R}))$, odd, solution* *of* (3.13) *and such that, if we define* $\tilde{u}_+$ *by* (5.15)*, i.e.*

$$\tilde{u}_+ = u_+ - \sum_{|I|=2} \mathrm{Op}(\tilde{m}_{0,I})(u_I) - u'^{\mathrm{app}}_+ - u''^{\mathrm{app}}_+, \tag{5.34}$$

then $\tilde{u}_+$ *satisfies for* $t \in [1, T]$ *the bounds*

$$\begin{aligned} \|\tilde{u}_+(t,\cdot)\|_{H^s} &\le D\varepsilon t^{\delta}, \\ \|\tilde{u}_+(t,\cdot)\|_{W^{\rho,\infty}} &\le D\frac{(\varepsilon^2\sqrt{t})^{\theta'}}{\sqrt{t}}, \\ \|L_+\tilde{u}_+(t,\cdot)\|_{L^2} &\le Dt^{\frac14}(\varepsilon^2\sqrt{t})^{\theta} \end{aligned} \tag{5.35}$$

for some constant D*. Then* $\tilde{u}_+$ *solves the equation*

$$\begin{aligned} &\big(D_t - p(D_x)\big)\tilde{u}_+ \\ &\quad = \sum_{\substack{3\le|I|\le 4\\ I=(I',I'')}} \mathrm{Op}(\tilde{m}_I)(\tilde{u}_{I'}, u^{\mathrm{app}}_{I''}) + \sum_{\substack{|I|=2\\ I=(I',I'')}} \mathrm{Op}(m'_{0,I})(\tilde{u}_{I'}, u^{\mathrm{app}}_{I''}) \\ &\qquad + \underline{a}^{\mathrm{app}}(t) \sum_{|I|=1} \mathrm{Op}(m'_{1,I})(\tilde{u}_I) \\ &\qquad + \frac13\Big(e^{it\frac{\sqrt3}{2}}g(t) + e^{-it\frac{\sqrt3}{2}}\overline{g(t)}\Big)^2 \sum_{|I|=1} \mathrm{Op}(m'_{0,I})(\tilde{u}_I) + R(t,x), \end{aligned} \tag{5.36}$$

where for some $\nu \in \mathbb{N}$, $\tilde{m}_I$ *are symbols in* $\tilde{S}_{1,0}(\prod_{j=1}^{|I|}\langle\xi_j\rangle^{-1}M_0(\xi)^{\nu}, |I|)$, $3 \le |I| \le 4$*, where* $m'_{0,I}, \tilde{m}'_{1,I}$ *are in* $\tilde{S}'_{1,0}(\prod_{j=1}^{|I|}\langle\xi_j\rangle^{-1}M_0(\xi)^{\nu}, |I|)$*, all these symbols satisfying* (3.7)*, and where*

$$\underline{a}^{\mathrm{app}}(t) = \frac{\sqrt3}{3}\big(\underline{a}^{\mathrm{app}}_+(t) - \underline{a}^{\mathrm{app}}_-(t)\big) \tag{5.37}$$

with $\underline{a}^{\mathrm{app}}_+(t)$ *being given by the first four terms on the right-hand side of* (4.8)*, namely*

$$\underline{a}^{\mathrm{app}}_+(t) = e^{it\frac{\sqrt3}{2}}g(t) + \omega_2 g(t)^2 e^{it\sqrt3} + \omega_0|g(t)|^2 + \omega_{-2}\overline{g(t)}^2 e^{-it\sqrt3} \tag{5.38}$$

and

$$\underline{a}^{\mathrm{app}}_-(t) = -\overline{\underline{a}^{\mathrm{app}}_+(t)},$$

and where $R(t,x)$ *satisfies the following bounds for* $t \in [1, T]$:

$$\|R(t,\cdot)\|_{H^s} \le \varepsilon t^{\delta-1} e(t,\varepsilon), \tag{5.39}$$

$$\|L_\pm R(t,\cdot)\|_{L^2} \le t^{-\frac34}(\varepsilon^2\sqrt{t})^{\theta} e(t,\varepsilon), \tag{5.40}$$

where

$$\lim_{\varepsilon\to0+} \sup_{1\le t\le \varepsilon^{-4+c}} e(t,\varepsilon) = 0. \tag{5.41}$$

As a preparation for the proof, let us rewrite equation (3.13) replacing in its left-hand side u_+ by the expression of that function that follows from (5.34), namely

$$\begin{aligned}\big(D_t - p(D_x)\big)\big(\tilde{u}_+ + u'^{\text{app}}_+ + u''^{\text{app}}_+\big) \\ = F_0^2[a] + F_0^3[a] + \sum_{3\le|I|\le4} \operatorname{Op}(m_{0,I})[u_I] + \sum_{|I|=2} \operatorname{Op}(m'_{0,I})[u_I] \\ + \sum_{j=1}^{3} a(t)^j \sum_{1\le|I|\le4-j} \operatorname{Op}(m'_{1,I})[u_I].\end{aligned} \tag{5.42}$$

Recall that we have written in (4.37) an expression for $(D_t - p(D_x))u_+^{\text{app}}$. Making the difference between (5.42) and (4.37), we get that $(D_t - p(D_x))\tilde{u}_+$ is equal to the sum of the following expressions:

$$F_0^2[a] - F_0^2[a^{\text{app}}] + F_0^3[a] - F_0^3[a^{\text{app}}], \tag{5.43}$$

$$\sum_{3\le|I|\le4} \operatorname{Op}(m_{0,I})[u_I], \tag{5.44}$$

$$\sum_{|I|=2} \operatorname{Op}(m'_{0,I})[u_I], \tag{5.45}$$

$$a(t) \sum_{|I|=1} \operatorname{Op}(m'_{1,I})[u_I] - a^{\text{app}}(t) \sum_{|I|=1} \operatorname{Op}(m'_{1,I})[u_I^{\text{app}}], \tag{5.46}$$

$$a(t) \sum_{2\le|I|\le3} \operatorname{Op}(m'_{0,I})[u_I], \tag{5.47}$$

$$a(t)^j \sum_{1\le|I|\le4-j} \operatorname{Op}(m'_{0,I})[u_I], \quad j = 2,3, \tag{5.48}$$

$$-R(t,x), \tag{5.49}$$

where R satisfies (4.38).

We shall analyze successively the expressions (5.43) to (5.49), using (5.34), in order to rewrite their sum as the right-hand side of (5.36) with a new remainder R.

We first write in a lemma some elementary inequalities that we shall refer to in the sequel.

Lemma 5.2.2. *We denote by $e(t,x)$ any real-valued function defined on the interval $[1, \varepsilon^{-4+c}]$, satisfying (5.41). We have then the following inequalities:*

$$t_\varepsilon^{-1} t^{-\gamma} = O(\varepsilon t^{-1} e(t,\varepsilon)) \quad \text{if } \gamma > \frac{1}{2}, \tag{5.50}$$

$$|\log\varepsilon| t_\varepsilon^{-\gamma} t^{-\frac{1}{2}} = O\big(t^{-\frac{3}{4}} (\varepsilon^2\sqrt{t})^\theta e(t,\varepsilon)\big) \quad \text{if } \gamma \ge \frac{1}{2},\ \theta < \frac{1}{2}, \tag{5.51}$$

$$\big(\varepsilon^\gamma + (\varepsilon^2\sqrt{t})^{\gamma'} t^{-1}\big)\varepsilon t^\delta = O\big(\varepsilon t^{\delta-1} e(t,\varepsilon)\big) \quad \text{if } \delta > 0,\ \gamma \ge 4,\ \gamma' > 0, \tag{5.52}$$

$$\begin{aligned}(\varepsilon^2\sqrt{t})^\gamma |\log\varepsilon|^4 t^{-\frac{3}{4}} (\varepsilon^2\sqrt{t})^\theta = O\big(t^{-\frac{3}{4}} (\varepsilon^2\sqrt{t})^\theta e(t,\varepsilon)\big) \\ \text{if } \gamma > 0,\ 0 < \theta < \frac{1}{2},\end{aligned} \tag{5.53}$$

$$(\varepsilon^2\sqrt{t})^\gamma|\log\varepsilon|t^{-\frac{3}{2}-\alpha}\big(t^{\frac{1}{4}}(\varepsilon^2\sqrt{t})^\theta\big) = O\big(\varepsilon t^{\delta-1}e(t,\varepsilon)\big)$$
$$\text{if } \frac{1}{2}-\theta<\gamma\le\frac{1}{2}-\theta+2\delta,\ \alpha\ge 0, \tag{5.54}$$

$$|\log\varepsilon|^2\varepsilon t^{-\frac{1}{2}} = O\big(t^{-\frac{3}{4}}(\varepsilon^2\sqrt{t})^\theta e(t,\varepsilon)\big) \quad \text{if } 0<\theta<\frac{1}{2}, \tag{5.55}$$

$$|\log\varepsilon|^2\varepsilon t_\varepsilon^{-\frac{1}{2}}t^{-\gamma} = O(\varepsilon t^{-1}e(t,\varepsilon)) \quad \text{if } \frac{1}{2}<\gamma<1, \tag{5.56}$$

$$\varepsilon^2 t_\varepsilon^{-1}t^{\frac{1}{4}} = O(\varepsilon t^{-1}e(t,\varepsilon)). \tag{5.57}$$

Proof of Proposition 5.2.1. Since $(D_t - p(D_x))\tilde{u}_+$ is given by (5.43) to (5.49), we have to write each of these terms as contributions to the right-hand side of (5.36). We study them successively.

Terms of the form (5.43). Recall that $a = \frac{\sqrt{3}}{3}(a_+ - a_-)$ with $a_- = -\bar{a}_+$ (see (2.33)) and that $a_+(t)$ is given by (4.96). Since by (4.99), $g(t)$ is $O(t_\varepsilon^{-1/2})$, it follows from (4.96), (4.98) that $a_+(t) - a_+^{\text{app}}(t) = O(t_\varepsilon^{-3/2})$. The definition (2.28) of $F_0^2[a]$, $F_0^3[a]$ implies that for any α, N integers

$$\big|\partial_x^\alpha\big(F_0^j[a] - F_0^j[a^{\text{app}}]\big)(t,x)\big| \le C_{\alpha,N}t_\varepsilon^{-2}\langle x\rangle^{-N},\ j=2,3. \tag{5.58}$$

Thus (5.50) implies that (5.39) holds (even with $\delta = 0$) and (5.51) implies that (5.40) is true for any $\theta < \frac{1}{2}$. So these terms contribute to R in (5.36).

Terms of the form (5.44). Notice that if $\tilde{u}_+$ satisfies estimates (5.35), then it satisfies bounds (5.21) (with a new constant D) in view of the definition of $E = H^s$, $F = W^{\rho,\infty}$ and (5.12) of $\tilde{E}$. Moreover, if we set $f_+ = u_+ - u''^{\text{app}}_+$, equation (5.34) may be written as (5.23). Then Proposition 5.1.3 implies that for ε small enough, there is a unique solution f_+ solving equation (5.23), and we have an expansion (5.24) for f_+ in terms of $\tilde{u}, u^{\text{app}}$. We may rewrite this as

$$u_+ = u_+^{\text{app}} + \tilde{u}_+ + \sum_{\substack{2\le|I|\le 4\\ I=(I',I'')}} \text{Op}(m_I)(\tilde{u}_{I'}, u_{I''}^{\text{app}}) + R \tag{5.59}$$

with symbols m_I in $\tilde{S}_{1,0}(\prod_{j=1}^{|I|}\langle\xi_j\rangle^{-1}M_0^\nu, |I|)$ and R satisfying (5.25) and (5.26). We plug expansion (5.59) inside (5.44). Recall that by Proposition 3.2.1, the symbols $m_{0,I}$ in (5.44) belong to $\tilde{S}_{1,0}(\prod_{j=1}^{|I|}\langle\xi_j\rangle^{-1}M_0, |I|)$. By Corollary B.2.6, we shall get terms of the following form:

$$\text{Op}(\tilde{m}_I)(\tilde{u}_{I'}, u_{I''}^{\text{app}}), \quad 3\le|I|\le 4,\ I=(I',I''), \tag{5.60}$$

where $\tilde{m}_I$ is some new symbol in $\tilde{S}_{1,0}(\prod_{j=1}^{|I|}\langle\xi_j\rangle^{-1}M_0^\nu, |I|)$ for some new ν;

$$\text{Op}(\tilde{m}_I)(U_1, U_2, \dots, U_k), \quad k = |I| \tag{5.61}$$

with $\tilde{m}_I$ as above and either

$$k\ge 5, \quad U_\ell \in \{\tilde{u}_\pm, u'^{\text{app}}_\pm, u''^{\text{app}}_\pm\} \tag{5.62}$$

or

$$k \geq 3, \quad U_\ell \in \{\tilde{u}_\pm, u'^{\text{app}}_\pm, u''^{\text{app}}_\pm, R\} \tag{5.63}$$

with R satisfying (5.25), (5.26), one of the U_ℓ at least being equal to R.

Terms of the form (5.60) are present on the right-hand side of (5.36). We have to show that (5.61) contributes to the remainder in that formula. By (D.32), under (5.62), the H^s norm of (5.61) is bounded from above by

$$C\left(\|\tilde{u}_+\|_{W^{\rho,\infty}} + \|u'^{\text{app}}_+\|_{W^{\rho,\infty}} + \|u''^{\text{app}}_+\|_{W^{\rho,\infty}}\right)^{k-1} \times \left(\|\tilde{u}_+\|_{H^s} + \|u'^{\text{app}}_+\|_{H^s} + \|u''^{\text{app}}_+\|_{H^s}\right).$$

By (5.35), (5.13), (5.14), and since $k \geq 5$, we obtain a bound in

$$C\left(\varepsilon^2|\log\varepsilon|^2 + \frac{(\varepsilon^2\sqrt{t})^{\theta'}}{\sqrt{t}}\right)^4 \varepsilon t^\delta \tag{5.64}$$

so that (5.52) implies that (5.39) holds. On the other hand, consider the action of $L_\pm$ on (5.61) and let us estimate the L^2 norm of the resulting expression by the right-hand side of (5.40). If we multiply (5.61) by x, we have to study

$$x\mathrm{Op}(\tilde{m}_I)(U_1, \dots, U_{k-1}, U_k). \tag{5.65}$$

Consider first the case when among the U_ℓ's in (5.61), at least one of them is equal to $\tilde{u}_\pm$ or $u'^{\text{app}}_\pm$, say U_k. We apply (D.36) (with $j = k$) and obtain thus for the L^2 norm of the relevant quantity at time τ a bound in

$$\begin{aligned} &C\left(\|\tilde{u}_+\|_{W^{\rho,\infty}} + \|u'^{\text{app}}_+\|_{W^{\rho,\infty}} + \|u''^{\text{app}}_+\|_{W^{\rho,\infty}}\right)^{k-1} \\ &\quad \times \left(\tau\|\tilde{u}_+\|_{L^2} + \|L_+\tilde{u}_+\|_{L^2} + \tau\|u'^{\text{app}}_+\|_{L^2} + \|L_+u'^{\text{app}}_+\|_{L^2}\right). \end{aligned} \tag{5.66}$$

By (5.35), (4.40), (4.44), (4.39), (4.41), and the fact that $k \geq 5$, we obtain a bound at time τ in

$$C\left(\varepsilon^2|\log\varepsilon|^2 + \frac{(\varepsilon^2\sqrt{\tau})^{\theta'}}{\sqrt{\tau}}\right)^4 \tau^{\frac{5}{4}}(\varepsilon^2\sqrt{\tau})^\theta. \tag{5.67}$$

By (5.53) we get a bound of the form (5.40) for (5.66).

Consider next the case when in (5.61), all the U_ℓ are equal to $u''^{\text{app}}_\pm$. In this case, we use (D.37) (with $\rho > \rho_0$) to estimate the L^2 norm of (5.65) at time τ. We get a bound by

$$C\|u''^{\text{app}}_+\|^{k-2}_{W^{\rho,\infty}}\left(\tau\|u''^{\text{app}}_+\|_{W^{\rho,\infty}} + \|L_+u''^{\text{app}}_+\|_{W^{\rho,\infty}}\right)\|u''^{\text{app}}_+\|_{L^2}. \tag{5.68}$$

By (4.43)–(4.45) we get an estimate by

$$C\varepsilon(\varepsilon^2\sqrt{\tau})^4|\log\varepsilon|^8\tau^{-1} + \varepsilon(\varepsilon^2\sqrt{\tau})^3|\log\varepsilon|^8\tau^{-\frac{3}{2}}$$

to which (5.53) largely applies.

On the other hand, the L^2 norm of the product of (5.61) by τ is estimated using (D.33) by (5.66) or (5.68) as well. We thus have obtained that, under condition (5.62), (5.61) forms part of the remainder in (5.36).

Let us study now case (5.63). If we compute the H^s norm of (5.61) applying (D.32), we obtain a bound in

$$\begin{aligned} &C\big(\|\tilde{u}_+\|_{W^{\rho,\infty}} + \|u'^{\text{app}}_+\|_{W^{\rho,\infty}} + \|u''^{\text{app}}_+\|_{W^{\rho,\infty}} + \|R\|_{W^{\rho,\infty}}\big)^{k-1}\|R\|_{H^s} \\ &\quad + C\big(\|\tilde{u}_+\|_{W^{\rho,\infty}} + \|u'^{\text{app}}_+\|_{W^{\rho,\infty}} + \|u''^{\text{app}}_+\|_{W^{\rho,\infty}} + \|R\|_{W^{\rho,\infty}}\big)^{k-2} \\ &\qquad \times \big(\|\tilde{u}_+\|_{H^s} + \|u'^{\text{app}}_+\|_{H^s} + \|u''^{\text{app}}_+\|_{H^s}\big)\|R\|_{W^{\rho,\infty}}. \end{aligned} \tag{5.69}$$

By (5.25), that allows to bound $\|R\|_{W^{\rho,\infty}}$ by Sobolev injection, (4.40), (4.44), (5.35), the first line is bounded by (5.25), so it satisfies (5.39). The second line of (5.69) is also estimated in that way. Notice that the assumption $k \geq 3$ is not used here, and that $k \geq 2$ suffices.

If we compute instead the L^2 norm of the product of (5.61) by x from an expression of the form (5.65) with U_k replaced by R and apply (D.36), we obtain an estimate at time τ in

$$\begin{aligned} &C\big(\|\tilde{u}_+\|_{W^{\rho,\infty}} + \|u'^{\text{app}}_+\|_{W^{\rho,\infty}} + \|u''^{\text{app}}_+\|_{W^{\rho,\infty}} + \|R\|_{W^{\rho,\infty}}\big)^{k-1} \\ &\quad \times \big(\tau\|R\|_{L^2} + \|xR\|_{L^2}\big). \end{aligned} \tag{5.70}$$

The first factor is $O(\varepsilon^{2\theta'})$ by (4.40), (4.44), (5.35) and (5.25) (coupled with Sobolev injection). The last one is bounded from above using (5.25) and (5.26), so that it satisfies (5.40) using (5.53). The L^2 norm of the product of (5.61) by τ is also estimated by (5.70). Again, only $k \geq 2$ is used.

Terms of the form (5.45). We plug in (5.45) expansion (5.59). By Corollary B.2.6, we get terms of the form

$$\operatorname{Op}(m'_{0,I})(\tilde{u}_{I'}, u^{\text{app}}_{I''}), \quad |I| = 2,\ I = (I', I'') \tag{5.71}$$

and terms of higher degree of homogeneity. We may thus write these terms as

$$\operatorname{Op}(\tilde{m}'_I)(U_1, \dots, U_k), \quad |I| = k, \tag{5.72}$$

where $\tilde{m}'_I$ is in $\tilde{S}'_{1,0}(\prod_{j=1}^{|I|}\langle\xi_j\rangle^{-1}M_0^\nu, |I|)$ for some ν and where either

$$k \geq 3, \quad U_\ell \in \{\tilde{u}_\pm, u'^{\text{app}}_\pm, u''^{\text{app}}_\pm\} \tag{5.73}$$

or

$$k \geq 2, \quad U_\ell \in \{\tilde{u}_\pm, u'^{\text{app}}_\pm, u''^{\text{app}}_\pm, R\} \tag{5.74}$$

with at least one factor equal to R. Terms (5.72) under condition (5.74) provide remainders satisfying (5.39) and (5.40), as it has been seen in (5.69) and (5.70). (The fact that $k \geq 3$ there has not been used.)

Terms (5.71) are present on the right-hand side of (5.36). Let us show that terms (5.72) under condition (5.73), provide contributions to R in (5.36). To estimate the H^s norm of (5.72), we may first split the symbols in new ones satisfying the support condition of Corollary D.2.12, i.e. for instance $|\xi_1| + \cdots + |\xi_{k-1}| \leq K(1 + |\xi_k|)$. We shall apply estimate (D.78) with $n = k$, $\ell = k - 1$. Let ℓ' be the number of indices j between 1 and $k - 1$ such that in (5.72), U_j is equal to $\tilde{u}_\pm$ or $u'^{\text{app}}_\pm$. Then by (D.78)

$$
\begin{aligned}
&\|\mathrm{Op}(\tilde{m}'_I)(U_1, \dots, U_k)\|_{H^s} \\
&\quad \leq C t^{-(k-1)+\sigma} \big(\|L_+ \tilde{u}_+\|_{L^2} + \|L_+ u'^{\text{app}}_+\|_{L^2} + \|\tilde{u}_+\|_{H^s} + \|u'^{\text{app}}_+\|_{H^s} \big)^{\ell'} \\
&\qquad \times \big(\|L_+ u''^{\text{app}}_+\|_{W^{\rho_0,\infty}} + \|u''^{\text{app}}_+\|_{W^{\rho_0,\infty}} + t^{-\frac{1}{2}} \|u''^{\text{app}}_+\|_{H^s} \big)^{k-1-\ell'} \\
&\qquad \times \big(\|\tilde{u}_+\|_{H^s} + \|u'^{\text{app}}_+\|_{H^s} + \|u''^{\text{app}}_+\|_{H^s} \big).
\end{aligned} \tag{5.75}
$$

Since $k \geq 3$, we obtain from (4.39)–(4.41), (4.43)–(4.45) and (5.35) a bound in

$$
C t^{\sigma-2} \big(t^{\frac{1}{4}} (\varepsilon^2 \sqrt{t})^\theta |\log \varepsilon|^2 \big)^2 \varepsilon t^\delta \leq C t^{-1} e(t, \varepsilon) \varepsilon t^\delta
$$

if σ is taken small enough, so that (5.39) holds.

We consider next the L^2 norm of (5.72) multiplied by x or t. The rapid decay of symbols in the $S'_{\kappa,0}$ class relatively to $M_0(\xi)^{-\kappa}|y|$ given by (B.13) implies that the product of $\tilde{m}'_I$ by x is still a symbol of the form $\tilde{m}'_I$ (with a new value of ν). We thus have to estimate just

$$
t \|\mathrm{Op}(\tilde{m}'_I)(U_1, \dots, U_k)\|_{L^2} \tag{5.76}
$$

with U_ℓ satisfying (5.73). If at least one U_j is equal to $\tilde{u}_\pm$ or $u'^{\text{app}}_\pm$, we use (D.71) with that value of j. We get a bound of (5.76) in

$$
\begin{aligned}
&C \big(\|\tilde{u}_+\|_{W^{\rho_0,\infty}} + \|u'^{\text{app}}_+\|_{W^{\rho_0,\infty}} + \|u''^{\text{app}}_+\|_{W^{\rho_0,\infty}} \big)^{k-1} \\
&\qquad \times \big(\|L_+ \tilde{u}_+\|_{L^2} + \|L_+ u'^{\text{app}}_+\|_{L^2} + \|\tilde{u}_+\|_{L^2} + \|u'^{\text{app}}_+\|_{L^2} \big).
\end{aligned} \tag{5.77}
$$

If all U_j are equal to $u''^{\text{app}}_\pm$, we use (D.72) in order to obtain a bound in

$$
C \|u''^{\text{app}}_+\|^{k-2}_{W^{\rho_0,\infty}} \big(\|L_+ u''^{\text{app}}_+\|_{W^{\rho_0,\infty}} + \|u''^{\text{app}}_+\|_{W^{\rho_0,\infty}} \big) \|u''^{\text{app}}_+\|_{L^2}. \tag{5.78}
$$

By (4.39)–(4.41), (4.43)–(4.45) and (5.35), the sum of (5.77) and (5.78) is estimated at time τ (since $k \geq 3$) by

$$
C \Big(\frac{(\varepsilon^2 \sqrt{\tau})^{\theta'}}{\sqrt{\tau}} + \varepsilon^2 |\log \varepsilon|^2 \Big)^2 \tau^{\frac{1}{4}} (\varepsilon^2 \sqrt{\tau})^\theta + \varepsilon^3 |\log \varepsilon|^4. \tag{5.79}
$$

By (5.53), the first term is smaller than the right-hand side of (5.40). The same holds true trivially for the last term in (5.79). This finishes the proof that terms (5.45) contributes to the remainder in (5.36).

Terms of the form (5.46). We need to prove that (5.46) contributes to the remainder and to the $\underline{a}^{\text{app}} \sum_{|I|=1} \mathrm{Op}(\tilde{m}'_{0,I})(u_I)$ terms on the right-hand side of (5.36). Substitute (5.59) in (5.46). We get the following terms:

$$\big(a(t)-a^{\mathrm{app}}(t)\big)\sum_{|I|=1}\mathrm{Op}(m'_{1,I})(u_I^{\mathrm{app}})+\big(a(t)-\underline{a}^{\mathrm{app}}(t)\big)\sum_{|I|=1}\mathrm{Op}(m'_{1,I})(\tilde{u}_I),\tag{5.80}$$

$$\underline{a}^{\mathrm{app}}(t)\sum_{|I|=1}\mathrm{Op}(m'_{1,I})(\tilde{u}_I),\tag{5.81}$$

$$a(t)\sum_{|I|=1}\sum_{\substack{2\le|\tilde{I}|\le 4\\ \tilde{I}=(\tilde{I}',\tilde{I}'')}}\mathrm{Op}(m'_{1,I})\mathrm{Op}(m_{\tilde{I}})(\tilde{u}_{\tilde{I}'},u^{\mathrm{app}}_{\tilde{I}''}),\tag{5.82}$$

$$a(t)\sum_{|I|=1}\mathrm{Op}(m'_{1,I})(R),\tag{5.83}$$

where R satisfies (5.25), (5.26).

By (5.38), (4.8), (4.6), (4.3) and (4.96), (4.98),

$$a^{\mathrm{app}}(t)-\underline{a}^{\mathrm{app}}(t)=O\big(t_\varepsilon^{-\frac{1}{2}}t^{-\frac{1}{2}}(\varepsilon^2\sqrt{t})^{\theta'}\big)$$

and

$$a(t)-a^{\mathrm{app}}(t)=O\big(t_\varepsilon^{-\frac{3}{2}}\big)=O\big(t_\varepsilon^{-\frac{1}{2}}t^{-\frac{1}{2}}(\varepsilon^2\sqrt{t})^{\theta'}\big).$$

By (D.31), the H^s norm of (5.80) is thus bounded from above at time τ by

$$C\tau_\varepsilon^{-\frac{1}{2}}\tau^{-\frac{1}{2}}(\varepsilon^2\sqrt{\tau})^{\theta'}\big(\|u'^{\mathrm{app}}_{+}\|_{H^s}+\|u''^{\mathrm{app}}_{+}\|_{H^s}+\|\tilde{u}_+\|_{H^s}\big)\le C\tau^{-1}(\varepsilon^2\sqrt{\tau})^{\theta'}\varepsilon\tau^{\delta}$$

using (4.39), (4.43), (5.35). This quantity satisfies (5.39). If we make act $L_\pm$ on (5.80) and use (D.71) to estimate the L^2 norm, we obtain a bound in

$$C\tau_\varepsilon^{-\frac{1}{2}}\tau^{-\frac{1}{2}}(\varepsilon^2\sqrt{\tau})^{\theta'}\big(\|L_+u'^{\mathrm{app}}_{+}\|_{L^2}+\|L_+\tilde{u}_+\|_{L^2}+\|u'^{\mathrm{app}}_{+}\|_{L^2}+\|\tilde{u}_+\|_{L^2}\big)$$

for the contribution of $u'^{\mathrm{app}}_{\pm}$ and $\tilde{u}_\pm$ to (5.80). Using (5.35) and (4.39), (4.41), we get by (5.53) the wanted estimate of the form (5.40). On the other hand, if we consider the contribution $(a(t)-a^{\mathrm{app}}(t))\mathrm{Op}(m'_{I,1})u''^{\mathrm{app}}_{\pm}$ to (5.80) on which acts $L_\pm$, we may estimate the L^2 norm from the L^∞ one, as $m'_{1,I}(x,\xi)$ is rapidly decaying in x. Then, by (D.77) with $\ell=n=1$, we obtain a bound in

$$\begin{aligned}Ct|a-a^{\mathrm{app}}|\Big(t^{-r}\big(\|u''^{\mathrm{app}}_{+}\|_{W^{\rho_0,\infty}}+t^{-\frac{1}{2}}\|u''^{\mathrm{app}}_{+}\|_{H^s}\big)\\ +t^{-1+\sigma}\big(\|u''^{\mathrm{app}}_{+}\|_{W^{\rho_0,\infty}}+\|L_+u''^{\mathrm{app}}_{+}\|_{W^{\rho_0,\infty}}\big)\Big).\end{aligned}\tag{5.84}$$

As $a-a^{\mathrm{app}}=O(t_\varepsilon^{-\frac{3}{2}})$, it follows, taking for instance $r=1$, and using (4.43), (4.44), (4.45) that (5.84) at time τ may be estimated, if σ is small enough, from

$$C\tau_\varepsilon^{-\frac{3}{2}}\tau^{\sigma}|\log\varepsilon|^2\le C\tau_\varepsilon^{-\frac{1}{2}}\tau^{-\frac{1}{2}}\varepsilon^{1-2\sigma}|\log\varepsilon|^2.$$

By (5.51), (5.40) will hold largely. We have thus obtained that (5.80) is a remainder.

Term (5.81) is present on the right-hand side of (5.36).

Consider next (5.82). By Corollary B.2.6, the composition $\mathrm{Op}(m'_{1,I}) \circ \mathrm{Op}(m_{\tilde{I}})$ may be written under the form $\mathrm{Op}(m'_{1,\tilde{I}})$ for new symbols $m'_{1,\tilde{I}}$ in

$$\tilde{S}'_{1,0}\left(\prod_{j=1}^{|\tilde{I}|}\langle\xi_j\rangle^{-1}M_0^{\nu}, |\tilde{I}|\right)$$

for some ν and $2 \le |\tilde{I}| \le 4$. Consequently, we write (5.82) under the form

$$a(t) \sum_{\substack{2\le|\tilde{I}|\le4 \\ \tilde{I}=(\tilde{I}',\tilde{I}'')}} \mathrm{Op}(m'_{1,\tilde{I}})(\tilde{u}_{\tilde{I}'}, u^{\mathrm{app}}_{\tilde{I}''}). \tag{5.85}$$

Since such expressions will appear also in the study of terms of the form (5.47), we postpone their study.

Finally, let us study (5.83). As $\mathrm{Op}(m'_{1,I})$ is bounded on H^s, the Sobolev norm of (5.83) is $O(t_\varepsilon^{-1/2}\|R(t,\cdot)\|_{H^s})$. Using (5.25), it satisfies (5.39). If we make act $L_\pm$ on (5.83), the rapid decay of $m'_{1,I}$ and (5.25), show that we obtain at time τ an expression whose L^2 norm is bounded from above by

$$C\tau_\varepsilon^{-\frac{1}{2}}(\varepsilon^2\sqrt{\tau})^{4\theta'}\tau^{-1+4\sigma}(\varepsilon\tau^\delta)$$

that trivially satisfies (5.40).

This concludes the study of terms of the form (5.46).

Terms of the form (5.47) *(and* (5.85)*).* We study now expressions of the form (5.47) and the related ones introduced in (5.85).

We plug expansion (5.59) in (5.47). By Corollary B.2.6, we get again terms of the form (5.85), with $2 \le |\tilde{I}| \le 6$ instead of $2 \le |\tilde{I}| \le 4$, and terms of the form

$$a(t)\mathrm{Op}(\tilde{m}'_{1,I})(U_1,\ldots,U_k), \quad |I| = k \ge 2 \tag{5.86}$$

with again $\tilde{m}'_{1,I}$ in $\tilde{S}'_{1,0}(\prod_{j=1}^{|I|}\langle\xi_j\rangle^{-1}M_0^{\nu}, |I|)$, U_ℓ belonging to

$$\{\tilde{u}_\pm, u'^{\mathrm{app}}_\pm, u''^{\mathrm{app}}_\pm, R\},$$

one of the arguments at least being equal to R satisfying (5.25) and (5.26). We have already checked that terms of this last form provide remainders (even without the pre-factor $a(t)$) (see (5.69) and (5.70), where the assumption $k \ge 3$ was not used). We are thus reduced to the study of terms of the form (5.85), with $|\tilde{I}| \ge 2$ in the sum. If $|\tilde{I}| \ge 3$, we get terms of the form (5.72) with conditions (5.73), that have been seen to be remainders. We must thus just study

$$a(t)\mathrm{Op}(\tilde{m}'_{1,I})(U_1, U_2) \tag{5.87}$$

with $|I| = 2$, $U_1, U_2 \in \{\tilde{u}_\pm, u'^{\mathrm{app}}_\pm, u''^{\mathrm{app}}_\pm\}$. Moreover, we may assume, in order to bound the Sobolev norm, that $\tilde{m}'_{1,I}$ is supported for $|\xi_1| \le K(1+|\xi_2|)$ for instance.

Applying (D.78) with $\ell' = \ell = 1$ if $U_1 = \tilde{u}_\pm$ or $u'^{\text{app}}_\pm$ and $\ell = 1, \ell' = 0$ if $U_1 = u''^{\text{app}}_\pm$, we bound the H^s norm of (5.87) by

$$\begin{aligned}|a(t)|t^{-1+\sigma}\big(\|L_+\tilde{u}_+\|_{L^2} + \|\tilde{u}_+\|_{H^s} + \|L_+u'^{\text{app}}_+\|_{L^2} + \|u'^{\text{app}}_+\|_{H^s}\\ + \|L_+u''^{\text{app}}_+\|_{W^{\rho_0,\infty}} + \|u''^{\text{app}}_+\|_{W^{\rho_0,\infty}} + t^{-\frac{1}{2}}\|u''^{\text{app}}_+\|_{H^s}\big)\\ \times \big(\|\tilde{u}_+\|_{H^s} + \|u'^{\text{app}}_+\|_{H^s} + \|u''^{\text{app}}_+\|_{H^s}\big).\end{aligned}$$

As $a(t) = O(t_\varepsilon^{-\frac{1}{2}})$, one gets at time τ a bound in $\varepsilon\tau^{\delta-1}e(\tau,\varepsilon)$ using (4.39)–(4.41), (4.43)–(4.45) and (5.35). It follows that (5.39) will hold. On the other hand, if we make act $L_\pm$ on (5.87) and compute the L^2 norm, we get a bound given by

$$|a(t)| = O(t_\varepsilon^{-\frac{1}{2}})$$

multiplied by (5.77) or (5.78) with $k = 2$. Using again (4.39)–(4.41), (4.43)–(4.45) and (5.35), we obtain at time τ an upper bound in

$$\begin{aligned}C\tau_\varepsilon^{-\frac{1}{2}}\Big(\Big(\frac{(\varepsilon^2\sqrt{\tau})^{\theta'}}{\sqrt{\tau}} + \varepsilon^2|\log\varepsilon|^2\Big)\tau^{\frac{1}{4}}(\varepsilon^2\sqrt{\tau})^{\theta}\\ + \log(1+\tau)\log(1+\tau\varepsilon^2)\varepsilon\Big(\frac{\tau\varepsilon^2}{\langle\tau\varepsilon^2\rangle}\Big)^{\frac{1}{2}}\Big).\end{aligned}$$

By (5.53), (5.55), (5.40) will hold true. This concludes the estimate of these terms.

Terms of the form (5.48). Terms (5.48) with $|I| \geq 2$ are of the same form as (5.47), with a smaller pre-factor $a(t)^j$, so they are remainders. We have thus to study

$$a(t)^j \sum_{|I|=1} \text{Op}(m'_{0,I})(u_I), \quad j = 2, 3. \tag{5.88}$$

By (4.96), (4.97), (4.98), (4.100) and the definition of $a(t) = \frac{\sqrt{3}}{3}(a_+ - a_-)$, one may write (5.88) from the term

$$\frac{1}{3}\sum_{|I|=1}\Big(e^{it\frac{\sqrt{3}}{2}}g(t) + e^{-it\frac{\sqrt{3}}{2}}\overline{g(t)}\Big)^2 \text{Op}(m'_{0,I})(u_I) \tag{5.89}$$

and from terms like

$$\tilde{a}(t)\sum_{|I|=1}\text{Op}(m'_{0,I})(u_I), \tag{5.90}$$

where

$$|\tilde{a}(t)| \leq Ct_\varepsilon^{-1}(t^{-\frac{1}{2}}(\varepsilon^2\sqrt{t})^{\theta'} + t_\varepsilon^{-\frac{1}{2}}). \tag{5.91}$$

Terms (5.89) are present on the right-hand side of (5.36). We have to show that (5.90) provides remainders. The H^s norm of these terms in bounded from above, using the Sobolev boundedness of $\text{Op}(m'_{0,I})$ and estimates (4.39), (4.43) and (5.35) by $C\varepsilon t^{\delta-1}\varepsilon^{2\theta'}$ so that (5.39) will hold.

On the other hand, if we make act $L_\pm$ on (5.90) and compute the L^2 norm, we have to estimate by (5.91) expressions of the form

$$t t_\varepsilon^{-1}\big(t^{-\frac{1}{2}}(\varepsilon^2\sqrt{t})^{\theta'} + t_\varepsilon^{-\frac{1}{2}}\big)\|\mathrm{Op}(\tilde{m}'_{0,I})U\|_{L^2}, \tag{5.92}$$

where $\tilde{m}'_{0,I}$ is of the same form as $m'_{0,I}$ and $U = \tilde{u}_\pm$ or $u'^{\mathrm{app}}_\pm$ or $u''^{\mathrm{app}}_\pm$.

When $U = \tilde{u}_\pm$ or $u'^{\mathrm{app}}_\pm$, we use (D.71) to bound (5.92) by

$$C t_\varepsilon^{-1}\big(t^{-\frac{1}{2}}(\varepsilon^2\sqrt{t})^{\theta'} + t_\varepsilon^{-\frac{1}{2}}\big)\big(\|L_+\tilde{u}_+\|_{L^2} + \|L_+ u'^{\mathrm{app}}_+\|_{L^2} + \|\tilde{u}_+\|_{L^2} + \|u'^{\mathrm{app}}_+\|_{L^2}\big).$$

Using (4.39), (4.41) and (5.35), we see from (5.53) that (5.40) will hold. On the other hand, if $U = u''^{\mathrm{app}}_+$, we estimate the L^2 norm in (5.92) from an L^∞ one, using the rapid decay of $\tilde{m}'_{0,I}$, and we use (D.77) with $\ell = n = 1$, $r = 1$, in order to obtain a bound in

$$t^\sigma t_\varepsilon^{-1}\big(t^{-\frac{1}{2}}(\varepsilon^2\sqrt{t})^{\theta'} + t_\varepsilon^{-\frac{1}{2}}\big)\big(\|u''^{\mathrm{app}}_+\|_{W^{\rho_0,\infty}} + \|L_+ u''^{\mathrm{app}}_+\|_{W^{\rho_0,\infty}} + t^{-\frac{1}{2}}\|u''^{\mathrm{app}}_+\|_{H^s}\big).$$

By (4.43)–(4.45), we bound this by

$$C|\log\varepsilon|^2 \varepsilon t^{-\frac{1}{2}}(t^\sigma\varepsilon)$$

so that, since $t \le \varepsilon^{-4}$ and σ may be taken as small as we want, (5.55) implies that (5.40) holds. This concludes the study of terms (5.48).

Terms of the form (5.49). These terms satisfy (4.38). It follows immediately from (5.50) that (5.39) holds. Using (5.51), we get as well (5.40).

This concludes the proof of Proposition 5.2.1. ■

The reduced equation (5.36) obtained in Proposition 5.2.1 still needs one more reduction before we are able to deal with it. Recall that in Proposition 4.1.2, we have decomposed u^{app}_+ under the form (4.48) $u^{\mathrm{app}}_+ = u^{\mathrm{app},1}_+ + \Sigma_+$, where $u^{\mathrm{app},1}_+$ was given by (4.49). We refined this decomposition in (4.54) as

$$\begin{aligned} u^{\mathrm{app},1}_+ &= u'^{\mathrm{app},1}_+ + u''^{\mathrm{app},1}_+, \\ u'^{\mathrm{app},1}_+ &= \sum_{j\in\{-2,0,2\}} U'_{j,+}(t,x), \\ u''^{\mathrm{app},1}_+ &= \sum_{j\in\{-2,0,2\}} U''_{j,+}(t,x), \end{aligned} \tag{5.93}$$

where $U'_{j,+}, U''_{j,+}$ are defined in (C.4) from the right-hand side of (4.50), namely

$$\begin{aligned} U'_{j,+}(t,x) &= i\int_1^{+\infty} e^{i(t-\tau)p(D_x)+ij\frac{\sqrt{3}}{2}}\chi\Big(\frac{\tau}{\sqrt{t}}\Big)M_j(\tau,\cdot)\,d\tau, \\ U''_{j,+}(t,x) &= i\int_{-\infty}^{t} e^{i(t-\tau)p(D_x)+ij\frac{\sqrt{3}}{2}}(1-\chi)\Big(\frac{\tau}{\sqrt{t}}\Big)M_j(\tau,\cdot)\,d\tau \end{aligned} \tag{5.94}$$

with M_j given by (4.21). Let us prove the following corollary of Proposition 5.2.1.

Corollary 5.2.3. *Under the assumptions of Proposition 5.2.1, $\tilde{u}_+$ solves an equation of the form*

$$
\begin{aligned}
&\big(D_t - p(D_x)\big)\tilde{u}_+ - \sum_{j=-2}^{2} e^{itj\frac{\sqrt{3}}{2}}\mathrm{Op}(b'_{j,+})\tilde{u}_+ - \sum_{j=-2}^{2} e^{itj\frac{\sqrt{3}}{2}}\mathrm{Op}(b'_{j,-})\tilde{u}_- \\
&\quad = \sum_{\substack{3\le |I|\le 4\\ I=(I',I'')}} \mathrm{Op}(\tilde{m}_I)(\tilde{u}_{I'}, u^{\mathrm{app}}_{I''}) + \sum_{|I|=2}\mathrm{Op}(m'_{0,I})(\tilde{u}_I) \\
&\qquad + \sum_{\substack{I=(I',I'')\\ |I'|=|I''|=1}} \mathrm{Op}(m'_{0,I})(\tilde{u}_{I'}, u'^{\mathrm{app},1}_{I''}) \\
&\qquad + \sum_{|I|=2}\mathrm{Op}(m'_{0,I})(u'^{\mathrm{app},1}_{I}) + R_+(t,x),
\end{aligned}
\tag{5.95}
$$

where $(\tilde{m}_I)_{3\le|I|\le4}$ is as in the statement of Proposition 5.2.1, where $(m'_{0,I})_{|I|=2}$ are symbols in the class $\tilde{S}'_{1,0}(\prod_{j=1}^{2}\langle\xi_j\rangle^{-1}M_0(\xi),2)$, where R_+ satisfies (5.39) and (5.40), and where the symbols $b'_{j,\pm}$ satisfy (3.7) and the following estimates for α, β, N in $\mathbb{N}$: If $j=-1$ or $j=1$,

$$
\begin{aligned}
|\partial_x^\alpha\partial_\xi^\beta b'_{j,\pm}(t,x,\xi)| &\le C_{\alpha,\beta,N} t_\varepsilon^{-\frac{1}{2}}\langle x\rangle^{-N}\langle\xi\rangle^{-1}, \\
|\partial_t\partial_x^\alpha\partial_\xi^\beta b'_{j,\pm}(t,x,\xi)| &\le C_{\alpha,\beta,N}\big(t_\varepsilon^{-\frac{3}{2}} + (\varepsilon^2\sqrt{t})^{\frac{3}{2}\theta'}t^{-\frac{3}{2}}\big)\langle x\rangle^{-N}\langle\xi\rangle^{-1},
\end{aligned}
\tag{5.96}
$$

and if $j=-2,0,2$,

$$
\begin{aligned}
|\partial_x^\alpha\partial_\xi^\beta b'_{j,\pm}(t,x,\xi)| &\le C_{\alpha,\beta,N} t_\varepsilon^{-1}\langle x\rangle^{-N}\langle\xi\rangle^{-1}, \\
|\partial_t\partial_x^\alpha\partial_\xi^\beta b'_{j,\pm}(t,x,\xi)| &\le C_{\alpha,\beta,N} t_\varepsilon^{-\frac{1}{2}}\big(t_\varepsilon^{-\frac{3}{2}} + (\varepsilon^2\sqrt{t})^{\frac{3}{2}\theta'}t^{-\frac{3}{2}}\big)\langle x\rangle^{-N}\langle\xi\rangle^{-1}.
\end{aligned}
\tag{5.97}
$$

Proof. Let us analyze the different terms on the right-hand side of (5.36). The first sum appears unchanged in (5.95).

By the definition (5.38) of $\underline{a}^{\mathrm{app}}_+$, the fact that $\underline{a}^{\mathrm{app}} = \frac{\sqrt{3}}{3}(\underline{a}^{\mathrm{app}}_+ + \overline{\underline{a}^{\mathrm{app}}_+})$ and (4.3), the $\underline{a}^{\mathrm{app}}(t)\sum_{|I|=1}\mathrm{Op}(m'_{1,I})(\tilde{u}_I)$ term in (5.36) contributes to the terms involving $b'_{j,\pm}$ on the left-hand side of (5.95). The same holds true for the last but one term in (5.36). We are thus left with studying

$$
\sum_{\substack{|I|=2\\ I=(I',I'')}} \mathrm{Op}(m'_{0,I})(\tilde{u}_{I'}, u^{\mathrm{app}}_{I''}).
\tag{5.98}
$$

First step. If $|I''|=0$, we get the $\sum_{|I|=2}\mathrm{Op}(m'_{0,I})(\tilde{u}_I)$ contribution in (5.95).

Second step. We consider next the contributions to (5.98) with $|I'|=1$, $|I''|=1$. As one may decompose

$$
u^{\mathrm{app}}_+ = u'^{\mathrm{app},1}_+ + u''^{\mathrm{app},1}_+ + \Sigma_+
$$

by (4.48) and (4.55), we shall get three type of terms:

$$\sum_{\substack{I=(I',I'')\\|I'|=|I''|=1}} \mathrm{Op}(m'_{0,I})(\tilde{u}_{I'}, u'^{\mathrm{app},1}_{I''}), \tag{5.99}$$

$$\sum_{\substack{I=(I',I'')\\|I'|=|I''|=1}} \mathrm{Op}(m'_{0,I})(\tilde{u}_{I'}, u''^{\mathrm{app},1}_{I''}), \tag{5.100}$$

$$\sum_{\substack{I=(I',I'')\\|I'|=|I''|=1}} \mathrm{Op}(m'_{0,I})(\tilde{u}_{I'}, \Sigma_{I''}). \tag{5.101}$$

Term (5.99) appears on the right-hand side of (5.95). From (5.93), we may rewrite (5.100) as a sum of expressions

$$\mathrm{Op}(m'_{0,I})(\tilde{u}_{I'}, U''_{j,I''}), \quad j = -2, 0, 2. \tag{5.102}$$

We shall apply Proposition C.2.2 with $\kappa = 1, \omega = 1$. Since $U''_{j,+}$ is defined by (5.94) from a M_j given by (4.21), thus satisfying by (4.3) inequalities (C.7) with $\omega = 1$, Assumption $(\mathrm{H1})_1$ of Proposition C.2.1 is satisfied, and so Proposition C.2.2 applies. It follows from (C.106), applied with $\lambda = j\frac{\sqrt{3}}{2}$, $j = -2, 0, 2$, that (5.102) may be written as

$$e^{ijt\frac{\sqrt{3}}{2}} \mathrm{Op}(b_1^j)\tilde{u}_{I'} + \mathrm{Op}(b_2^j)\tilde{u}_{I'} \tag{5.103}$$

where b_1^j (resp. b_2^j) satisfies (3.7) and the first two lines (resp. the last line) in (C.107) with $\omega = 1$. The first term in (5.103) brings thus contributions to the last two sums on the left-hand side of (5.95), for $j = -2, 0, 2$, with symbols satisfying (5.97) and (3.7).

We have to check next that the last term in (5.103) contributes to the remainders. By the last line in (C.107) and (D.32), (5.35)

$$\|\mathrm{Op}(b_2^j)\tilde{u}_{I'}\|_{H^s} \leq C\varepsilon^2 t^{-1}\log(1+t)\varepsilon t^{\delta}$$

from which a remainder estimate of the form (5.39) follows. If we make act $L_\pm$ on $\mathrm{Op}(b_2^j)\tilde{u}_{I'}$ and use (D.71) with $n = 1$ and the bounds (C.107) for the semi-norms of b_2^j (with $\omega = 1$), we obtain from (5.35)

$$\|L_\pm \mathrm{Op}(b_2^j)\tilde{u}_{I'}\|_{L^2} \leq C\varepsilon^2 t^{-1}\log(1+t) t^{\frac{1}{4}}(\varepsilon^2\sqrt{t})^{\theta} \tag{5.104}$$

so that a bound of form (5.40) holds.

It remains to study (5.101). Recall the definition of Σ_+ given after (4.50): this function is a sum

$$\sum_{j=-3}^{3} \underline{U}_j(t,x),$$

where $\underline{U}_j$ solves (4.50) with source term $e^{ijt\frac{\sqrt{3}}{2}}\underline{M}_j$, where $\underline{M}_j$ satisfies (4.51), i.e. the first inequality in (C.8). We may then decompose each $\underline{U}_j$ as $\underline{U}'_{j,1} + \underline{U}''_{j,1}$, according to (C.110) with $\lambda = j\frac{\sqrt{3}}{2}$ and rewrite the terms in (5.101) from

$$\mathrm{Op}(m'_{0,I})(\tilde{u}_{I'}, \underline{U}'_{j,1,I''}), \quad \mathrm{Op}(m'_{0,I})(\tilde{u}_{I'}, \underline{U}''_{j,1,I''}) \tag{5.105}$$

to which Proposition C.2.5 applies. This allows us to rewrite these terms in the form $\mathrm{Op}(b)(\tilde{u}_\pm)$, where b satisfies estimates (C.117), namely

$$|\partial_y^{\alpha'_0}\partial_\xi b(t,y,\xi)| \le C t_\varepsilon^{-\frac{1}{2}} t^{-1}\log(1+t)\langle y\rangle^{-N}\langle\xi\rangle^{-1}. \tag{5.106}$$

By (D.32) and (5.35), we thus get

$$\begin{aligned}\|\mathrm{Op}(b)(\tilde{u}_\pm)\|_{H^s} &\le C t_\varepsilon^{-\frac{1}{2}} t^{-1}\log(1+t)\|\tilde{u}_+\|_{H^s}\\ &\le C t_\varepsilon^{-\frac{1}{2}} t^{-1}\log(1+t)\varepsilon t^{\delta}.\end{aligned}$$

An estimate of the form (5.39) follows at once. If we make act $L_\pm$ on $\mathrm{Op}(b)(\tilde{u}_\pm)$, use the rapid decay in y of (5.106) and (D.71), we obtain an estimate of the L^2 norm by the right-hand side of (5.104), with ε^2 replaced by $t_\varepsilon^{-1/2} \le \varepsilon$. This suffices to imply that (5.40) holds, and thus shows that (5.101) is a remainder.

Third step. We study finally contributions to (5.98) where $|I'| = 0$. Again, we use (4.48) and (4.55) to write

$$u_+^{\mathrm{app}} = u'^{\mathrm{app},1}_+ + u''^{\mathrm{app},1}_+ + \Sigma_+.$$

Plugging this expression inside the terms (5.98) with $|I'| = 0$, we shall get expressions given by

$$\mathrm{Op}(m'_{0,I})\big(u'^{\mathrm{app},1}_I\big), \qquad |I| = 2, \tag{5.107}$$

$$\mathrm{Op}(m'_{0,I})\big(\Sigma_{I'}, u'^{\mathrm{app},1}_{I''}\big), \qquad |I'| = |I''| = 1,\ I = (I', I''), \tag{5.108}$$

$$\mathrm{Op}(m'_{0,I})(\Sigma_I), \qquad |I| = 2, \tag{5.109}$$

$$\mathrm{Op}(m'_{0,I})\big(u''^{\mathrm{app},1}_I\big), \qquad |I| = 2, \tag{5.110}$$

$$\mathrm{Op}(m'_{0,I})\big(\Sigma_{I'}, u''^{\mathrm{app},1}_{I''}\big), \qquad |I'| = |I''| = 1,\ I = (I', I''), \tag{5.111}$$

$$\mathrm{Op}(m'_{0,I})\big(u'^{\mathrm{app},1}_{I'}, u''^{\mathrm{app},1}_{I''}\big), \qquad |I'| = |I''| = 1,\ I = (I', I''), \tag{5.112}$$

where $m'_{0,I}$ are still elements of $\tilde{S}'_{1,0}(\prod_{j=1}^{|I|}\langle\xi_j\rangle^{-1}M_0^{\nu}, |I|)$.

Term (5.107) appears on the right-hand side of (5.95).

Term (5.108) is treated as (5.101): actually, $u'^{\mathrm{app},1}_+$ satisfies (4.39)–(4.41) as has been established after (4.54), and these bounds are better than inequalities (5.35) for $\tilde{u}_+$.

Term (5.109) may be treated in the same way: we have seen in the study of (5.101) that $\mathrm{Op}(m'_{0,I})(\,\cdot\,, \Sigma_{I''})$ may be written as $\mathrm{Op}(b)\,\cdot$ for b satisfying (5.106) (see (5.105)). By (4.52), we shall get for any N,

$$\begin{aligned}\|x^N\mathrm{Op}(m'_{0,I})(\Sigma_I)\|_{H^s} &\le C\|x^N\mathrm{Op}(b)(\Sigma_\pm)\|_{H^s}\\ &\le C t_\varepsilon^{-\frac{1}{2}} t^{-1}(\log(1+t))^2\big(t_\varepsilon^{-\frac{3}{2}} + t^{-1}t_\varepsilon^{-\frac{1}{2}} + t^{-1}\varepsilon^2\big).\end{aligned} \tag{5.113}$$

By (5.56), we see that (5.39) will hold. Estimating the action of $L_{\pm}$ on $\mathrm{Op}(m'_{0,I})(\Sigma_I)$ in L^2, we get an upper bound by the right-hand side of (5.113) multiplied by t. Then (5.55) shows that (5.40) holds.

To study (5.110), we recall that $u''^{\mathrm{app},1}_{+}$ is given by (4.54), where $U''_{j,+}$ is given by the second formula (C.4) in terms of an M that satisfies (4.13), i.e. such that (C.7) with $\omega = 1$ (Assumption $(\mathrm{H1})_1$) holds. We may thus apply Corollary C.2.3 with $\omega = 1$. It follows that the H^s norm of (5.110) is bounded from above by

$$C\big(t_\varepsilon^{-2} + \varepsilon^4 t^{-2}(\log(1+t))^2\big).$$

This largely implies (5.39). On the other hand, the L^2 norm of the action of $L_{\pm}$ on (5.110) is bounded by

$$C\big(t t_\varepsilon^{-2} + \varepsilon^4 t^{-1}(\log(1+t))^2\big).$$

Then (5.55) implies that (5.40) largely holds.

Terms (5.111) may be treated in a similar way as (5.109): we have seen that $\mathrm{Op}(\tilde{m}'_I)(\Sigma_{I'}, u''^{\mathrm{app},1}_{I''})$ may be written as $\mathrm{Op}(b)u''^{\mathrm{app},1}_{\pm}$ with b satisfying (5.106). By the expression (4.54) of

$$u''^{\mathrm{app},1}_{+} = \sum_{j\in\{-2,0,2\}} U''_{j,+},$$

where $U''_{j,+}$ is defined by the second formula (C.4) with $\lambda = j\frac{\sqrt{3}}{2}$ and $M = M_j$ given by (4.21), we see that we may apply Proposition C.2.1 with $\omega = 1$. Taking into account the time decaying factor on the right-hand side of (5.106), it follows from (C.89)–(C.91) that

$$\begin{aligned} &|\partial_x^\alpha \mathrm{Op}(m'_{0,I})(\Sigma_{I'}, u''^{\mathrm{app},1}_{I''})| \\ &\quad \le C t_\varepsilon^{-\frac{1}{2}} t^{-1}(\log(1+t))\big(t_\varepsilon^{-1} + \varepsilon^2 t^{-1}\log(1+t)\big)\langle x\rangle^{-N}. \end{aligned} \tag{5.114}$$

Thus the H^s norm of (5.111) is bounded from above by the t-depending factor in (5.114). By (5.56), we get that (5.39) largely holds. If we make act $L_{\pm}$ on (5.111) and estimate the L^2 norm, we get a bound in

$$C t_\varepsilon^{-\frac{1}{2}} \log(1+t)\big(t_\varepsilon^{-1} + \varepsilon^2 t^{-1}\log(1+t)\big).$$

Thus (5.55) implies (5.40).

It just remains to treat (5.112). Notice that (5.112) is of the same form as (5.100) with $\tilde{u}_{I'}$ replaced by $u'^{\mathrm{app},1}_{I'}$, so that may be written under a similar form as (5.103), namely

$$e^{ijt\frac{\sqrt{3}}{2}}\mathrm{Op}(b_1^j)u'^{\mathrm{app},1}_{I'} + \mathrm{Op}(b_2^j)u'^{\mathrm{app},1}_{I'}, \tag{5.115}$$

where b_1^j (resp. b_2^j) satisfies the first two lines (resp. the last line) in (C.107) with $\omega = 1$. We have checked after (5.103) that the second term in that formula is a remainder. Since as seen above, $u'^{\mathrm{app},1}_{+}$ satisfies (4.39)–(4.41), which are better estimates than those verified by $\tilde{u}_+$, it follows that the last term in (5.115) is also a remainder. Let us prove that, because of the better bounds satisfied by $u'^{\text{app},1}_{+}$ versus $\tilde{u}_+$, the first term in (5.115) is a remainder as well. By the estimates of b_1 in (C.107) and (D.32),

$$\|\mathrm{Op}(b_1^j)u'^{\text{app},1}_{+}\|_{H^s} \leq C t_\varepsilon^{-1}\|u'^{\text{app},1}_{+}\|_{H^s} \leq C t_\varepsilon^{-1}\varepsilon^2 t^{\frac{1}{4}}$$

according to (4.39) written for $u'^{\text{app},1}_{+}$. By (5.57), we conclude that (5.39) holds. To estimate $\|L_\pm \mathrm{Op}(b_1^j)u'^{\text{app},1}_{+}\|_{L^2}$, we are reduced, by the fact that b_1^j is rapidly decaying in x, to bounding $t\|\mathrm{Op}(b_1^j)u'^{\text{app},1}_{+}\|_{L^2}$. According to (D.71) and the bounds (C.107) of b_1^j, we thus get an estimate in

$$t_\varepsilon^{-1}\big(\|u'^{\text{app},1}_{+}\|_{L^2} + \|L_+ u'^{\text{app},1}_{+}\|_{L^2}\big) \leq C t_\varepsilon^{-1} t^{\frac{1}{4}}\big((\varepsilon^2\sqrt{t}) + (\varepsilon^2\sqrt{t})^{\frac{7}{8}}\varepsilon^{\frac{1}{8}}\big)$$

by (4.41). As in (5.40) $\theta < \frac{1}{2}$, (5.53) shows that (5.40) holds.

This ends the study of term (5.112) and thus the proof of Corollary 5.2.3. ■