Chapter 6

Normal forms

This chapter is devoted to the completion of Step 5 of the proof of our main theorem, that is described in Section 2.7 of Chapter 2. We recall here some elements of the strategy. The preceding steps of the proof allowed us to reduce ourselves to an equation (5.95) for a new unknown $\tilde{u}_+$. In this chapter, we first write a system made of that equation and of the one obtained by conjugation. In that way, if we set $\tilde{u}_- = -\overline{\tilde{u}_+}$ and $\tilde{u} = \left[\begin{smallmatrix}\tilde{u}_+\\ \tilde{u}_-\end{smallmatrix}\right]$, the system we get on $\tilde{u}$ may be written (see equation (6.17) below)

$$\big(D_t - P_0 - \mathcal{V}\big)\tilde{u} = \mathcal{M}_3(\tilde{u}, u^{\mathrm{app}}) + \mathcal{M}_4(\tilde{u}, u^{\mathrm{app}}) + \mathcal{M}_2'(\tilde{u}, u'^{\mathrm{app},1}) + \mathcal{R}, \tag{6.1}$$

where $\mathcal{R}$ is a remainder and the other terms in the equation have the following structure:

- Operator P_0 is just
$$P_0 = \begin{bmatrix} p(D_x) & 0 \\ 0 & -p(D_x)\end{bmatrix}. \tag{6.2}$$
- Operator $\mathcal{V}$ is a 2×2 matrix of linear operators acting on $\tilde{u}$.

Each of these operators is a pseudo-differential operator of order -1, whose coefficients depend on the approximate solution u^{app} constructed in Chapter 4. The main contribution to $\mathcal{V}$ has thus entries of the following simplified form:

$$e^{\pm it\frac{\sqrt{3}}{2}} t_\varepsilon^{-\frac{1}{2}} c(x)\langle D_x\rangle^{-1}, \tag{6.3}$$

where $c(x)$ is in $\mathcal{S}(\mathbb{R})$ and again $t_\varepsilon^{-\frac{1}{2}} = \frac{\varepsilon}{(1+t\varepsilon^2)^{1/2}}$. The left-hand side of (6.1) is thus a vectorial version of the scalar operator

$$D_t - p(D_x) - t_\varepsilon^{-\frac{1}{2}} \operatorname{Re}\big(c(x)\langle D_x\rangle^{-1} e^{it\frac{\sqrt{3}}{2}}\big). \tag{6.4}$$

We get thus a perturbation of the constant coefficients operator $p(D_x) = \sqrt{1 + D_x^2}$ by a potential term, rapidly decaying in x. We already encountered such a perturbation in Chapter 2, except that there the potential was *autonomous*. Here, it is time dependent and has some decay when t goes to infinity. Because of that, we cannot apply the results of Chapter 2 or of Appendix A to eliminate term $\mathcal{V}$ in (6.1) through conjugation. Nevertheless, one may construct by hand some wave operators for a time depending perturbation of $D_t - p(D_x)$ like the one in (6.4). That construction is made on the Fourier transform side: we introduce in Lemma 6.1.1 below a class of operators, obtained composing at the left and the right the last term in (6.4) by (inverse) Fourier transform. In Appendix E below we design "by hand" wave operators for such perturbations of $p(D_x)$, so that, conjugating (6.1) through them, we may eliminate $\mathcal{V}$ from that equation, exactly as we got rid of potential $2V$ in the second equation of (2.9) in Section 2.1 of Chapter 2 (see equation (2.17)).

The second part of this chapter is devoted to a normal form procedure allowing one to eliminate non-characteristic contributions to the quadratic, cubic and quartic terms $\mathcal{M}'_2, \mathcal{M}_3, \mathcal{M}_4$ in (6.1). Characteristic contributions are terms like $|\tilde{u}_+|^2\tilde{u}_+$ that obey a Leibniz type rule of the form

$$\|L_+(|\tilde{u}_+|^2\tilde{u}_+)\|_{L^2} \le C\|\tilde{u}_+\|^2_{W^{\rho_0,\infty}}\|L_+\tilde{u}_+\|_{L^2}$$

up to remainders. These contributions may be safely kept on the right-hand side of (6.1). The non-characteristic terms are those that do not satisfy such a Leibniz rule, and that have to be eliminated by normal form. We explained this idea on a simple model in Section 1.6 of the introduction, and gave more details in Section 2.7. In the present chapter, we apply this method to $\mathcal{M}_3, \mathcal{M}_4$ that have essentially the same structure as the models discussed there.

We have also to eliminate the quadratic term $\mathcal{M}'_2(\tilde{u}, u'^{\mathrm{app},1})$ on the right-hand side of (6.1). Since the arguments $\tilde{u}, u'^{\mathrm{app},1}$ are odd, and $\mathcal{M}'_2$ is morally of the form $a(x)\tilde{u}_\pm\tilde{u}_\pm$, with $a(x)$ rapidly decaying, one may express each factor $\tilde{u}_\pm$ using (2.65) in terms of $L_\pm\tilde{u}_\pm$ gaining a t^{-1} decay for each factor. Nevertheless, this gain is not sufficient to be able to consider $\mathcal{M}'_2$ as a remainder. One get operators of the form (2.68)–(2.69), and we explained at the end of Section 2.7 how to eliminate these expressions performing again some elementary normal form.

6.1 Expression of the equation as a system

Let us first fix some notation. From $\tilde{u}_+$, $\tilde{u}_- = -\overline{\tilde{u}_+}$, u^{app}_+, $u^{\mathrm{app}}_- = -\overline{u^{\mathrm{app}}_+}$, u'^{app}_+ and $u'^{\mathrm{app}}_- = -\overline{u'^{\mathrm{app}}_+}$, we introduce the vector-valued functions

$$\tilde{u} = \begin{bmatrix}\tilde{u}_+\\ \tilde{u}_-\end{bmatrix}, \quad u^{\mathrm{app}} = \begin{bmatrix}u^{\mathrm{app}}_+\\ u^{\mathrm{app}}_-\end{bmatrix}, \quad u'^{\mathrm{app}} = \begin{bmatrix}u'^{\mathrm{app}}_+\\ u'^{\mathrm{app}}_-\end{bmatrix}. \tag{6.5}$$

In order to write (5.95) as a system on $\tilde{u}$, let us define, when $I = \pm$,

$$b'_I(t,x,\xi) = \sum_{j=-2}^{2} e^{itj\frac{\sqrt{3}}{2}} b'_{j,I}(t,x,\xi), \tag{6.6}$$

where $b'_{j,\pm}$ satisfies (5.96), (5.97). Denoting $\bar{b}'^{\vee}_{\pm}(t,x,\xi) = \overline{b'_\pm(t,x,-\xi)}$, we define the matrix of symbols

$$M'(t,x,\xi) = \begin{bmatrix} b'_+(t,x,\xi) & b'_-(t,x,\xi)\\ -\bar{b}'^{\vee}_-(t,x,\xi) & -\bar{b}'^{\vee}_+(t,x,\xi)\end{bmatrix}. \tag{6.7}$$

As $\overline{\mathrm{Op}(b'_\pm)w} = \mathrm{Op}(\bar{b}'^{\vee}_{\pm})\bar{w}$, if we denote by $\mathrm{Op}(M')$ the quantization of M' defined entry by entry, and define $\overline{\mathrm{Op}(M')}$ by

$$\overline{\mathrm{Op}(M')}\tilde{u} = \overline{\mathrm{Op}(M')\bar{\tilde{u}}},$$

the form of M' shows that

$$\mathrm{Op}(M') = \begin{bmatrix} \mathrm{Op}(b'_+) & \mathrm{Op}(b'_-) \\ -\overline{\mathrm{Op}(b'_-)} & -\overline{\mathrm{Op}(b'_+)} \end{bmatrix} \tag{6.8}$$

or equivalently, if $N_0 = \left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right]$,

$$\overline{\mathrm{Op}(M')}N_0 + N_0 \mathrm{Op}(M') = 0. \tag{6.9}$$

If we define for $j = -2, \dots, 2$,

$$M'_j(t,x,\xi) = \begin{bmatrix} b'_{j,+}(t,x,\xi) & b'_{j,-}(t,x,\xi) \\ -\bar{b}'^{\vee}_{-j,-}(t,x,\xi) & -\bar{b}'^{\vee}_{-j,+}(t,x,\xi) \end{bmatrix},$$

we have

$$\begin{aligned} M'(t,x,\xi) &= \sum_{j=-2}^{2} e^{ijt\frac{\sqrt{3}}{2}} M'_j(t,x,\xi), \\ \overline{\mathrm{Op}(M'_j)}N_0 &+ N_0 \mathrm{Op}(M'_{-j}) = 0. \end{aligned} \tag{6.10}$$

We shall set also, if $m(x,\xi_1,\dots,\xi_n)$ is a multilinear symbol,

$$\overline{m}^{\vee}(x,\xi_1,\dots,\xi_n) = \overline{m(x,-\xi_1,\dots,-\xi_n)} \tag{6.11}$$

so that $\overline{\mathrm{Op}(m)} = \mathrm{Op}(\overline{m}^{\vee})$ if we set again

$$\overline{\mathrm{Op}(m)}(w_1,\dots,w_n) = \overline{\mathrm{Op}(m)(\overline{w}_1,\dots,\overline{w}_n)}.$$

If $I = (i_1,\dots,i_n) \in \{-,+\}^n$ and $u_I = (u_{i_1},\dots,u_{i_n})$, we denote $\bar{I} = (-i_1,\dots,-i_n)$

$$u_{\bar{I}} = (u_{-i_1},\dots,u_{-i_n}) = -(\bar{u}_{i_1},\dots,\bar{u}_{i_n}) = -\overline{u_I} \tag{6.12}$$

according to our definition $u_- = -\bar{u}_+$. Then if m_I is in $S_{\kappa,0}(M,|I|)$, we shall get that

$$\overline{\mathrm{Op}(m_I)(u_I)} = \overline{\mathrm{Op}(m_I)}(\overline{u_I}) = (-1)^{|I|}\overline{\mathrm{Op}(m_I)}(u_{\bar{I}}) = (-1)^{|I|}\mathrm{Op}(\bar{m}_I^{\vee})(u_{\bar{I}}). \tag{6.13}$$

Let us use this notation to express nonlinear quantities constructed from (5.95). We define first the quadratic terms, that will come from the right-hand side of (5.95), namely

$$\begin{aligned} \mathcal{M}'_2(\tilde{u}, u'^{\mathrm{app},1}) = & \sum_{\substack{I=(I',I'') \\ |I'|=0,\,|I''|=2}} \begin{bmatrix} \mathrm{Op}(m'_{0,I})(u'^{\mathrm{app},1}_{I''}) \\ \mathrm{Op}(\bar{m}'^{\vee}_{0,I})(u'^{\mathrm{app},1}_{\bar{I}''}) \end{bmatrix} \\ & + \sum_{\substack{I=(I',I'') \\ |I'|=|I''|=1}} \begin{bmatrix} \mathrm{Op}(m'_{0,I})(\tilde{u}_{I'}, u'^{\mathrm{app},1}_{I''}) \\ \mathrm{Op}(\bar{m}'^{\vee}_{0,I})(\tilde{u}_{\bar{I}'}, u'^{\mathrm{app},1}_{\bar{I}''}) \end{bmatrix} \\ & + \sum_{\substack{I=(I',I'') \\ |I'|=2,\,|I''|=0}} \begin{bmatrix} \mathrm{Op}(m'_{0,I})(\tilde{u}_{I'}) \\ \mathrm{Op}(\bar{m}'^{\vee}_{0,I})(\tilde{u}_{\bar{I}'}) \end{bmatrix} \end{aligned} \tag{6.14}$$

and the cubic and quartic expressions, given for $j = 3, 4$ by

$$\mathcal{M}_j(\tilde{u}, u^{\text{app}}) = \begin{bmatrix} \sum_{I=(I',I''),|I|=j} \operatorname{Op}(\tilde{m}_I)(\tilde{u}_{I'}, u^{\text{app}}_{I''}) \\ (-1)^j \sum_{I=(I',I''),|I|=j} \operatorname{Op}(\overline{\tilde{m}}^{\vee}_I)(\tilde{u}_{\bar{I}'}, u^{\text{app}}_{\bar{I}''}) \end{bmatrix}. \tag{6.15}$$

We also set

$$\mathcal{R}(t,x) = \begin{bmatrix} R_+(t,x) \\ \overline{R_+(t,x)} \end{bmatrix} \tag{6.16}$$

where R_+ is the last term in (5.95).

The system obtained taking equation (5.95) and the conjugated equation may be written as follows, denoting $\mathcal{V}$ the operator $\operatorname{Op}(M')$ given by (6.8) and P_0 the matrix of operators given by (6.2):

$$\big(D_t - P_0 - \mathcal{V}\big)\tilde{u} = \mathcal{M}_3(\tilde{u}, u^{\text{app}}) + \mathcal{M}_4(\tilde{u}, u^{\text{app}}) + \mathcal{M}_2'(\tilde{u}, u'^{\text{app},1}) + \mathcal{R}. \tag{6.17}$$

In order to apply the results of Appendix E below, we need to re-express operator $\mathcal{V}$ on the Fourier transform side.

Lemma 6.1.1. *For $j = -2, \dots, 2$, there are two by two matrices*

$$Q_j(t,\xi,\eta) = \Big[\frac{\xi}{\langle\xi\rangle}\frac{\eta}{\langle\eta\rangle} q_{j,(k,\ell)}(t,\xi,\eta)\Big]_{1\le k,\ell\le 2}$$

whose entries satisfy estimates

$$\begin{aligned} |\partial_\xi^\alpha \partial_\eta^\beta q_{j,(k,\ell)}| &\le C_N t_\varepsilon^{-\frac{1}{2}} \langle |\xi| - |\eta| \rangle^{-N} \langle\eta\rangle^{-1}, \\ |\partial_\xi^\alpha \partial_\eta^\beta \partial_t q_{j,(k,\ell)}| &\le C_N \big(t_\varepsilon^{-\frac{3}{2}} + (\varepsilon^2\sqrt{t})^{\frac{3}{2}\theta'} t^{-\frac{3}{2}}\big) \langle |\xi| - |\eta| \rangle^{-N} \langle\eta\rangle^{-1} \end{aligned} \tag{6.18}$$

for any α, β, N if $j = -1, 1$, and

$$\begin{aligned} |\partial_\xi^\alpha \partial_\eta^\beta q_{j,(k,\ell)}| &\le C_N t_\varepsilon^{-1} \langle |\xi| - |\eta| \rangle^{-N} \langle\eta\rangle^{-1}, \\ |\partial_\xi^\alpha \partial_\eta^\beta \partial_t q_{j,(k,\ell)}| &\le C_N t_\varepsilon^{-\frac{1}{2}} \big(t_\varepsilon^{-\frac{3}{2}} + (\varepsilon^2\sqrt{t})^{\frac{3}{2}\theta'} t^{-\frac{3}{2}}\big) \langle |\xi| - |\eta| \rangle^{-N} \langle\eta\rangle^{-1} \end{aligned} \tag{6.19}$$

for any α, β, N if $j = -2, 0, 2$, such that, if we define the operator K_{Q_j} by

$$\widehat{K_{Q_j} f}(\xi) = \int Q_j(t,\xi,\eta) \hat{f}(\eta)\, d\eta \tag{6.20}$$

for f a $\mathbb{C}^2$-valued function, the operator $\mathcal{V}$ acting on odd *functions may be written as*

$$\mathcal{V} = \sum_{j=-2}^{2} e^{itj\frac{\sqrt{3}}{2}} K_{Q_j}. \tag{6.21}$$

Moreover, one has $\overline{\mathcal{V}} N_0 = -N_0 \mathcal{V}$.

Proof. If $f = \left[\begin{smallmatrix} f_+ \\ f_- \end{smallmatrix}\right]$, we have according to the definition (6.8) of $\mathcal{V} = \mathrm{Op}(M')$ and (6.10)

$$\mathrm{Op}(M')f = \sum_{j=-2}^{2} e^{itj\frac{\sqrt{3}}{2}} \mathrm{Op}(M'_j)f, \tag{6.22}$$

$$\mathrm{Op}(M'_j)f = \begin{bmatrix} \mathrm{Op}(b'_{j,+})f_+ + \mathrm{Op}(b'_{j,-})f_- \\ -\mathrm{Op}(\bar{b}'^{\vee}_{-j,-})f_+ - \mathrm{Op}(\bar{b}'^{\vee}_{-j,+})f_- \end{bmatrix}. \tag{6.23}$$

The Fourier transform of the first line of (6.23) may be written

$$\int \hat{b}'_{j,+}(t,\xi-\eta,\eta)\hat{f}_+(\eta)\,d\eta + \int \hat{b}'_{j,-}(t,\xi-\eta,\eta)\hat{f}_-(\eta)\,d\eta, \tag{6.24}$$

where $\hat{b}'_{j,\pm}$ is the Fourier transform relatively to the first variable. Since $b'_{j,\pm}$ satisfies (3.7), if we set

$$\tilde{q}_{j,(1,1)}(t,\xi,\eta) = \hat{b}'_{j,+}(t,\xi-\eta,\eta),\ \tilde{q}_{j,(1,2)}(t,\xi,\eta) = \hat{b}'_{j,-}(t,\xi-\eta,\eta),$$

we see that $\tilde{q}_{j,(k,\ell)}(t,-\xi,-\eta) = \tilde{q}_{j,(k,\ell)}(t,\xi,\eta)$. If we make act (6.24) on odd functions f_+, f_-, we may rewrite this expression as the sum for $(k,\ell) = (1,1)$ or $(1,2)$ of

$$\frac{1}{2}\int \big(\tilde{q}_{j,(k,\ell)}(t,\xi,\eta) - \tilde{q}_{j,(k,\ell)}(t,\xi,-\eta)\big)\hat{f}_\pm(\eta)\,d\eta$$

(with f_+ if $(k,\ell) = (1,1)$ and f_- if $(k,\ell) = (1,2)$). In other words, we may assume that $\tilde{q}_{j,(1,1)}(t,\xi,\eta)$ is odd in η. Since that function is even in (ξ,η), it has also to be odd in ξ. By (5.96)–(5.97), $x \mapsto b'_j(t,x,\eta)$ is in $\mathcal{S}(\mathbb{R})$, and the function is C^∞ in η. It follows that the Fourier transform in x of these functions satisfies

$$|\partial_\xi^\alpha \partial_\eta^\beta \partial_t^{\ell-1} \hat{b}'_{j,I}(t,\xi-\eta,\eta)| \leq C_{\alpha,\beta,N} \mathcal{T}_j^\ell(t,\varepsilon)\langle|\xi|-|\eta|\rangle^{-N}\langle\eta\rangle^{-1}$$

for any α, β, N, $\ell = 1,2$, where $\mathcal{T}_j^\ell(t,\varepsilon)$ is the time dependent pre-factor in the ℓ-th equation in (5.96) (resp. (5.97)). After the preceding reductions, it follows that $\tilde{q}_{j,(k,\ell)}$ satisfies for all $\alpha, \beta, N \in \mathbb{N}$, $\ell = 1,2$,

$$|\partial_\xi^\alpha \partial_\eta^\beta \partial_t^{\ell-1} \tilde{q}_{j,(k,\ell)}(t,\xi,\eta)| \leq C_{\alpha,\beta,N} \mathcal{T}_j^\ell(t,\varepsilon)\langle|\xi|-|\eta|\rangle^{-N}\langle\eta\rangle^{-1}.$$

Since we have seen that this function is odd in ξ and odd in η, we may write it as $\frac{\xi}{\langle\xi\rangle}\frac{\eta}{\langle\eta\rangle} q_{j,(k,\ell)}(t,\xi,\eta)$, where $q_{j,(k,\ell)}$ satisfies (6.18)–(6.19). It follows that we have written the first component of the Fourier transform $\widehat{\mathcal{V}f}$ of (6.22) as the first component of $\sum_{j=-2}^{2} e^{itj\sqrt{3}/2}\widehat{K_{Q_j}f}(\xi)$. Since the reasoning is the same for the second component, we get (6.21).

The last statement of the lemma follows from (6.9). ■

We may now eliminate the operator $\mathcal{V}$ on the left-hand side of (6.17), using the results of Appendix E.

Proposition 6.1.2. *Fix m in $]0, \frac{1}{2}[$ close to $\frac{1}{2}$, and set as in the example following Definition E.1.1, $\iota = \min(1-2m, \frac{3}{4}c\theta') > 0$. There is $\varepsilon_0 > 0$ such that, for any $\mathcal{V}$ of the form (6.21), defined in terms of matrices Q_j whose coefficients satisfy (6.18) and (6.19), with $\varepsilon \in]0, \varepsilon_0[$, there are operators $B(t)$, $C(t)$, defined for $t \in [1, T]$ ($T \leq \varepsilon^{-4+c}$), bounded on $H^s(\mathbb{R})$, satisfying the properties of Propositions E.1.1 and E.1.3 of Appendix E, such that, if $\tilde{u}$ solves (6.17) and satisfies estimates (5.35), then $C(t)\tilde{u}$ solves*

$$(D_t - P_0)C(t)\tilde{u} = C(t)\mathcal{M}_3(\tilde{u}, u^{\mathrm{app}}) + C(t)\mathcal{M}_4(\tilde{u}, u^{\mathrm{app}}) + C(t)\mathcal{M}_2'(\tilde{u}, u'^{\mathrm{app},1}) + C(t)\mathcal{R} \tag{6.25}$$

with $\mathcal{R}$ satisfying for any $t \in [1, T]$,

$$\|\mathcal{R}(t, \cdot)\|_{H^s} \leq \varepsilon t^{\delta-1} e(t, \varepsilon), \tag{6.26}$$

$$\|L\mathcal{R}(t, \cdot)\|_{H^s} \leq t^{-\frac{3}{4}}(\varepsilon^2\sqrt{t})^\theta e(t, \varepsilon), \tag{6.27}$$

where e satisfies (5.41). Moreover, $C(t)\tilde{u}$ is odd if $\tilde{u}$ is odd and $N_0 C(t)\tilde{u} = -\overline{C(t)\tilde{u}}$.

Proof. By (E.9), it holds $(D_t - P_0 - \mathcal{V})B(t) = B(t)(D_t - P_0)$ and by (E.14), we have $\tilde{u} = B(t)C(t)\tilde{u}$. Replacing $\tilde{u}$ by this value on the left-hand side of (6.17), composing at the left with $C(t)$ and using again (E.14), we obtain (6.25). Since $\mathcal{V}(t)$ preserves odd functions and satisfies $\overline{\mathcal{V}(t)}N_0 = -N_0\mathcal{V}(t)$, the last statement of the proposition follows from (E.23) and the fact that $N_0\tilde{u} = -\overline{\tilde{u}}$. This concludes the proof, as estimates (6.26) and (6.27) are just rewriting of (5.39) and (5.40). ∎

6.2 Normal forms

Our next objective will be to eliminate by normal forms most of the contributions on the right-hand side of (6.25). We shall construct first the relevant operators in order to do so.

Let us fix some notation. Let n be in $\mathbb{N}^*$. Consider $\mathbb{C}^2$-valued test functions v_j, defined on $[1, T] \times \mathbb{R}$ for some T, of the form

$$(t, x) \mapsto v_j(t, x) = \begin{bmatrix} v_{j,+}(t, x) \\ v_{j,-}(t, x) \end{bmatrix} \tag{6.28}$$

with $v_{j,\pm}$ odd in x and satisfying $v_{j,-} = -\overline{v_{j,+}}$. If $n \geq 3$, we shall consider n-linear maps

$$(v_1, \dots, v_n) \mapsto \tilde{\mathcal{M}}_j(v_1, \dots, v_n) \tag{6.29}$$

sending $\mathbb{C}^2$-valued functions to $\mathbb{C}^2$-valued functions and having the following structure (using notation (B.17)):

$$\tilde{\mathcal{M}}_n(v_1, \dots, v_n) = \begin{bmatrix} \sum_{|I|=n} \mathrm{Op}^t(\tilde{m}_I)(v_{1,i_1}, \dots, v_{n,i_n}) \\ (-1)^n \sum_{|I|=n} \mathrm{Op}^t(\overline{\tilde{m}_I^\vee})(v_{1,-i_1}, \dots, v_{n,-i_n}) \end{bmatrix}, \tag{6.30}$$

where $I = (i_1, \dots, i_n) \in \{-,+\}^n$, $\tilde{m}_I$ is in $S_{1,\beta}(M_0^\nu \prod_{j=1}^n \langle\xi_j\rangle^{-1}, n)$ for some $\beta > 0$ small, $\nu \in \mathbb{N}$, where $\overline{\tilde{m}_I^\vee}$ is defined by (6.11), and where the form of the second line of (6.30) versus the first one just reflects the fact that $\mathcal{M}_n(v_1, \dots, v_n)$ will have a structure with respect to conjugation similar to the one in (6.14) or (6.15) (see (6.13)). Moreover, we assume that $\tilde{m}_I$ satisfies

$$\tilde{m}(y, x, \xi_1, \dots, \xi_n) = (-1)^{n-1}\tilde{m}(-y, -x, -\xi_1, \dots, -\xi_n) \tag{6.31}$$

so that the associated operator preserves odd functions (see (3.7)).

Proposition 6.2.1. *Let $n \geq 3$. For any I with $|I| = n$ one may find symbols $\hat{m}_I$ in $S_{4,\beta}(M_0^\nu \prod_{j=1}^n \langle\xi_j\rangle^{-1}\langle x\rangle^{-\infty}, n)$ such that, if one sets*

$$\hat{\tilde{\mathcal{M}}}_n(v_1, \dots, v_n) = \begin{bmatrix} \sum_{|I|=n} \mathrm{Op}^t(\hat{m}_I)(v_{1,i_1}, \dots, v_{n,i_n}) \\ (-1)^n \sum_{|I|=n} \mathrm{Op}^t(\overline{\hat{m}_I^\vee})(v_{1,-i_1}, \dots, v_{n,-i_n}) \end{bmatrix} \tag{6.32}$$

one may write

$$\begin{aligned} R_n(v_1, \dots, v_n) &\stackrel{\text{def}}{=} (D_t - P_0)\hat{\tilde{\mathcal{M}}}_n(v_1, \dots, v_n) - \tilde{\mathcal{M}}(v_1, \dots, v_n) \\ &\quad - \sum_{j=1}^n \hat{\tilde{\mathcal{M}}}_n(v_1, \dots, (D_t - P_0)v_j, \dots, v_n) \end{aligned} \tag{6.33}$$

under the following form:

$$R_n(v_1, \dots, v_n) = \begin{bmatrix} R_{n,+}(v_1, \dots, v_n) \\ R_{n,-}(v_1, \dots, v_n) \end{bmatrix} \tag{6.34}$$

with $R_{n,-} = \overline{R_{n,+}}$, and $R_{n,+}$ satisfies the following: One may write $R_{n,+}(v_1, \dots, v_n)$ as a sum

$$R_{n,+}(v_1, \dots, v_n) = \sum_{|I|=n} \mathrm{Op}^t(r_I)(v_{1,i_1}, \dots, v_{n,i_n}) \tag{6.35}$$

with symbols r_I in the class $S_{4,\beta}(M_0^\nu \prod_{j=1}^n \langle\xi_j\rangle^{-1}, n)$ for some $\nu \in \mathbb{N}$. Moreover, $L_+ R_{n,+}(v_1, \dots, v_n)$ may be written as a sum of terms of the following form:

$$\sum_{|I|=n} \sum_{j=1}^n \mathrm{Op}^t(r_{I,j})(v_{1,i_1}, \dots, L_{i_j} v_{j,i_j}, \dots, v_{n,i_n}) \tag{6.36}$$

with $r_{I,j}$ in $S_{4,\beta}(M_0^\nu \prod_{j=1}^n \langle\xi_j\rangle^{-1}, n)$,

$$\sum_{|I|=n} \mathrm{Op}^t(r_I)(v_{1,i_1}, \dots, v_{n,i_n}) \tag{6.37}$$

for symbols r_I in $S_{4,\beta}(M_0^\nu \prod_{j=1}^n \langle\xi_j\rangle^{-1}, n)$, and

$$t \sum_{|I|=n} \mathrm{Op}^t(r'_I)(v_{1,i_1}, \dots, v_{n,i_n}) \tag{6.38}$$

for symbols r_I' in $S'_{4,\beta}(M_0^\nu \prod_{j=1}^n \langle\xi_j\rangle^{-1}, n)$. Moreover, $\hat{m}_I$ satisfies

$$\hat{m}_I(-y, -x, -\xi_1, \dots, -\xi_n) = (-1)^{n-1}\hat{m}_I(y, x, \xi_1, \dots, \xi_n) \tag{6.39}$$

if $\tilde{m}_I$ does so in (6.30).

We shall prove the proposition expressing (6.33) in terms of the semiclassical quantization of symbols introduced in (B.14) in Appendix B. If $h = \frac{1}{t}$, we introduce for any function v_j, $j = 1, \dots, n$, the function $\underline{v}_j$ defined by

$$v_j(t,x) = \frac{1}{\sqrt{t}}\underline{v}_j\Big(t, \frac{x}{t}\Big) = \Theta_t \underline{v}_j(t,x) \tag{6.40}$$

according to (B.15). By (B.16), each term on the first line of (6.30) may be written

$$\mathrm{Op}^t(\tilde{m}_I)(v_{1,i_1}, \dots, v_{n,i_n})(t,x) = h^{\frac{n}{2}}\mathrm{Op}_h(\tilde{m}_I)(\underline{v}_{1,i_1}, \dots, \underline{v}_{n,i_n})\Big(t, \frac{x}{t}\Big) \tag{6.41}$$

and similarly for the first line of (6.32). The first line on the right-hand side of (6.33) may be written as the sum in I of

$$\begin{aligned} &(D_t - p(D_x))\mathrm{Op}^t(\hat{m}_I)(v_{1,i_1}, \dots, v_{n,i_n}) - \mathrm{Op}^t(\tilde{m}_I)(v_{1,i_1}, \dots, v_{n,i_n}) \\ &\quad - \sum_{j=1}^n \mathrm{Op}^t(\hat{m}_I)(v_{1,i_1}, \dots, (D_t - i_j p(D_x))v_{j,i_j}, \dots, v_{n,i_n}). \end{aligned} \tag{6.42}$$

It follows from (6.41) that the first term in (6.42) may be written as

$$h^{\frac{n}{2}}\Big(D_t - \mathrm{Op}_h\Big(x\xi + p(\xi) - i\frac{n}{2}h\Big)\Big)(\mathrm{Op}_h(\hat{m}_I)(\underline{v}_{1,i_1}, \dots, \underline{v}_{n,i_n}))\Big(t, \frac{x}{t}\Big).$$

The other terms in (6.42) admit analogous expressions, so that (6.42) may be rewritten as $h^{\frac{n}{2}}\underline{R}^I_{n,+}(\underline{v}_{1,i_1}, \dots, \underline{v}_{n,i_n})(t, \frac{x}{t})$ with

$$\begin{aligned} &\underline{R}^I_{n,+}(\underline{v}_{1,i_1}, \dots, \underline{v}_{n,i_n})(t,x) \\ &\quad = \Big(D_t - \mathrm{Op}_h\Big(x\xi + p(\xi) - i\frac{n}{2}h\Big)\Big)(\mathrm{Op}_h(\hat{m}_I)(\underline{v}_{1,i_1}, \dots, \underline{v}_{n,i_n})) \\ &\qquad - \mathrm{Op}_h(\tilde{m}_I)(\underline{v}_{1,i_1}, \dots, \underline{v}_{n,i_n}) \\ &\qquad - \sum_{j=1}^n \mathrm{Op}_h(\hat{m}_I)\Big[\underline{v}_{1,i_1}, \dots, \Big(D_t - \mathrm{Op}_h\Big(x\xi + i_j p(\xi) - i\frac{h}{2}\Big)\Big)\underline{v}_{i,i_j}, \\ &\qquad\qquad \dots, \underline{v}_{n,i_n}\Big]. \end{aligned} \tag{6.43}$$

We shall study (6.43) both when I is characteristic and I is non-characteristic, according to the terminology introduced in Definition F.1.1, that we recall in the statements of the following two lemmas.

Lemma 6.2.2. *Let $I = (i_1, \dots, i_n)$ be characteristic, i.e. $i_1 + \dots + i_n = 1$, and take $\hat{m}_I = 0$ in* (6.43)*. Then if $\mathcal{L}_\pm = \frac{1}{h}\mathrm{Op}_h(x \pm p'(\xi))$, the term $\mathcal{L}_\pm \underline{R}^I_{n,+}(\underline{v}_{1,i_1}, \dots, \underline{v}_{n,i_n})$*

may be written as a sum of the following expressions:

$$\begin{aligned} &\mathrm{Op}_h(r_{I,j})(\underline{v}_{1,i_1},\dots,\mathcal{L}_{i_j}\underline{v}_{j,i_j},\dots,\underline{v}_{n,i_n}),\\ &\mathrm{Op}_h(r_I)(\underline{v}_{1,i_1},\dots,\underline{v}_{n,i_n}),\\ &\frac{1}{h}\mathrm{Op}_h(r'_I)(\underline{v}_{1,i_1},\dots,\underline{v}_{n,i_n}) \end{aligned} \tag{6.44}$$

with $r_{I,j}, r_I$ *in* $S_{4,\beta}(M_0^\nu \prod_{j=1}^n \langle\xi_j\rangle^{-1}, n)$ *and* r'_I *in* $S'_{4,\beta}(M_0^\nu \prod_{j=1}^n \langle\xi_j\rangle^{-1}, n)$ *for some* ν.

Proof. We just have to apply Proposition F.2.1 of Appendix F. ∎

We shall consider next the case of non-characteristic indices.

Lemma 6.2.3. *Let* $I = (i_1,\dots,i_n)$ *be non-characteristic, i.e.* $i_1+\dots+i_n \neq 1$. *Then one may find a symbol* $\hat{m}_I$ *in* $S_{4,\beta}(M_0^\nu \prod_{j=1}^n \langle\xi_j\rangle^{-1}\langle x\rangle^{-\infty}, n)$, *for some* ν, *such that* $\underline{R}^I_{n,+}(\underline{v}_{1,i_1},\dots,\underline{v}_{n,i_n})$ *given by* (6.43) *may be written as a sum of terms*

$$\begin{aligned} &\mathrm{Op}_h(r_I^1)(\underline{v}_{1,i_1},\dots,\underline{v}_{n,i_n}),\\ &h\mathrm{Op}_h(r_I)(\underline{v}_{1,i_1},\dots,\underline{v}_{n,i_n}),\\ &\mathrm{Op}_h(r'_I)(\underline{v}_{1,i_1},\dots,\underline{v}_{n,i_n}) \end{aligned} \tag{6.45}$$

with symbols r_I^1 *in* $S_{4,\beta}(M_0^\nu \prod_{j=1}^n \langle\xi_j\rangle^{-1}, n)$, r_I *in* $S_{4,\beta}(M_0^\nu \prod_{j=1}^n \langle\xi_j\rangle^{-1}\langle x\rangle^{-1}, n)$, *and* r'_I *in* $S'_{4,\beta}(M_0^\nu \prod_{j=1}^n \langle\xi_j\rangle^{-1}, n)$. *Moreover,* $\mathcal{L}_+\underline{R}^I_{n,+}(\underline{v}_{1,i_1},\dots,\underline{v}_{n,i_n})$ *may be written under the form* (6.44) *and* $\hat{m}_I$ *satisfies* (6.39) *if* $\tilde{m}_I$ *does so.*

Proof. We apply Proposition F.3.1 and define $\hat{m}_I$ to be the symbol a_I of that statement, that satisfies (F.7). According to (F.20) (with m_I replaced by $\tilde{m}_I$ in its right-hand side), (6.43) may be written as the sum of (F.22) and of the last two lines in (F.21). This gives (6.45).

To get the last statement of the lemma, we use that $\underline{R}^I_{n,+}$ is also given by (F.21). We have thus to show that the action of $\mathcal{L}_+ = \frac{1}{h}\mathrm{Op}_h(x + p'(\xi))$ on the three terms in (F.21) may be rewritten under the form (6.44). For $\frac{1}{h}\mathrm{Op}_h(p'(\xi))$ this follows from the composition result of Proposition B.2.1. For the product of $\frac{x}{h}$ by (F.21), this is a consequence of the fact that in these formulas $m_{I,j}$ and r_I are in classes $S_{4,\beta}(M_0^\nu \prod_{j=1}^n \langle\xi_j\rangle^{-1}\langle x\rangle^{-1}, n)$. In the case of r'_I, the fact that the symbol belongs to the class $S'_{4,\beta}(M_0^\nu \prod_{j=1}^n \langle\xi_j\rangle^{-1}, n)$ means that it is rapidly decaying in $M_0(\xi)^{-4}|y|$, so may be multiplied by x (and even by x/h), up to a loss on the exponent ν. This concludes the proof since the definition (F.9) of a_I (with m_I replaced by $\tilde{m}_I$) shows that it satisfies (6.39) if $\tilde{m}_I$ does (taking the cut-off γ even). ∎

Proof of Proposition 6.2.1. We just have to translate the above two lemmas going back to functions $v_1,\dots,v_n$ from $\underline{v}_1,\dots,\underline{v}_n$ through (6.40). The first component $R_{n,+}$ of (6.33) is then $h^{\frac{n}{2}}\underline{R}^I_{n,+}(\underline{v}_{1,i_1},\dots,\underline{v}_{n,i_n})$ with $\underline{R}^I_{n,+}$ given by (6.43). In the characteristic case, (6.43) with $\hat{m}_I = 0$ and (6.41) show that equation (6.35) holds, and Lemma 6.2.2 implies that $L_+ R_{n,+}$ is of the form (6.36). In the non-characteristic case, these properties follow from Lemma 6.2.3. ■

Proposition 6.2.1 will allow us to treat by normal form the contributions $\mathcal{M}_3, \mathcal{M}_4$ on the right-hand side of (6.25). We need also a result that will allow us to treat $\mathcal{M}_2'$.

We consider a bilinear map $(v_1, v_2) \mapsto \tilde{\mathcal{M}}_2'(v_1, v_2)$ of the form

$$\tilde{\mathcal{M}}_2'(v_1, v_2) = \begin{bmatrix} \sum_{|I|=2} \mathrm{Op}(m'_{0,I})(v_{1,i_1}, v_{2,i_2}) \\ \sum_{|I|=2} \mathrm{Op}(\bar{m}'^{\vee}_{0,I})(v_{1,-i_1}, v_{2,-i_2}) \end{bmatrix}, \tag{6.46}$$

where $m'_{0,I}$ is in $\tilde{S}'_{1,0}(\prod_{j=1}^2 \langle \xi_j \rangle^{-1} M_0(\xi), 2)$ and satisfies (3.7). Our goal is to prove:

Proposition 6.2.4. *One may find an operator* $(v_1, v_2) \mapsto \hat{\tilde{\mathcal{M}}}_2'(v_1, v_2)$*, that may be written*

$$\hat{\tilde{\mathcal{M}}}_2'(v_1, v_2) = \begin{bmatrix} \sum \sum_{(i_1,i_2)\in\{-,+\}^2} Q_{i_1,i_2}(v_{1,i_1}, v_{2,i_2}) \\ \sum \sum_{(i_1,i_2)\in\{-,+\}^2} \overline{Q_{i_1,i_2}(v_{1,i_1}, v_{2,i_2})} \end{bmatrix} \tag{6.47}$$

with operators $Q_{i_1,i_2}(v_{1,i_1}, v_{2,i_2})$ *of the form* (F.35)*, preserving the space of odd functions, such that, if we set*

$$\begin{aligned} R_2(v_1, v_2) = (D_t - P_0)\hat{\tilde{\mathcal{M}}}_2'(v_1, v_2) - \tilde{\mathcal{M}}_2'(v_1, v_2) - \hat{\tilde{\mathcal{M}}}_2'\big((D_t - P_0)v_1, v_2\big) \\ - \hat{\tilde{\mathcal{M}}}_2'\big(v_1, (D_t - P_0)v_2\big) \end{aligned} \tag{6.48}$$

and if v_1, v_2 *are odd functions, then* $R_2 = \left[\begin{smallmatrix} R_{2,+} \\ R_{2,-} \end{smallmatrix}\right]$ *with* $R_{2,-} = \overline{R_{2,+}}$ *and* $R_{2,+}$ *being a sum*

$$R_{2,+}(v_1, v_2) = t^{-2} \sum_{(i_1,i_2)\in\{-,+\}^2} \sum_{\ell_1=0}^{1} \sum_{\ell_2=0}^{1} K^{\ell_1,\ell_2}_{L,i_1,i_2}\big(L^{\ell_1}_{i_1} v_{1,i_1}, L^{\ell_2}_{i_2} v_{2,i_2}\big) \tag{6.49}$$

with $K^{\ell_1,\ell_2}_{L,i_1,i_2}$ *in the class* $\mathcal{K}'_{1,\frac{1}{2}}(1, i_1, i_2)$ *of Definition* F.4.1.

Proof. We just have to apply Corollary F.4.4 to the first component of equality (6.48) changing the definition of the notation $K^{\ell_1,\ell_2}_{L,i_1,i_2}$ on the right-hand side of (6.49). ■

We shall use the results established so far in that section in order to rewrite equation (6.25). Recall first that by (E.8), (E.9), (E.14), where $\mathcal{V}$ is the operator (6.21), we have

$$(D_t - P_0)C(t) = C(t)(D_t - P_0 - \mathcal{V}) \tag{6.50}$$

when both sides of these equalities act on odd functions.

Recall the form of operators $\mathcal{M}_j$ in (6.15): these operators may be written as

$$\mathcal{M}_j(\tilde{u}, u^{\mathrm{app}}) = \sum_{\ell=0}^{j} \mathcal{M}_j^{\ell}(\underbrace{\tilde{u}, \dots, \tilde{u}}_{\ell}, \underbrace{u^{\mathrm{app}}, \dots, u^{\mathrm{app}}}_{j-\ell}), \quad j = 3, 4, \tag{6.51}$$

where

$$\mathcal{M}_j^\ell(v_1,\dots,v_j) = \begin{bmatrix} \sum_{\substack{I'=(i_1,\dots,i_\ell)\\ I''=(i_{\ell+1},\dots,i_j)}} \mathrm{Op}(\tilde{m}_{I',I''})(v_{1,i_1},\dots,v_{j,i_j}) \\ \sum_{\substack{I'=(i_1,\dots,i_\ell)\\ I''=(i_{\ell+1},\dots,i_j)}} (-1)^j \mathrm{Op}(\overline{\tilde{m}}^\vee_{I',I''})(v_{1,-i_1},\dots,v_{j,-i_j}) \end{bmatrix} \tag{6.52}$$

and the symbols $\tilde{m}_{I',I''}$ are in $\tilde{S}_{1,0}(\prod_{j=1}^{|I|}\langle\xi_j\rangle^{-1}M_0(\xi)^\nu,|I|)$, with $3\le|I|=j\le 4$, according to Proposition 5.2.1. According to Corollary D.1.7, each of these symbols may be replaced by a symbol in $S_{1,\beta}(\prod_{j=1}^{|I|}\langle\xi_j\rangle^{-1}M_0(\xi)^\nu,|I|)$, for $\beta>0$ small, up to adding to (6.51) some remainder satisfying (D.35) for an arbitrary r. In other words, we may rewrite (6.51) under the form

$$\mathcal{M}_j(\tilde{u},u^{\mathrm{app}}) = \sum_{\ell=0}^{j} \mathcal{M}_j^\ell(\tilde{u},\dots,\tilde{u},u^{\mathrm{app}},\dots,u^{\mathrm{app}}) + \tilde{\mathcal{R}}_j(\tilde{u},u^{\mathrm{app}}), \tag{6.53}$$

where $\mathcal{M}_j^\ell$ is of the form (6.52) with symbols $\tilde{m}_{I',I''}$ in

$$S_{1,\beta}\Bigg(\prod_{j=1}^{|I|}\langle\xi_j\rangle^{-1}M_0(\xi)^\nu,|I|\Bigg),$$

with $\beta>0$ and where $\tilde{\mathcal{R}}_j$ satisfies

$$\|\tilde{\mathcal{R}}_j(\tilde{u},u^{\mathrm{app}})\|_{H^s} \le Ct^{-2}\big(\|\tilde{u}\|_{H^s}+\|u^{\mathrm{app}}\|_{H^s}\big)^j \tag{6.54}$$

and setting $L=\left[\begin{smallmatrix} L_+ & 0\\ 0 & L_-\end{smallmatrix}\right]$,

$$\begin{aligned}\|L\tilde{\mathcal{R}}_j(\tilde{u},u^{\mathrm{app}})\|_{L^2} \le Ct^{-2}&\big(\|\tilde{u}\|_{H^s}+\|u^{\mathrm{app}}\|_{H^s}\big)^{j-1}\\ &\times\big(\|\tilde{u}\|_{H^s}+\|u^{\mathrm{app}}\|_{H^s}+\|L\tilde{u}\|_{L^2}\\ &\qquad+\|Lu'^{\mathrm{app}}\|_{L^2}+\|Lu''^{\mathrm{app}}\|_{W^{\rho_0,\infty}}\big),\end{aligned} \tag{6.55}$$

where in (6.55), we decomposed the factor u^{app} that eventually replaces v_n in (D.35) as $u^{\mathrm{app}}=u'^{\mathrm{app}}+u''^{\mathrm{app}}$, and used the second (resp. third) of these estimates if v_n is substituted by u'^{app} (resp. u''^{app}).

In the same way, operators $\mathcal{M}_2'$ in (6.14) may be written as

$$\mathcal{M}_2'(\tilde{u},u'^{\mathrm{app},1}) = \mathcal{M}'^0_2(u'^{\mathrm{app},1},u'^{\mathrm{app},1}) + \mathcal{M}'^1_2(\tilde{u},u'^{\mathrm{app},1}) + \mathcal{M}'^2_2(\tilde{u},\tilde{u}), \tag{6.56}$$

where $\mathcal{M}'^\ell_2$ is given by the $(\ell+1)$-st contribution in (6.14). Applying again Corollary D.1.7, we may assume that

$$\mathcal{M}'^\ell_2(v_1,v_2) = \begin{bmatrix} \sum_{\substack{I'=(i_1,\dots,i_\ell)\\ I''=(i_{\ell+1},\dots,i_j)}} \mathrm{Op}(m'_{0,I',I''})(v_{1,i_1},v_{2,i_2}) \\ \sum_{\substack{I'=(i_1,\dots,i_\ell)\\ I''=(i_{\ell+1},\dots,i_j)}} \mathrm{Op}(\bar{m}'^\vee_{0,I',I''})(v_{1,-i_1},v_{2,-i_2}) \end{bmatrix} \tag{6.57}$$

up to replacing (6.56) by

$$\mathcal{M}_2'(\tilde{u}, u'^{\mathrm{app},1}) = \mathcal{M}'^0_2(u'^{\mathrm{app},1}, u'^{\mathrm{app},1}) + \mathcal{M}'^1_2(\tilde{u}, u'^{\mathrm{app},1}) + \mathcal{M}'^2_2(\tilde{u}, \tilde{u}) + \tilde{\mathcal{R}}_2(\tilde{u}, u'^{\mathrm{app},1}), \tag{6.58}$$

where $\tilde{\mathcal{R}}_2$ satisfies

$$\begin{aligned} \|\tilde{\mathcal{R}}_2(\tilde{u}, u'^{\mathrm{app},1})\|_{H^s} &\le Ct^{-2}\big(\|\tilde{u}\|_{H^s} + \|u'^{\mathrm{app},1}\|_{H^s}\big)^2, \\ \|L\tilde{\mathcal{R}}_2(\tilde{u}, u'^{\mathrm{app},1})\|_{L^2} &\le Ct^{-2}\big(\|\tilde{u}\|_{H^s} + \|u'^{\mathrm{app},1}\|_{H^s}\big) \\ &\quad \times \big(\|\tilde{u}\|_{H^s} + \|u'^{\mathrm{app},1}\|_{H^s} \\ &\quad + \|L\tilde{u}\|_{L^2} + \|Lu'^{\mathrm{app},1}\|_{L^2}\big) \end{aligned} \tag{6.59}$$

and where the symbols $m'_{0,I',I''}$ in (6.57) are now in $S'_{1,\beta}\Big(\prod_{j=1}^2 \langle \xi_j \rangle^{-1} M_0(\xi), 2\Big)$ for some $\beta > 0$.

Let us apply to each $\mathcal{M}_j^\ell$ on the right-hand side of (6.53) Proposition 6.2.1 setting $\tilde{\mathcal{M}}_j = \mathcal{M}_j^\ell$ in order to define by (6.32) an operator $\hat{\tilde{\mathcal{M}}}_j$ that we denote just by $\hat{\mathcal{M}}_j^\ell$, $0 \le \ell \le j$, $j = 3, 4$. In the same way, apply to each $\mathcal{M}'^\ell_2$, $\ell = 0, 1, 2$ Proposition 6.2.4 in order to define operators $\hat{\mathcal{M}}'^\ell_2$, $\ell = 0, 1, 2$. Denote

$$\begin{aligned} \hat{\mathcal{M}}_j(\tilde{u}, u^{\mathrm{app}}) &= \sum_{\ell=0}^{j} \hat{\mathcal{M}}_j^\ell(\underbrace{\tilde{u}, \dots, \tilde{u}}_{\ell}, \underbrace{u^{\mathrm{app}}, \dots, u^{\mathrm{app}}}_{j-\ell}), \quad j = 3, 4, \\ \hat{\mathcal{M}}_2'(\tilde{u}, u'^{\mathrm{app},1}) &= \sum_{\ell=0}^{2} \hat{\mathcal{M}}'^\ell_2(\underbrace{\tilde{u}, \dots, \tilde{u}}_{\ell}, \underbrace{u'^{\mathrm{app},1}, \dots, u'^{\mathrm{app},1}}_{2-\ell}). \end{aligned} \tag{6.60}$$

Let us prove:

Corollary 6.2.5. *Let $\tilde{u}$ satisfy the assumptions of Proposition 6.1.2, so that equation (6.25) holds. Then, with the above notation,*

$$(D_t - P_0)\bigg(C(t)\Big(\tilde{u} - \sum_{j=3}^{4} \hat{\mathcal{M}}_j(\tilde{u}, u^{\mathrm{app}})\Big) - \hat{\mathcal{M}}_2'(\tilde{u}, u'^{\mathrm{app},1})\bigg) = \hat{\mathcal{R}}, \tag{6.61}$$

where $\hat{\mathcal{R}}$ is the sum of contributions of the following form:

$$C(t)\mathcal{V}(t)\hat{\mathcal{M}}_j^\ell(\underbrace{\tilde{u}, \dots, \tilde{u}}_{\ell}, \underbrace{u^{\mathrm{app}}, \dots, u^{\mathrm{app}}}_{j-\ell}), \qquad j = 3, 4,\ 0 \le \ell \le j \tag{6.62}$$

$$\big(C(t) - \mathrm{Id}\big)\mathcal{M}'^\ell_2(\underbrace{\tilde{u}, \dots, \tilde{u}}_{\ell}, \underbrace{u'^{\mathrm{app},1}, \dots, u'^{\mathrm{app},1}}_{2-\ell}), \quad 0 \le \ell \le 2, \tag{6.63}$$

$$\begin{aligned} &-C(t)\hat{\mathcal{M}}_j^\ell(\underbrace{\tilde{u}, \dots, \tilde{u}, (D_t - P_0)\tilde{u}, \dots, \tilde{u}}_{\ell}, u^{\mathrm{app}}, \dots, u^{\mathrm{app}}), \\ &-C(t)\hat{\mathcal{M}}_j^\ell(\underbrace{\tilde{u}, \dots, \tilde{u}}_{\ell}, u^{\mathrm{app}}, \dots, u^{\mathrm{app}}, (D_t - P_0)u^{\mathrm{app}}, \dots, u^{\mathrm{app}}) \end{aligned} \tag{6.64}$$

for $j = 3, 4$, $0 \le \ell \le j$,

$$\begin{aligned} &-C(t)\hat{\mathcal{M}}'^{\ell}_2(\underbrace{\tilde{u}, \dots, (D_t - P_0)\tilde{u}, \dots, \tilde{u}}_{\ell}, u'^{\text{app},1}, \dots, u'^{\text{app},1}),\\ &-C(t)\hat{\mathcal{M}}'^{\ell}_2(\underbrace{\tilde{u}, \dots, \tilde{u}}_{\ell}, u'^{\text{app},1}, \dots, (D_t - P_0)u'^{\text{app},1}, \dots, u'^{\text{app},1}) \end{aligned} \tag{6.65}$$

for $0 \le \ell \le 2$, *of remainders of type*

$$C(t)R_j(\underbrace{\tilde{u}, \dots, \tilde{u}}_{\ell}, \underbrace{u^{\text{app}}, \dots, u^{\text{app}}}_{j-\ell}), \quad j = 3, 4, \ 0 \le \ell \le j, \tag{6.66}$$

where R_j *is of the form* (6.34) *and*

$$R_2(\underbrace{\tilde{u}, \dots, \tilde{u}}_{\ell}, \underbrace{u'^{\text{app},1}, \dots, u'^{\text{app},1}}_{2-\ell}), \quad 0 \le \ell \le 2, \tag{6.67}$$

where $R_2 = \begin{bmatrix} R_{2,+} \\ R_{2,-} \end{bmatrix}$ *with* $R_{2,-} = \overline{R_{2,+}}$, *and* $R_{2,+}$ *given by* (6.49), *and of contributions*

$$C(t)\big(\mathcal{R}(t,x) + \tilde{\mathcal{R}}_3 + \tilde{\mathcal{R}}_4\big) + \tilde{\mathcal{R}}_2, \tag{6.68}$$

where $\mathcal{R}$ *is given by equation* (6.16) *and satisfies* (6.26)–(6.27) *and with* $\tilde{\mathcal{R}}_2$ *(resp.* $\tilde{\mathcal{R}}_3$, *resp.* $\tilde{\mathcal{R}}_4$*) satisfying* (6.59) *(resp.* (6.54)*, resp.* (6.55)*).*

Proof. We write, using (6.50), for $j = 3, 4$,

$$\begin{aligned} (D_t - P_0)C(t)\hat{\mathcal{M}}_j(\tilde{u}, u^{\text{app}}) = &-C(t)\mathcal{V}(t)\hat{\mathcal{M}}_j(\tilde{u}, u^{\text{app}})\\ &+ C(t)(D_t - P_0)\hat{\mathcal{M}}_j(\tilde{u}, u^{\text{app}}). \end{aligned} \tag{6.69}$$

We plug in the right-hand side of this equality (6.33) with $\tilde{\mathcal{M}}$ (resp. $\hat{\tilde{\mathcal{M}}}_n$) replaced by $\mathcal{M}^{\ell}_j$ (resp. $\hat{\mathcal{M}}^{\ell}_j$) according to the notation defined before (6.60). In the same way, we express

$$(D_t - P_0)\hat{\mathcal{M}}'_2(\tilde{u}, u'^{\text{app},1})$$

from (6.48) with $\tilde{\mathcal{M}}'_2$ (resp. $\hat{\tilde{\mathcal{M}}}'_2$) replaced by $\mathcal{M}'^{\ell}_2$ (resp. $\hat{\tilde{\mathcal{M}}}'^{\ell}_2$). Making the difference between (6.25) (where we substitute (6.53) and (6.58)) and these expressions, we obtain the contributions (6.62) to (6.68). This concludes the proof. ∎