Chapter 7

Bootstrap: L^2 estimates

The proof of the main theorem relies on a bootstrap argument of the type described in Sections 1.4 and 1.5 of the introduction (see estimates (1.28), (1.29) and (1.39)). In our setting, the bounds to be bootstrapped will be actually (2.45), (2.46), (2.47) of Section 2.5 in Chapter 2 (see (7.3) below). In the present chapter our objective is to bootstrap the first and last estimates (7.3) (see Proposition 7.3.7 below). We have thus to bound the Sobolev norm of the solution $\tilde{u}$ of (6.61), and the L^2 norm of $L\tilde{u}$. This is done by energy inequality, and the main task is to estimate the right-hand side of (6.61) in Sobolev spaces or the action of L on that right-hand side in L^2. We do that first for cubic and quartic terms, then for quadratic ones, and finally for terms of higher order.

7.1 Estimates for cubic and quartic terms

We consider $\mathbb{C}$-valued functions $u'^{\text{app}}_{+}, u''^{\text{app}}_{+}$, defined on some interval $[1, T]$, with $T \le \varepsilon^{-4+c}$ for some given $c > 0$, and that satisfy on that interval, for a given large r in $\mathbb{N}$ and some constant $C(A, A')$ bounds (4.39)–(4.41) and (4.43)–(4.45) that we recall below:

$$\begin{aligned} \|u'^{\text{app}}_{+}(t,\cdot)\|_{H^r} &\le C(A,A')\varepsilon^2 t^{\frac14},\\ \|u'^{\text{app}}_{+}(t,\cdot)\|_{W^{r,\infty}} &\le C(A,A')\varepsilon^2,\\ \|L_+ u'^{\text{app}}_{+}(t,\cdot)\|_{H^r} &\le C(A,A')t^{\frac14}\big((\varepsilon^2\sqrt{t}) + (\varepsilon^2\sqrt{t})^{\frac78}\varepsilon^{\frac18}\big) \end{aligned} \tag{7.1}$$

and

$$\begin{aligned} \|u''^{\text{app}}_{+}(t,\cdot)\|_{H^r} &\le C(A,A')\varepsilon\bigg(\frac{t\varepsilon^2}{\langle t\varepsilon^2\rangle}\bigg)^{\frac12},\\ \|u''^{\text{app}}_{+}(t,\cdot)\|_{W^{r,\infty}} &\le C(A,A')\varepsilon^2\log(1+t)^2,\\ \|L_+ u''^{\text{app}}_{+}(t,\cdot)\|_{W^{r,\infty}} &\le C(A,A')\log(1+t)\log(1+\varepsilon^2 t). \end{aligned} \tag{7.2}$$

Moreover, we shall assume that the solution $\tilde{u} = [\begin{smallmatrix}\tilde{u}_+\\ \tilde{u}_-\end{smallmatrix}]$ (with $\tilde{u}_- = -\overline{\tilde{u}_+}$) of (6.61) satisfies a priori estimates (5.35), i.e. having fixed $c > 0$, $\theta' < \theta < \frac12$ with θ' close to $\frac12$, and $\delta > 0$ small, for some $1 \ll \rho \ll s$, we have

$$\begin{aligned} \|\tilde{u}_+(t,\cdot)\|_{H^s} &\le D\varepsilon t^{\delta},\\ \|\tilde{u}_+(t,\cdot)\|_{W^{\rho,\infty}} &\le D\frac{(\varepsilon^2\sqrt{t})^{\theta'}}{\sqrt{t}},\\ \|L_+\tilde{u}_+(t,\cdot)\|_{L^2} &\le Dt^{\frac14}(\varepsilon^2\sqrt{t})^{\theta}. \end{aligned} \tag{7.3}$$

We recall also that we have defined from u_+^{app} the function $u_+^{\text{app},1}$ in (4.48), that we decomposed in (4.55) as $u'^{\text{app},1}_+ + u''^{\text{app},1}_+$ and we have seen after (4.54) that $u'^{\text{app},1}_+$ satisfies the same estimates as u'^{app}_+, so that we shall have

$$\begin{aligned} \|u'^{\text{app},1}_+(t,\cdot)\|_{H^r} &\le C(A,A')\varepsilon^2 t^{\frac14},\\ \|u'^{\text{app},1}_+(t,\cdot)\|_{W^{r,\infty}} &\le C(A,A')\varepsilon^2,\\ \|L_+u'^{\text{app},1}_+(t,\cdot)\|_{H^r} &\le C(A,A')t^{\frac14}\big((\varepsilon^2\sqrt{t}) + (\varepsilon^2\sqrt{t})^{\frac78}\varepsilon^{\frac18}\big). \end{aligned} \tag{7.4}$$

We may assume that r in (7.1) and (7.4) is as large as we want since the smoothness of the approximate solution u^{app} is independent of s: these functions are actually C^∞, since their x dependence comes only from stationary solution to our initial problem.

Our goal in that section is to deduce from (7.1) to (7.4) bounds for the cubic and quartic terms on the left-hand side of (6.61) and in (6.62) and (6.64).

Proposition 7.1.1. *Let $\hat{\mathcal{M}}_j(\tilde u, u^{\text{app}})$, $j=3,4$, be given by the first line in (6.60). There is a function $(t,\varepsilon)\mapsto e(t,\varepsilon)$, depending on the constants A, A', D in (7.1)–(7.3), satisfying $\lim_{\varepsilon\to 0+}\sup_{1\le t\le \varepsilon^{-4+c}} e(t,\varepsilon)=0$, such that the following bounds hold:*

$$\|C(t)\hat{\mathcal{M}}_j(\tilde u, u^{\text{app}})\|_{H^s} \le C\varepsilon t^\delta\big((\varepsilon^2\sqrt{t})^{2\theta'}t^{-1} + \varepsilon^4 t^\sigma\big) \le \varepsilon t^\delta e(t,\varepsilon), \tag{7.5}$$

$$\|LC(t)\hat{\mathcal{M}}_j(\tilde u, u^{\text{app}})\|_{L^2} \le t^{\frac14}(\varepsilon^2\sqrt{t})^\theta e(t,\varepsilon) \tag{7.6}$$

for any $t\in[1,\varepsilon^{-4+c}]$, any $\sigma>0$.

Proof. We prove first (7.5). By (E.19), $C(t)$ is bounded on H^s, uniformly in t staying in the wanted interval. By (6.60) we have thus to bound

$$\|\hat{\mathcal{M}}_j^\ell(\underbrace{\tilde u,\dots,\tilde u}_{\ell},\underbrace{u^{\text{app}},\dots,u^{\text{app}}}_{j-\ell})\|_{H^s},\quad 0\le \ell\le j,\ j=3,4 \tag{7.7}$$

(where each $\hat{\mathcal{M}}_j^\ell$ has form (6.32)) by the right-hand side of (7.5). By (D.32), (7.7) is bounded from above by

$$C\big[\|\tilde u\|_{H^s}\|\tilde u\|^{\ell-1}_{W^{\rho_0,\infty}}\|u^{\text{app}}\|^{j-\ell}_{W^{\rho_0,\infty}} + \|u^{\text{app}}\|_{H^s}\|u^{\text{app}}\|^{j-\ell-1}_{W^{\rho_0,\infty}}\|\tilde u\|^{\ell}_{W^{\rho_0,\infty}}\big] \tag{7.8}$$

with the convention that the first (resp. second) term in the bracket should be replaced by zero if $\ell=0$ (resp. $\ell=j$). As

$$u^{\text{app}}_\pm = u'^{\text{app}}_\pm + u''^{\text{app}}_\pm,\quad u^{\text{app}} = \begin{bmatrix} u^{\text{app}}_+\\ u^{\text{app}}_-\end{bmatrix},$$

it follows from (7.1) and (7.2) that

$$\begin{aligned} \|u^{\text{app}}\|_{H^s} &\le \tilde C(A,A')\varepsilon\Big(\frac{t\varepsilon^2}{\langle t\varepsilon^2\rangle}\Big)^{\frac12},\\ \|u^{\text{app}}\|_{W^{\rho_0,\infty}} &\le \tilde C(A,A')\varepsilon^2(\log(1+t))^2 \end{aligned} \tag{7.9}$$

for $t \leq \varepsilon^{-4}$. Using also (7.3), we bound (7.8) by

$$C\varepsilon t^{\delta}\Big(\big(\varepsilon^2(\log(1+t))^2\big)^{j-1} + \Big(\frac{(\varepsilon^2\sqrt{t})^{\theta'}}{\sqrt{t}}\Big)^{j-1}\Big). \tag{7.10}$$

Since $j \geq 3$, we have obtained a bound by the right-hand side of (7.5).

Let us prove (7.6). By (E.20)–(E.22), it suffices to bound by the right-hand side of (7.6) the quantities

$$\|L\hat{\mathcal{M}}_j(\tilde{u}, u^{\mathrm{app}})\|_{L^2},\ \|\hat{\mathcal{M}}_j(\tilde{u}, u^{\mathrm{app}})\|_{L^2} t^{\frac{1}{2}-m}\varepsilon^{\iota},$$

where m is close to $\frac{1}{2}$. The estimate of the second term is a consequence of (7.5). To study the first one, we recall that $L = \left[\begin{smallmatrix} L_+ & 0 \\ 0 & L_- \end{smallmatrix}\right]$ with $L_\pm = x \pm tp'(D_x)$, so that we have to estimate

$$t\|\hat{\mathcal{M}}_j(\tilde{u}, u^{\mathrm{app}})\|_{L^2},\ \|x\hat{\mathcal{M}}_j(\tilde{u}, u^{\mathrm{app}})\|_{L^2}. \tag{7.11}$$

By (7.10), the first term is estimated by (as $j \geq 3$)

$$t^{\frac{1}{4}}(\varepsilon^2\sqrt{t})^{\theta} e(t,\varepsilon) \tag{7.12}$$

with

$$e(t,\varepsilon) = O\big(\varepsilon^2 t^{\delta}(\log(1+t))^4(\varepsilon^2\sqrt{t})^{\frac{3}{2}-\theta} + \varepsilon t^{-\frac{1}{4}+\delta}(\varepsilon^2\sqrt{t})^{2\theta'-\theta}\big).$$

If $t \leq \varepsilon^{-4}$, $\theta' < \theta < \frac{1}{2}$ is close enough to $\frac{1}{2}$, so that $2\theta' - \theta \geq 0$, and if δ is small enough, one gets that e satisfies the condition in the statement. This concludes the proof of (7.6) for the first term in (7.11). To study the second one, we have to bound by $t^{\frac{1}{4}}(\varepsilon^2\sqrt{t})^{\theta} e$ the norm $\|x\hat{\mathcal{M}}_j^{\ell}(\tilde{u}, \dots, \tilde{u}, u^{\mathrm{app}}, \dots, u^{\mathrm{app}})\|_{L^2}$, $\ell = 0, \dots, j$. Consider first the case $\ell > 0$, so that at least one of the arguments is equal to $\tilde{u}$. By the form (6.32) of $\hat{\mathcal{M}}_j^{\ell}$, we may apply (D.36), putting the L^2 norm on that argument equal to $\tilde{u}$, i.e. we obtain a bound in

$$C\big[\|\tilde{u}\|^{j-1}_{W^{\rho_0,\infty}} + \|u^{\mathrm{app}}\|^{j-1}_{W^{\rho_0,\infty}}\big]\big[t\|\tilde{u}\|_{L^2} + \|L\tilde{u}\|_{L^2}\big]. \tag{7.13}$$

The contribution of the first term in the last bracket has already been estimates by (7.12) in the study of the first term (7.11). The second term gives rise, according to (7.9) and (7.3), to a quantity bounded by

$$Ct^{\frac{1}{4}}(\varepsilon^2\sqrt{t})^{\theta}\Big(\frac{(\varepsilon^2\sqrt{t})^{\theta'}}{\sqrt{t}} + \varepsilon^2(\log(1+t))^2\Big)^2$$

which is also of the form (7.12). It just remains to study the term

$$\|x\hat{\mathcal{M}}_j^{\ell}(u^{\mathrm{app}}, \dots, u^{\mathrm{app}})\|_{L^2}.$$

We decompose one of the arguments u^{app}, say the last one, as $u^{\mathrm{app}} = u'^{\mathrm{app}} + u''^{\mathrm{app}}$. We estimate then the L^2 norm of the function $x\hat{\mathcal{M}}_j^{\ell}(u^{\mathrm{app}}, \dots, u^{\mathrm{app}}, u'^{\mathrm{app}})$ (resp. of $x\hat{\mathcal{M}}_j^{\ell}(u^{\mathrm{app}}, \dots, u^{\mathrm{app}}, u''^{\mathrm{app}})$) using (D.36) with $n = j$ (resp. (D.37) with $n = j$). We obtain a bound in

$$C\|u^{\text{app}}\|_{W^{\rho_0,\infty}}^{j-1}\big(t\|u'^{\text{app}}\|_{L^2}+\|Lu'^{\text{app}}\|_{L^2}\big) \\ +C\|u^{\text{app}}\|_{W^{\rho_0,\infty}}^{j-2}\|u^{\text{app}}\|_{L^2}\big(t\|u''^{\text{app}}\|_{W^{\rho_0,\infty}}+\|Lu''^{\text{app}}\|_{W^{\rho_0,\infty}}\big). \tag{7.14}$$

Using (7.9), (7.1), (7.2), we obtain a bound in

$$C\varepsilon^4(\log(1+t))^4\big(\varepsilon^2 t^{\frac{5}{4}}+t^{\frac{1}{4}}\big(\varepsilon^2\sqrt{t}+(\varepsilon^2\sqrt{t})^{\frac{7}{8}}\varepsilon^{\frac{1}{8}}\big)\big) \\ +C\varepsilon^2(\log(1+t))^2\varepsilon\big(\varepsilon^2 t(\log(1+t))^2+\log(1+t)\log(1+\varepsilon^2 t)\big) \tag{7.15}$$

which is largely of form (7.12). This concludes the proof. ■

We shall study next term (6.62).

Proposition 7.1.2. *With notation (5.41) for $e(t,\varepsilon)$, one has the following bounds for $0\le\ell\le j$, $j=3,4$:*

$$\|C(t)\mathcal{V}(t)\hat{\mathcal{M}}_j^{\ell}(\underbrace{\tilde u,\dots,\tilde u}_{\ell},u^{\text{app}},\dots,u^{\text{app}})\|_{H^s}\le t^{-1}\varepsilon t^{\delta}e(t,\varepsilon), \tag{7.16}$$

$$\|LC(t)\mathcal{V}(t)\hat{\mathcal{M}}_j^{\ell}(\underbrace{\tilde u,\dots,\tilde u}_{\ell},u^{\text{app}},\dots,u^{\text{app}})\|_{H^s}\le t^{-1}\big(t^{\frac{1}{4}}(\varepsilon^2\sqrt{t})^{\theta}\big)e(t,\varepsilon). \tag{7.17}$$

Proof. Recall that $\hat{\mathcal{M}}_j$ is given by (6.60) in terms of operators $\hat{\mathcal{M}}_j^{\ell}$ defined in (6.32). Moreover, recall that $\mathcal{V}(t)$ in (6.17) is by definition the operator $\mathrm{Op}(M')$ given by (6.8), in function of symbols $b'_\pm$ satisfying (5.96)–(5.97). This means that in particular $t_\varepsilon^{1/2}b'_\pm$ are elements of the class $\tilde S'_{\kappa,\beta}(\langle\xi\rangle^{-1},1)$ (for any κ,β as these symbols depend only on one frequency variable). Moreover, the symbols $\hat m_I$ in (6.32) belong to $S_{4,\beta}(M_0^{\nu}\prod_{\ell=1}^{j}\langle\xi_\ell\rangle^{-1},j)$. It follows from the composition result of Corollary B.2.6 that the components of $\mathcal{V}(t)\hat{\mathcal{M}}_j^{\ell}(\tilde u,\dots,u^{\text{app}})$ may be written under the form

$$t_\varepsilon^{-\frac{1}{2}}\mathrm{Op}^t(m')(\tilde u_\pm,\dots,\tilde u_\pm,u_\pm^{\text{app}},\dots,u_\pm^{\text{app}}) \tag{7.18}$$

for some symbol m' in the class $S'_{4,\beta}(M_0^{\nu}\prod_{\ell=1}^{j}\langle\xi_\ell\rangle^{-1},j)$ (for some new ν), and any choice of the signs $\pm$. We use (D.32) together with the boundedness of $C(t)$ on H^s, to estimate the left-hand side of (7.16) by

$$Ct_\varepsilon^{-\frac{1}{2}}\big(\|u^{\text{app}}\|_{W^{\rho,\infty}}+\|\tilde u\|_{W^{\rho,\infty}}\big)^{j-1}\big(\|u^{\text{app}}\|_{H^s}+\|\tilde u\|_{H^s}\big). \tag{7.19}$$

Using estimates (7.9), (7.3) and $j\ge 3$, we bound this largely by the right-hand side of (7.16).

Let us prove (7.17). By (E.20)–(E.22) it is enough to estimate

$$\varepsilon^{\prime}t^{\frac{1}{2}-m}\|\mathcal{V}(t)\hat{\mathcal{M}}_j^{\ell}(\tilde u,\dots,u^{\text{app}})\|_{L^2},\ \|L\mathcal{V}(t)\hat{\mathcal{M}}_j^{\ell}(\tilde u,\dots,u^{\text{app}})\|_{L^2}$$

by the right-hand side of (7.17). The first term satisfies the wanted bound as a consequence of (7.19), since the exponent $\frac{1}{2}-m$ is close to zero. By (7.18), the study of the second one is reduced to

$$t_\varepsilon^{-\frac{1}{2}}\|L_\pm \mathrm{Op}^t(m')(\tilde{u}_\pm,\dots,\tilde{u}_\pm,u_\pm^{\mathrm{app}},\dots,u_\pm^{\mathrm{app}})\|_{L^2} \tag{7.20}$$

for m' in the class $S'_{4,\beta}(M_0^\nu\prod_{\ell=1}^j\langle\xi_\ell\rangle^{-1},j)$. As $L_\pm = x \pm tp'(\xi)$, and the symbol $m'(y,x,\xi_1,\dots,\xi_j)$ is decaying like $\langle M_0(\xi)^{-\kappa}y\rangle^{-N}$ for any N, we are reduced to bounding by the right-hand side of (7.17) the quantity

$$tt_\varepsilon^{-\frac{1}{2}}\|\mathrm{Op}^t(m')(\tilde{u}_\pm,\dots,\tilde{u}_\pm,u_\pm^{\mathrm{app}},\dots,u_\pm^{\mathrm{app}})\|_{L^2} \tag{7.21}$$

for a new m'. If there is at least one argument equal to $\tilde{u}_\pm$ in (7.21), we use estimate (D.71), making play the special role devoted to v_j there to such an $\tilde{u}_\pm$ argument. We obtain a bound of (7.21) in

$$Ct_\varepsilon^{-\frac{1}{2}}\big(\|\tilde{u}\|_{W^{\rho,\infty}}+\|u^{\mathrm{app}}\|_{W^{\rho,\infty}}\big)^{j-1}\big(\|\tilde{u}\|_{L^2}+\|L\tilde{u}\|_{L^2}\big). \tag{7.22}$$

By (7.9) and (7.3), this is bounded by

$$Ct_\varepsilon^{-\frac{1}{2}}\bigg(\frac{(\varepsilon^2\sqrt{t})^{\theta'}}{\sqrt{t}}+\varepsilon^2(\log(1+t))^2\bigg)^2\big(t^{\frac{1}{4}}(\varepsilon^2\sqrt{t})^{\theta}\big) \tag{7.23}$$

since $j \geq 3$. Again this is largely bounded by the right-hand side of (7.17).

Consider next the case when all arguments in (7.21) are equal to u^{app}. Decompose one of these arguments, say the last one, as $u^{\mathrm{app}} = u'^{\mathrm{app}} + u''^{\mathrm{app}}$. By linearity, we get a contribution in $\mathrm{Op}^t(m')(u_\pm^{\mathrm{app}},\dots,u_\pm^{\mathrm{app}},u'^{\mathrm{app}}_\pm)$ for which (7.21) may be estimated by (7.22) with $\tilde{u}$ replaced by u'^{app} in the last factor. As by (7.1) the L^2 bounds of u'^{app} and Lu'^{app} are better than the corresponding ones for $\tilde{u}$, $L\tilde{u}$ in (7.3), we get that (7.23) holds again. We are thus left with

$$tt_\varepsilon^{-\frac{1}{2}}\|\mathrm{Op}^t(m')(u''^{\mathrm{app}}_\pm,\dots,u''^{\mathrm{app}}_\pm)\|_{L^2}.$$

We use then (D.72) to estimate this by

$$Ct_\varepsilon^{-\frac{1}{2}}\|u''^{\mathrm{app}}\|_{W^{\rho_0,\infty}}^{j-2}\|u''^{\mathrm{app}}\|_{L^2}\big(\|u''^{\mathrm{app}}\|_{W^{\rho_0,\infty}}+\|Lu''^{\mathrm{app}}\|_{W^{\rho_0,\infty}}\big). \tag{7.24}$$

By (7.2), we thus get a bound in

$$t_\varepsilon^{-\frac{1}{2}}\varepsilon^2(\log(1+t))^2\varepsilon\bigg(\frac{t\varepsilon^2}{\langle t\varepsilon^2\rangle}\bigg)^{\frac{1}{2}}\log(1+t)\log(1+t\varepsilon^2).$$

Distinguishing the cases $t\varepsilon^2 \leq 1$, $t\varepsilon^2 \geq 1$, one checks that this is smaller than

$$t^{-\frac{3}{4}}(\varepsilon^2\sqrt{t})^{\frac{1}{2}}e(t,\varepsilon),$$

so than the right-hand side of (7.17). This concludes the proof. ∎

7.2 Estimates for quadratic terms

We shall study in this section the quadratic term in (6.61) and (6.63).

Proposition 7.2.1. *Let $\hat{\mathcal{M}}_2'$ be given by the second line in (6.60). One has the following bounds:*

$$\|\hat{\mathcal{M}}_2'(\tilde{u}, u^{\mathrm{app},1})\|_{H^s} \leq \varepsilon t^{\delta} e(t,\varepsilon), \tag{7.25}$$

$$\|L\hat{\mathcal{M}}_2'(\tilde{u}, u^{\mathrm{app},1})\|_{L^2} \leq t^{\frac{1}{4}}(\varepsilon^2\sqrt{t})^{\theta} e(t,\varepsilon) \tag{7.26}$$

for any $t \in [1, \varepsilon^{-4+c}]$, where $e(t,\varepsilon)$ satisfies (5.41).

To prove the proposition, we shall study the three terms in the definition of $\hat{\mathcal{M}}_2'$.

Lemma 7.2.2. *One has the following estimates:*

$$\|\hat{\mathcal{M}}'^2_2(\tilde{u},\tilde{u})\|_{H^s} \leq C\varepsilon t^{\delta}\big(t^{-\frac{1}{2}+\sigma}(\varepsilon^2\sqrt{t})^{\theta}\big), \tag{7.27}$$

$$\|L\hat{\mathcal{M}}'^2_2(\tilde{u},\tilde{u})\|_{L^2} \leq t^{\frac{1}{4}}(\varepsilon^2\sqrt{t})^{\theta} e(t,\varepsilon) \tag{7.28}$$

for any t in $[1, \varepsilon^{-1+c}]$, any $\sigma > 0$, if s is large enough relatively to $\frac{1}{\sigma}$.

Proof. By definition, $\hat{\mathcal{M}}'^2_2$ is obtained applying Proposition 6.2.4 to $\mathcal{M}'^2_2$ given by the first term on the right-hand side of the second line in (6.60). It has structure (6.47). We thus have to study

$$\|Q'_{i_1,i_2}(\tilde{u}_{i_1},\tilde{u}_{i_2})\|_{H^s}, \tag{7.29}$$

$$\|L_{\pm}Q'_{i_1,i_2}(\tilde{u}_{i_1},\tilde{u}_{i_2})\|_{L^2} \tag{7.30}$$

to obtain respectively (7.27) and (7.28), where Q'_{i_1,i_2} are operators of the form (F.35), preserving the space of odd functions. To bound (7.29), we thus have to study

$$t^{-\frac{3}{2}}\|K^{\ell_1,\ell_2}_{H,i_1,i_2}(L^{\ell_1}_{i_1}\tilde{u}_{i_1}, L^{\ell_2}_{i_2}\tilde{u}_{i_2})\|_{H^s}, \tag{7.31}$$

where $0 \leq \ell_1, \ell_2 \leq 1$.

If $\ell_1 = \ell_2 = 0$, we apply inequality (F.46) of Corollary F.5.2, with $\omega = \frac{1}{2}$. We obtain a bound of (7.31) in

$$Ct^{-\frac{7}{4}}\|\tilde{u}_+\|^2_{H^s}. \tag{7.32}$$

If $\ell_1 = 0, \ell_2 = 1$ (or the symmetric case), we apply (F.58), which gives for (7.31) an estimate in

$$Ct^{-\frac{3}{4}}\|\tilde{u}_+\|^2_{H^s}. \tag{7.33}$$

If $\ell_1 = \ell_2 = 1$, we use (F.57) in order to bound (7.31) by

$$Ct^{-\frac{3}{4}+\sigma}\big(\|L_+\tilde{u}_+\|_{L^2} + \|\tilde{u}_+\|_{H^s}\big)\|\tilde{u}_+\|_{H^s} \tag{7.34}$$

where $\sigma > 0$ is as small as we want (if s is large enough). Plugging in these estimates (7.3), we obtain a bound in

$$C\varepsilon t^{-\frac{3}{4}+\sigma+\delta}t^{\frac{1}{4}}(\varepsilon^2\sqrt{t})^{\theta}, \tag{7.35}$$

which gives (7.27).

Consider next (7.30) and decompose $L_\pm = x \pm tp'(D_x)$. The action of $tp'(D_x)$ on $Q'_{i_1,i_2}(\tilde{u}_{i_1}, \tilde{u}_{i_2})$ has L^2 norm bounded from above, according to (F.35), by

$$t^{-\frac{1}{2}} \| K^{\ell_1,\ell_2}_{H,i_1,i_2}(L^{\ell_1}_{i_1}\tilde{u}_{i_1}, L^{\ell_2}_{i_2}\tilde{u}_{i_2})\|_{L^2}. \tag{7.36}$$

When $\ell_1 = \ell_2 = 0$ (resp. $(\ell_1,\ell_2) = (1,0)$ or $(0,1)$), we apply (F.46) with $s = 0$ (resp. (F.50) and (F.51)) to bound this by

$$Ct^{-\frac{3}{4}+\sigma}\big(\|\tilde{u}_+\|_{H^s} + \|L_+\tilde{u}_+\|_{L^2}\big)\|\tilde{u}_+\|_{H^s}$$

for any $\sigma > 0$, so by (7.35), which is better that what we want.

On the other hand, if $\ell_1 = \ell_2 = 1$ in (7.36), we apply (F.50) or (F.51) with f_2 or f_1 replaced by $L_+\tilde{u}_+$. We obtain for (7.36) an estimate in

$$Ct^{-\frac{3}{4}+\sigma}\big(\|L_+\tilde{u}_+\|_{L^2} + \|\tilde{u}_+\|_{H^s}\big)^2. \tag{7.37}$$

Using (7.3), we obtain a better bound than (7.28). We are left with studying

$$t^{-\frac{3}{2}} \|xK^{\ell_1,\ell_2}_{H,i_1,i_2}(L^{\ell_1}_{i_1}\tilde{u}_{i_1}, L^{\ell_2}_{i_2}\tilde{u}_{i_2})\|_{L^2}. \tag{7.38}$$

We noticed at the end of the proof of Proposition F.5.1 that an operator xK may be written as an operator K_1 of the same type as K, up to the loss of a factor t^ω (here $t^{\frac{1}{2}}$). It follows that (7.38) will be bounded by $t^{-\frac{1}{2}}$ times (7.36), which is better than the estimate already obtained for the other contribution to (7.30). This concludes the proof. ∎

Proof of Proposition 7.2.1. We remark first that the conclusion of Lemma 7.2.2 holds for the three terms on the right-hand side of the second formula in (6.60) that defines $\hat{\mathcal{M}}'_2$. We have seen it for the last one. It holds for the other two terms as, by the end of the statement in Proposition 4.1.2, $u'^{\mathrm{app},1}_+$ satisfies the same estimates (7.1) as u'^{app}. Since these bounds are better than inequalities (7.3) satisfied by $\tilde{u}$ (for $t \le \varepsilon^{-4}$), the proof of Lemma 7.2.2 thus applies as well to $\hat{\mathcal{M}}'^0_2$, $\hat{\mathcal{M}}'^1_2$ in (6.60). Consequently, (7.25) and (7.26) hold. ∎

We want next to study quadratic terms on the right-hand side of (6.61), i.e. terms of the form (6.63).

Proposition 7.2.3. *Let $\mathcal{M}'_2$ be given by* (6.14) *and denote by $e(t,\varepsilon)$ a function satisfying* (5.41). *We have bounds*

$$\|(C(t) - \mathrm{Id})\mathcal{M}'_2(\tilde{u}, u'^{\mathrm{app},1})\|_{H^s} \le t^{-1}\varepsilon t^\delta e(t,\varepsilon), \tag{7.39}$$

$$\|L(C(t) - \mathrm{Id})\mathcal{M}'_2(\tilde{u}, u'^{\mathrm{app},1})\|_{L^2} \le t^{-1}t^{\frac{1}{4}}(\varepsilon^2\sqrt{t})^\theta e(t,\varepsilon). \tag{7.40}$$

Proof. We write the proof for the component of $\mathcal{M}'_2$ that is quadratic in $\tilde{u}$. This implies the general case, as $u'^{\mathrm{app},1}$ satisfies better estimates than those holding true for $\tilde{u}$.

Recall that by (6.14), the components of $\mathcal{M}_2'$ are of the form $\mathrm{Op}(m'_{0,I})(\tilde{u}_I)$ with $m'_{0,I}$ in $\tilde{S}'_{1,0}(\prod_{j=1}^{2}\langle\xi_j\rangle^{-1}M_0, 2)$. If we apply estimate (D.78) with $\ell' = \ell = 1$ and $n = 2$, we obtain

$$\|\mathcal{M}_2'(\tilde{u},\tilde{u})\|_{H^s} \le Ct^{-1+\sigma}\big(\|L\tilde{u}\|_{L^2} + \|\tilde{u}\|_{H^s}\big)\|\tilde{u}\|_{H^s}.$$

Plugging there (7.3), we get a bound in

$$C(\varepsilon t^{\delta})t^{-\frac{3}{4}+\sigma}(\varepsilon^2\sqrt{t})^{\theta}. \tag{7.41}$$

Since $\|C(t) - \mathrm{Id}\|_{\mathcal{L}(L^2)} = O(\varepsilon^{\iota}t^{-m+\delta'+\frac{1}{4}})$ by (E.19), we obtain an estimate in

$$C\varepsilon t^{\delta-1}\big[\varepsilon^{\iota}t^{\frac{1}{2}-m+\delta'+\sigma}(\varepsilon^2\sqrt{t})^{\theta}\big].$$

Since m may be taken as close to $\frac{1}{2}$ as we want (see the example following Definition E.1.1 where m is introduced), and since δ', σ may also be taken as small as wanted (in function of the fixed parameters c, θ, θ'), for $t \le \varepsilon^{-4+c}$, the factor between brackets is of the form $e(t,\varepsilon)$ in (7.39).

To prove (7.40), we write by (E.20)

$$L(C(t) - \mathrm{Id})\mathcal{M}_2' = (\tilde{C}(t) - \mathrm{Id})L\mathcal{M}_2' + \tilde{C}_1(t)\mathcal{M}_2'. \tag{7.42}$$

Since $\|\mathcal{M}_2'(\tilde{u},\tilde{u})\|_{L^2}$ is estimated by (7.41), and since $\|\tilde{C}_1(t)\|_{\mathcal{L}(L^2)}$ is bounded by (E.22) with m close to $\frac{1}{2}$, we see that the L^2 norm of the last term in (7.42) is smaller than the right-hand side of (7.40) (for $t \le \varepsilon^{-4}$).

On the other hand, by the definition of L, $\|L\mathcal{M}_2'(\tilde{u},\tilde{u})\|_{L^2}$ is bounded from above by $t\|\mathrm{Op}(m'_{0,I})(\tilde{u}_I)\|_{L^2}$, with $m'_{0,I}$ in $\tilde{S}'_{1,0}(\prod_{j=1}^{2}\langle\xi_j\rangle^{-1}, 2)$. Using (D.76), we estimate this by

$$Ct^{-1+\sigma}\big(\|L_+\tilde{u}_+\|_{L^2} + \|\tilde{u}_+\|_{H^s}\big)^2 \le Ct^{-1+\sigma}\big(t^{\frac{1}{4}}(\varepsilon^2\sqrt{t})^{\theta}\big)^2.$$

Since $\|\tilde{C}(t) - \mathrm{Id}\|_{\mathcal{L}(L^2)} = O(\varepsilon^{\iota}t^{-m+\delta'+\frac{1}{4}})$ with m close to $\frac{1}{2}$ by (E.21), we see that the L^2 norm of the first term on the right-hand side of (7.42) is bounded from above by

$$Ct^{-1}t^{\frac{1}{4}}(\varepsilon^2\sqrt{t})^{\theta}\big[(\varepsilon^2\sqrt{t})^{\theta}t^{\frac{1}{2}-m+\delta'+\sigma}\varepsilon^{\iota}\big]$$

and again, if $\frac{1}{2} - m, \delta', \sigma$ have been taken small enough, the bracket is of the form $e(t,\varepsilon)$, whence a bound by the right-hand side of (7.40). This concludes the proof. ■

7.3 Higher-order terms

In this section, we shall bound expressions of the form (6.64)–(6.65) that appear as contributions of higher order of homogeneity if one replaces $(D_t - P_0)\tilde{u}$ by its expression coming from (6.17). We study first the first line in (6.64).

Proposition 7.3.1. *Denote for* $1 \le \ell \le j$, $j = 3, 4$,

$$F(t) = C(t)\hat{\mathcal{M}}_j^\ell\big(\tilde{u}, \dots, (D_t - P_0)\tilde{u}, \dots, \tilde{u}, u^{\mathrm{app}}, \dots, u^{\mathrm{app}}\big). \tag{7.43}$$

Then under a priori assumptions (7.1) *and* (7.3), *one has the following bounds:*

$$\|F(t)\|_{H^s} \le t^{-1}\varepsilon t^{\delta} e(t,\varepsilon), \tag{7.44}$$

$$\|LF(t)\|_{L^2} \le t^{-1}\big(t^{\frac{1}{4}}(\varepsilon^2\sqrt{t})^{\theta}\big)e(t,\varepsilon) \tag{7.45}$$

with e *satisfying* (5.41).

To prove the proposition, we first re-express $F(t)$ replacing on the right-hand side $(D_t - P_0)\tilde{u}$ by its value.

Lemma 7.3.2. *The components of*

$$\hat{\mathcal{M}}_j^\ell\big(\tilde{u}, \dots, (D_t - P_0)\tilde{u}, \dots, \tilde{u}, u^{\mathrm{app}}, \dots, u^{\mathrm{app}}\big)$$

may be written as sums of terms of the following form:

$$t_\varepsilon^{-\frac{1}{2}}\mathrm{Op}^t(m')(\tilde{u}_{I'}, u^{\mathrm{app}}_{I''}), \quad j = |I'| + |I''| \ge 3, \tag{7.46}$$

where m' *is in* $S'_{4,\beta}(M_0^\nu \prod_{\ell=1}^{j} \langle\xi_\ell\rangle^{-1}, j)$,

$$\mathrm{Op}^t(m)(\tilde{u}_{I'}, u^{\mathrm{app}}_{I''}), \quad j = |I'| + |I''| \ge 5, \tag{7.47}$$

where m *is in* $S_{4,\beta}(M_0^\nu \prod_{\ell=1}^{j} \langle\xi_\ell\rangle^{-1}, j)$,

$$\mathrm{Op}^t(m)\big(\mathcal{R}_{j'}(\tilde{u}, u^{\mathrm{app}}), \tilde{u}_{I'}, u^{\mathrm{app}}_{I''}\big), \quad j = |I'| + |I''|, \tag{7.48}$$

where $j' \ge 3$, $j \ge 2$, m *is in* $S_{4,\beta}(M_0^\nu \prod_{\ell=1}^{j+1} \langle\xi_\ell\rangle^{-1}, j+1)$ *and* $\mathcal{R}_{j'}$ *satisfies* (6.54) *and* (6.55),

$$\mathrm{Op}^t(m')\big(\tilde{u}_{I'}, u'^{\mathrm{app},1}_{I''}, u^{\mathrm{app}}_{I'''}\big), \quad j = |I'| + |I''| + |I'''| \ge 4, \tag{7.49}$$

where m' *is in* $S'_{4,\beta}(M_0^\nu \prod_{\ell=1}^{j} \langle\xi_\ell\rangle^{-1}, j)$,

$$\mathrm{Op}^t(m)\big(\tilde{\mathcal{R}}_2(\tilde{u}, u'^{\mathrm{app},1}), \tilde{u}_{I'}, u^{\mathrm{app}}_{I''}\big), \quad j = |I'| + |I''|, \tag{7.50}$$

with $j \ge 2$, m *is in* $S_{4,\beta}(M_0^\nu \prod_{\ell=1}^{j+1} \langle\xi_\ell\rangle^{-1}, j+1)$, $\tilde{\mathcal{R}}_2$ *satisfying* (6.59),

$$\mathrm{Op}^t(m)(\mathcal{R}, \tilde{u}_{I'}, u^{\mathrm{app}}_{I''}), \quad j = |I'| + |I''| \ge 2, \tag{7.51}$$

where $\mathcal{R}$ *satisfies estimates* (5.39) *and* (5.40) *and where* m *is a symbol in the class* $S_{4,\beta}(M_0^\nu \prod_{\ell=1}^{j+1} \langle\xi_\ell\rangle^{-1}, j+1)$.

Proof. Recall that by (6.17)

$$(D_t - P_0)\tilde{u} = \mathcal{V}(t)\tilde{u} + \mathcal{M}_3(\tilde{u}, u^{\mathrm{app}}) + \mathcal{M}_4(\tilde{u}, u^{\mathrm{app}}) + \mathcal{M}_2'(\tilde{u}, u'^{\mathrm{app},1}) + \mathcal{R}. \tag{7.52}$$

Recall that $\hat{\mathcal{M}}_j^\ell$ is an operator of the form (6.32), so that its components computed at $(\tilde{u}, \dots, \tilde{u}, u^{\text{app}}, \dots, u^{\text{app}})$ may be written

$$\text{Op}^t(m)(\tilde{u}_{i_1}, \dots, \tilde{u}_{i_\ell}, u^{\text{app}}_{i_{\ell+1}}, \dots, u^{\text{app}}_{i_j}) \tag{7.53}$$

with $i_j = \pm$ and m element of $S_{4,\beta}(M_0^\nu \prod_{\ell=1}^j \langle\xi_\ell\rangle^{-1}, j)$ for some $\beta > 0$. We have to compute (7.53) when one of its $\tilde{u}$ arguments, say the first one, is replaced by $(D_t - P_0)\tilde{u}$, so by the right-hand side of (7.52). If we replace $(D_t - P_0)\tilde{u}$ by $\mathcal{V}(t)\tilde{u}$ and use that $\mathcal{V}(t)$ is constructed from operators $\text{Op}(b'_\pm)$ in (6.8) that satisfy (5.96) and (5.97), i.e. are such that $t_\varepsilon^{1/2} b'_\pm = c'_\pm$ is in $S'_{\kappa,\beta}(\langle\xi\rangle^{-1}, 1)$ (for any κ, β), we get a contribution

$$t_\varepsilon^{-\frac{1}{2}} \text{Op}^t(m)\big(\text{Op}(c'_{i_1})\tilde{u}_{i_1}, \tilde{u}_{i_2}, \dots, \tilde{u}_{i_\ell}, u^{\text{app}}_{i_{\ell+1}}, \dots, u^{\text{app}}_{i_j}\big).$$

By the composition result of Corollary B.2.6, we get a term of the form (7.46).

Let us study next (7.53) with the first argument replaced by

$$\mathcal{M}_3(\tilde{u}, u^{\text{app}}) + \mathcal{M}_4(\tilde{u}, u^{\text{app}})$$

coming from (7.52). According to definition (6.15) of $\mathcal{M}_j$ and to (6.53), we shall get contributions

$$\text{Op}^t(m)\big(\text{Op}(\tilde{m}_I)(\tilde{u}_{I'}, u^{\text{app}}_{I''}), \tilde{u}_{i_2}, \dots, \tilde{u}_{i_\ell}, u^{\text{app}}_{i_{\ell+1}}, \dots, u^{\text{app}}_{i_j}\big) \tag{7.54}$$

with $|I| = 3$ or 4 and $\tilde{m}$ in $\tilde{S}_{1,\beta}(M_0(\xi)^\nu \prod_{j=1}^{|I|}\langle\xi_j\rangle^{-1}, |I|)$, with $\beta > 0$ and

$$\text{Op}^t(m)\big(\tilde{\mathcal{R}}_{j',\pm}(\tilde{u}, u^{\text{app}}), \tilde{u}_{i_2}, \dots, u^{\text{app}}_{i_j}\big) \tag{7.55}$$

for

$$\tilde{R}_{j'} = \begin{bmatrix} \tilde{\mathcal{R}}_{j',+} \\ \tilde{\mathcal{R}}_{j',-} \end{bmatrix}$$

satisfying (6.54) and (6.55) with $j' = 3$ or 4. By Corollary (B.19), (7.54) may be written as a term homogeneous of degree larger than or equal to 5 that has the structure (7.47). Moreover, (7.55) provides terms of the form (7.48).

We have to study then the term (7.53) where the first argument is replaced by the $\mathcal{M}'_2(\tilde{u}, u'^{\text{app},1})$ term in (7.52). By (6.58) and (6.57), we get contributions of the form

$$\text{Op}^t(m)\big[\text{Op}(m'_{0,I',I''})(\tilde{u}_{I'}, u'^{\text{app},1}_{I''}), \tilde{u}_{i_2}, \dots, \tilde{u}_{i_\ell}, u^{\text{app}}_{i_{\ell+1}}, \dots, u^{\text{app}}_{i_j}\big] \tag{7.56}$$

with $|I'| + |I''| = 2$, $j \geq 3$, and

$$\text{Op}^t(m)\big[\tilde{\mathcal{R}}_{2,\pm}(\tilde{u}, u'^{\text{app},1}), \tilde{u}_{i_2}, \dots, u^{\text{app}}_{i_j}\big]. \tag{7.57}$$

Again by Corollary B.2.6, (7.56) brings a contribution of the form (7.49) and (7.57) an expression of type (7.50).

Finally, we have to replace one argument of (7.53) by the last term $\mathcal{R}$ in (7.52). This brings (7.51). This concludes the proof of the lemma. ∎

Proof of Proposition 7.3.1. Let us prove (7.44) and (7.45). We have to estimate all contributions from (7.46) to (7.51). As already seen, (E.19) to (E.22) allow us to ignore the action of operator $C(t)$ on the definition (7.43) of $F(t)$, so that we need to study only the Sobolev norm of (7.46) to (7.51), and the L^2 norm of the action of L on these two quantities.

Term (7.46). This term is of the form (7.18) and has already been estimated by the wanted quantities.

Term (7.47). The Sobolev norm of this term may be bounded from above, according to (D.32), by

$$C\big(\|\tilde{u}\|_{W^{\rho_0,\infty}} + \|u^{\mathrm{app}}\|_{W^{\rho_0,\infty}}\big)^4\big(\|\tilde{u}\|_{H^s} + \|u^{\mathrm{app}}\|_{H^s}\big).$$

Using (7.1) and (7.3), we bound this by

$$Ct^{-2}(\varepsilon^2\sqrt{t})^{4\theta'}\varepsilon t^{\delta} \tag{7.58}$$

which is better than the right-hand side of (7.44). If we make act $L_\pm$ on (7.47) and compute the L^2 norm, we get on the one hand the product of (7.58) by t, which is smaller than the right-hand side of (7.45) and $\|x\mathrm{Op}^t(m)(\tilde{u}_{I'}, u^{\mathrm{app}}_{I''})\|_{L^2}$. This is a quantity of the same form as the second term in (7.11), except that $j \geq 5$. We thus obtain a bound by (7.13), when at least one of the arguments in (7.47) is equal to $\tilde{u}$. By (7.1)–(7.3) and $j \geq 5$, this is controlled by the right-hand side of (7.45). If all the arguments are equal to u^{app}, we get instead a bound by (7.14) with $j \geq 5$, so by (7.15) multiplied by $\|u^{\mathrm{app}}\|^2_{W^{\rho_0,\infty}} \leq Ct^{-1}$ when $t \leq \varepsilon^{-4+c}$ by (7.1) and (7.2). Since (7.15) was controlled by (7.12), we get again a bound of the form (7.45).

Term (7.48). By (D.32), the H^s norm of (7.48) is bounded by

$$\begin{aligned} C\|\tilde{\mathcal{R}}_{j'}(\tilde{u}, u^{\mathrm{app}})\|_{H^s}\big(\|\tilde{u}\|_{W^{\rho_0,\infty}} + \|u^{\mathrm{app}}\|_{W^{\rho_0,\infty}}\big)^2 \\ + \|\tilde{\mathcal{R}}_{j'}(\tilde{u}, u^{\mathrm{app}})\|_{W^{\rho_0,\infty}}\big(\|\tilde{u}\|_{W^{\rho_0,\infty}} + \|u^{\mathrm{app}}\|_{W^{\rho_0,\infty}}\big) \\ \times\big(\|\tilde{u}\|_{H^s} + \|u^{\mathrm{app}}\|_{H^s}\big) \end{aligned} \tag{7.59}$$

since $j \geq 2$ in (7.48). Using Sobolev injection, we may bound $\|\tilde{\mathcal{R}}_{j'}\|_{W^{\rho_0,\infty}}$ from $\|\tilde{\mathcal{R}}_{j'}\|_{H^s}$. By (6.54) and (7.1)–(7.3), we largely get an estimate of the form (7.44).

If we make act $L_\pm$ on (7.48), and use that

$$x\mathrm{Op}^t(m)(v_1, \dots, v_n) - \mathrm{Op}^t(m)(xv_1, \dots, v_n)$$

is of the form $\mathrm{Op}^t(m_1)(v_1, \dots, v_n)$ for a new symbol m_1 of the same form as m, we reduce the estimate of the L^2 norm of the action of $L_\pm$ on (7.48) to bounding

$$\begin{aligned} &t\|\mathrm{Op}^t(m)\big(\tilde{\mathcal{R}}_{j',\pm}(\tilde{u}, u^{\mathrm{app}}), \tilde{u}_{I'}, u^{\mathrm{app}}_{I''}\big)\|_{L^2}, \\ &\|\mathrm{Op}^t(m)\big(L\tilde{\mathcal{R}}_{j',\pm}(\tilde{u}, u^{\mathrm{app}}), \tilde{u}_{I'}, u^{\mathrm{app}}_{I''}\big)\|_{L^2}. \end{aligned}$$

By (D.33), we get an estimate in

$$\big(t\|\tilde{\mathcal{R}}_{j'}(\tilde{u},u^{\mathrm{app}})\|_{L^2}+\|L_\pm\tilde{\mathcal{R}}_{j'}(\tilde{u},u^{\mathrm{app}})\|_{L^2}\big)\big(\|\tilde{u}\|_{W^{\rho_0,\infty}}+\|u^{\mathrm{app}}\|_{W^{\rho_0,\infty}}\big)^2. \quad (7.60)$$

By (6.54), (6.55), (7.1)–(7.3), this is largely estimated by the right-hand side of (7.45).

Term (7.49). This term is of the form (7.18), except that there is no $t_\varepsilon^{-1/2}$ factor, that we may have an argument $u'^{\mathrm{app},1}$ instead of u^{app}, and that the number of arguments is larger than or equal to 4. By (7.19), the H^s norm of (7.49) is bounded from above by

$$C\big(\|u'^{\mathrm{app},1}\|_{W^{\rho_0,\infty}}+\|u^{\mathrm{app}}\|_{W^{\rho_0,\infty}}+\|\tilde{u}\|_{W^{\rho_0,\infty}}\big)^3 \\ \times\big(\|u^{\mathrm{app}}\|_{H^s}+\|\tilde{u}\|_{H^s}+\|u'^{\mathrm{app},1}\|_{H^s}\big).$$

Using (7.1)–(7.4) we get a better estimate than (7.44). If we make act $L_\pm$ on (7.49) and compute the L^2 norm, we obtain a quantity of the form (7.20), without the pre-factor $t_\varepsilon^{-1/2}$. We obtain thus an upper bound given by (7.22) or (7.24) without the $t_\varepsilon^{-1/2}$ factor, but with $j\geq 4$ and an argument $u'^{\mathrm{app},1}$ replacing eventually an u^{app}. By (7.1)–(7.4),

$$\big(\|u'^{\mathrm{app},1}\|_{W^{\rho_0,\infty}}+\|u^{\mathrm{app}}\|_{W^{\rho_0,\infty}}+\|\tilde{u}\|_{W^{\rho_0,\infty}}\big)^3\big(\|\tilde{u}\|_{L^2}+\|L\tilde{u}\|_{L^2}\big)$$

is smaller than the right-hand side of (7.44). On the other hand, the contribution of the form (7.24) is bounded from above by

$$C\|u''^{\mathrm{app}}\|^2_{W^{\rho_0,\infty}}\|u''^{\mathrm{app}}\|_{L^2}\big(\|u''^{\mathrm{app}}\|_{W^{\rho_0,\infty}}+\|Lu''^{\mathrm{app}}\|_{W^{\rho_0,\infty}}\big)\leq C\varepsilon^5(\log(1+t))^6$$

by (7.2). As $t\leq\varepsilon^{-4+c}$, we estimate this by $\frac{1}{t}\varepsilon e(t,\varepsilon)$, so by the right-hand side of (7.45).

Term (7.50). This is a term of form (7.48). The H^s norm may be bounded by (7.59), with $\tilde{\mathcal{R}}_{j'}$ replaced by $\tilde{\mathcal{R}}_2$. It follows from (6.59), Sobolev injection and (7.1)–(7.4) that we largely get a bound of the form (7.44). If we make act $L_\pm$ and estimate the L^2 norm, we get a bound of the form (7.60), with $\tilde{\mathcal{R}}_{j'}$ replaced by $\tilde{\mathcal{R}}_2$. Again, by (6.59), (7.1)–(7.4), we obtain the conclusion.

Term (7.51). This is a term of the form (7.48), with $\tilde{\mathcal{R}}_{j'}$ replaced by $\mathcal{R}$. Again, we may apply (7.59) to bound the H^s norm. According to (5.39), we obtain a bound by the right-hand side of (7.44). To study the L^2 norm of the action of $L_\pm$ on (7.51), we use that we have again a bound of the form (7.60) with $\tilde{\mathcal{R}}_{j'}$ replaced by $\mathcal{R}$. As the last factor in (7.60) is $O(t^{-1})$ by (7.1)–(7.3), we conclude that we get an upper bound by (7.45) using (5.39), (5.40). This concludes the proof of Proposition 7.3.1 ∎

Our next task is to study the second line in (6.64).

Proposition 7.3.3. *Denote now*

$$F(t)=C(t)\hat{\mathcal{M}}^\ell_j(\tilde{u},\dots,\tilde{u},u^{\mathrm{app}},\dots,(D_t-P_0)u^{\mathrm{app}},\dots,u^{\mathrm{app}}). \quad (7.61)$$

Then under assumptions (7.1)–(7.4)

$$\|F(t)\|_{H^s} \le t^{-1}\varepsilon t^{\delta} e(t,\varepsilon), \tag{7.62}$$

$$\|LF(t)\|_{H^s} \le t^{-1}t^{\frac{1}{4}}(\varepsilon^2\sqrt{t})^{\theta} e(t,\varepsilon). \tag{7.63}$$

Proof. Recall that $(D_t - p(D_x))u_+^{\mathrm{app}}$ is given by (4.37). Together with the definition (2.28) of F_0^2, F_0^3, with the fact that by (4.3), (4.6), (4.8), a^{app} is $O(t_\varepsilon^{-1/2})$, and with estimates (4.38), this implies that

$$(D_t - p(D_x))u_+^{\mathrm{app}} = Z(t,x) + a^{\mathrm{app}}(t)\sum_{|I|=1}\mathrm{Op}(m'_{1,I})(u_I^{\mathrm{app}}), \tag{7.64}$$

where $m'_{1,I}$ is in $\tilde{S}'_{1,0}(\langle\xi\rangle^{-1},1)$ and $Z(t,x)$ satisfies for any α, N,

$$|\partial_x^{\alpha} Z(t,x)| \le C_{\alpha,N} t_\varepsilon^{-1}\langle x\rangle^{-N}. \tag{7.65}$$

Notice that we may consider as well $m'_{1,I}$ as an element of $S'_{1,\beta}(\langle\xi\rangle^{-1},1)$ for $\beta > 0$, since for symbols depending only on one frequency variable, this does not make any difference. We plug (7.64) inside (7.61). Using the form (6.32) of $\hat{\mathcal{M}}_j^{\ell}$ and the composition result of Corollary B.2.6, we write (7.61), where we forget factor $C(t)$ that does not affect the estimates, as a sum of terms (up to permutations of the arguments)

$$t_\varepsilon^{-\frac{1}{2}}\mathrm{Op}^t(m')(\tilde{u}_\pm,\dots,u_\pm^{\mathrm{app}}), \tag{7.66}$$

$$\mathrm{Op}^t(m)(Z,\tilde{u}_\pm,\dots,u_\pm^{\mathrm{app}}), \tag{7.67}$$

where the number of arguments $(\tilde{u}_\pm,\dots,u_\pm^{\mathrm{app}})$ in term (7.66) (resp. term (7.67)) is j (resp. $j-1$) with $j \ge 3$, and m' belongs to $S'_{4,\beta}(M_0^{\nu}\prod_{\ell=1}^{j}\langle\xi_\ell\rangle^{-1}, j)$, m to $S_{4,\beta}(M_0^{\nu}\prod_{\ell=1}^{j}\langle\xi_\ell\rangle^{-1}, j)$ for some ν. Expression (7.66) is of the form (7.46), so satisfies the wanted bounds (7.62)–(7.63) by the first point in the proof of Proposition 7.3.1. The H^s norm of (7.67) is bounded by (D.32) by

$$\begin{aligned} C\big(\|\tilde{u}\|_{H^s} + \|u^{\mathrm{app}}\|_{H^s}\big)\big(\|\tilde{u}\|_{W^{\rho_0,\infty}} + \|u^{\mathrm{app}}\|_{W^{\rho_0,\infty}}\big)\|Z\|_{W^{\rho_0,\infty}} \\ + C\big(\|\tilde{u}\|_{W^{\rho_0,\infty}} + \|u^{\mathrm{app}}\|_{W^{\rho_0,\infty}}\big)^2\|Z\|_{H^s} \end{aligned}$$

so by the right-hand side of (7.62), by (7.1)–(7.3) and (7.65).

Let us bound next the L^2 norm of the action of $L_\pm$ on (7.67). We decompose each factor $u_\pm^{\mathrm{app}} = u'^{\mathrm{app}}_\pm + u''^{\mathrm{app}}_\pm$. Consider first the case of the resulting expression where at least one of the last $j-1$ arguments in (7.67) is equal to $\tilde{u}_\pm$ or $u'^{\mathrm{app}}_\pm$, say the last one. We have to estimate

$$\begin{aligned} &t\|\mathrm{Op}^t(m)(Z,\tilde{u}_\pm,\dots,u_\pm^{\mathrm{app}},w)\|_{L^2},\\ &\|x\mathrm{Op}^t(m)(Z,\tilde{u}_\pm,\dots,u_\pm^{\mathrm{app}},w)\|_{L^2} \end{aligned} \tag{7.68}$$

with $w = \tilde{u}_\pm$ or $u'^{\mathrm{app}}_\pm$. Up to commuting x to $\mathrm{Op}^t(m)$ in order to put it against Z, it is enough to bound the first expression. We use (D.73) with the special index j equal to the last one. Recalling the t_ε^{-1} factor in (7.65), we get a bound in

$$\begin{aligned} C t_\varepsilon^{-1} \big(\|\tilde u\|_{W^{\rho_0,\infty}} + \|u^{\mathrm{app}}\|_{W^{\rho_0,\infty}} \big)^{j-2} \\ \times \big(\|\tilde u\|_{L^2} + \|L\tilde u\|_{L^2} + \|u'^{\mathrm{app}}\|_{L^2} + \|L_\pm u'^{\mathrm{app}}\|_{L^2} \big) \end{aligned} \tag{7.69}$$

which by (7.1)–(7.3) is smaller than the right-hand side of (7.63) (as $j-2 \geq 1$). On the other hand, if we consider (7.68) with all arguments $(\tilde u_\pm, \dots, u_\pm^{\mathrm{app}}, w)$ replaced by $u''^{\mathrm{app}}_\pm$, we use (D.74) and get instead of (7.69), by (7.2)

$$\begin{aligned} &C t_\varepsilon^{-1} \|u''^{\mathrm{app}}\|^{j-3}_{W^{\rho_0,\infty}} \big(\|L u''^{\mathrm{app}}\|_{W^{\rho_0,\infty}} + \|u''^{\mathrm{app}}\|_{W^{\rho_0,\infty}} \big) \|u''^{\mathrm{app}}\|_{L^2} \\ &\leq C t_\varepsilon^{-1} \varepsilon \log(1+t) \log(1+t\varepsilon^2). \end{aligned}$$

This is much better than (7.63). This concludes the proof. ∎

Let us move now to the study of (6.65).

Proposition 7.3.4. *Denote*

$$\begin{aligned} F(t) = C(t)\hat{\mathcal M}'^0_2\big((D_t - P_0)u'^{\mathrm{app},1}, u'^{\mathrm{app},1}\big) \\ + C(t)\hat{\mathcal M}'^0_2\big(u'^{\mathrm{app},1}, (D_t - P_0)u'^{\mathrm{app},1}\big) \\ + C(t)\hat{\mathcal M}'^1_2\big((D_t - P_0)\tilde u, u'^{\mathrm{app},1}\big) \\ + C(t)\hat{\mathcal M}'^1_2\big(\tilde u, (D_t - P_0)u'^{\mathrm{app},1}\big) \\ + C(t)\hat{\mathcal M}'^2_2\big((D_t - P_0)\tilde u, \tilde u\big) + C(t)\hat{\mathcal M}'^2_2\big(\tilde u, (D_t - P_0)\tilde u\big). \end{aligned} \tag{7.70}$$

Then

$$\|F(t)\|_{H^s} \leq t^{-1} \varepsilon t^\delta e(t,\varepsilon), \tag{7.71}$$

$$\|L_\pm F(t)\|_{L^2} \leq t^{-1} \big(t^{\frac14} (\varepsilon^2 \sqrt t)^\theta \big) e(t,\varepsilon). \tag{7.72}$$

Before starting the proof, we recall some estimates for $(D_t - P_0)\tilde u$.

Lemma 7.3.5. *Under a priori assumptions* (7.43)–(7.45) *we have the following estimates:*

$$\|(D_t - P_0)\tilde u\|_{H^s} \leq C\varepsilon t^{\delta - \frac12}, \tag{7.73}$$

$$L(D_t - P_0)\tilde u = f_1 + x f_2 \tag{7.74}$$

with

$$\|f_1\|_{L^2} \leq C t^{-\frac12} \big(t^{\frac14} (\varepsilon^2 \sqrt t)^\theta \big), \tag{7.75}$$

$$\|f_2\|_{L^2} \leq C t^{-1} (\varepsilon^2 \sqrt t)^{2\theta'} \varepsilon t^\delta. \tag{7.76}$$

Proof. Recall that $(D_t - P_0)\tilde u$ is given by (7.52) and that $\mathcal V(t)$ may be expressed, according to (6.8), from operators $t_\varepsilon^{-1/2} \mathrm{Op}^t(c'_\pm)$ with $c'_\pm$ in the class $S'_{\kappa,\beta}(\langle\xi\rangle^{-1}, 1)$. By boundedness of these operators on H^s and (7.3), we get for $\|\mathcal V(t)\tilde u\|_{H^s}$ a bound by the right-hand side of (7.73).

The action of L on $\mathcal{V}(t)\tilde{u}$ will have L^2 norm bounded from above by

$$t_\varepsilon^{-\frac{1}{2}}\|x\mathrm{Op}^t(c'_\pm)\tilde{u}\|_{L^2} + t t_\varepsilon^{-\frac{1}{2}}\|\mathrm{Op}^t(c'_\pm)\tilde{u}\|_{L^2}.$$

By (D.71) with $n = 1$ and (7.3), we get a bound by the right-hand side of (7.75).

Consider next the $\mathcal{M}_j(\tilde{u}, u^{\mathrm{app}})$ terms, $j = 3, 4$, on the right-hand side of (7.52). By (6.53), these terms are given on the one hand by the contributions $\tilde{\mathcal{R}}_j$, which by (6.54) are largely bounded in H^s by the right-hand side of (7.73), and which by (6.55) contribute to f_1 in (7.74) if we apply L on them. On the other hand, the main terms in (6.53) are of the form $\mathrm{Op}^t(\tilde{m}_{I',I''})(\tilde{u}_{I'}, u_{I''}^{\mathrm{app}})$. By (D.32) and (7.1)–(7.3), they satisfy (7.73). Let us study $L_\pm \mathrm{Op}^t(\tilde{m}_{I',I''})(\tilde{u}_{I'}, u_{I''}^{\mathrm{app}})$. We apply Proposition F.2.1 and Corollary F.2.2 (translated in the non-semiclassical framework). This allows us to re-express this quantity from

$$\mathrm{Op}^t(\tilde{m})\big(L_\pm v_1, v_2, \dots, v_j\big), \tag{7.77}$$
$$\mathrm{Op}^t(\tilde{r})(v_1, \dots, v_j), \tag{7.78}$$
$$t\mathrm{Op}^t(\tilde{r}')(v_1, \dots, v_j), \tag{7.79}$$
$$x\mathrm{Op}^t(\tilde{r})(v_1, \dots, v_j) \tag{7.80}$$

where $v_\ell = \tilde{u}_\pm$ or $v_\ell = u'^{\mathrm{app}} + u''^{\mathrm{app}}$, where $\tilde{m}, \tilde{r}$ are in $S_{4,\beta}(M_0^\nu \prod_{\ell=1}^j \langle\xi_\ell\rangle^{-1}, j)$ and $\tilde{r}'$ is in $S'_{4,\beta}(M_0^\nu \prod_{\ell=1}^j \langle\xi_\ell\rangle^{-1}, j)$.

We estimate the L^2 norm of (7.77) using (D.33) with the special index equal to the first one, when v_1 is replaced either by $\tilde{u}_\pm$ or $u'^{\mathrm{app}}_\pm$. We largely get a bound by (7.75) as $j \geq 3$ using (7.1)–(7.3). If v_1 is replaced by $u''^{\mathrm{app}}_\pm$, we still use (D.33), but make play the special role to the second argument. We obtain a bound in

$$\|L_+ u''^{\mathrm{app}}_+\|_{W^{\rho_0,\infty}}\big(\|u_+^{\mathrm{app}}\|_{W^{\rho_0,\infty}} + \|\tilde{u}\|_{W^{\rho_0,\infty}}\big)\big(\|u_+^{\mathrm{app}}\|_{L^2} + \|\tilde{u}_+\|_{L^2}\big) \tag{7.81}$$

which is largely controlled by (7.75) by (7.1)–(7.3).

The L^2 norm of (7.78) (or of the coefficient of x in (7.80)) is bounded from above by the right-hand side of (7.75) (or (7.76)) again by (D.33), (7.1)–(7.3) and the fact that $j \geq 3$.

Consider (7.79). If at least one v_ℓ is replaced by $\tilde{u}_\pm$ or $u'^{\mathrm{app}}_\pm$, we use (D.71), with the special index equal to this ℓ. By (7.1)–(7.3) we largely get an estimate (7.75). If all v_ℓ are equal to $u''^{\mathrm{app}}_\pm$, we use instead (D.72), from which (7.75) largely follows.

To finish the proof of the lemma, we still have to study the last two terms on the right-hand side of (7.52). Contribution $\mathcal{M}'(\tilde{u}, u'^{\mathrm{app}})$ has structure (6.58). The remainders $\mathcal{R}_2$ largely satisfy bounds (7.73), (7.75). The other terms are, by (6.57), of the form $\mathrm{Op}^t(\tilde{m}')(v_1, v_2)$ with $\tilde{m}'$ in $S'_{1,\beta}(M_0(\xi)\prod_{j=1}^2\langle\xi_j\rangle^{-1}, 2)$ and v_1, v_2 equal to $\tilde{u}_\pm$ or $u'^{\mathrm{app},1}_\pm$. By (D.32) and (7.3)–(7.4), the Sobolev estimate (7.73) holds. On the other hand, by (D.76) (and the rapid decay in x of symbols in $S'_{1,\beta}(M_0(\xi)\prod_{j=1}^2\langle\xi_j\rangle^{-1}, 2)$), we have

$$\|L_\pm \mathrm{Op}^t(\tilde{m}')(v_1, v_2)\|_{L^2} \leq C t^{-1+\sigma}\big(\|L_+\tilde{u}_\pm\|_{L^2} + \|L_+ u'^{\mathrm{app},1}_+\|_{L^2} + \|\tilde{u}_+\|_{H^s} + \|u'^{\mathrm{app},1}_+\|_{H^s}\big)^2$$

if $s\sigma$ is large enough. Using (7.3)–(7.4) and taking $\sigma < \frac{1}{4}$, we estimate this by the right-hand side of (7.75).

Finally, the last term $\mathcal{R}$ in (7.52) satisfies (5.39)–(5.40), so also (7.73) and (7.75) for the action of L on it. This concludes the proof of the lemma. ∎

Proof of Proposition 7.3.4. We shall prove successively (7.71) and (7.72).

Step 1: Proof of (7.71). Since $C(t)$ is bounded on H^s, we may ignore it. We thus need to study $\|\hat{\mathcal{M}}_2'(v_1, v_2)\|_{H_h^s}$, where (up to symmetries)

$$v_1 = (D_t - P_0)\tilde{u} \text{ or } (D_t - P_0)u'^{\text{app},1}, \quad v_2 = \tilde{u} \text{ or } u'^{\text{app},1}. \tag{7.82}$$

Recall that $\hat{\mathcal{M}}_2'$ is given by (6.47) in term of operators Q_{i_1,i_2} of the form (F.35). We have thus to bound

$$t^{-\frac{3}{2}}\|K_{H,i_1,i_2}^{\ell_1,\ell_2}(L_{i_1}^{\ell_1}v_{1,i_1}, L_{i_2}^{\ell_2}v_{2,i_2})\|_{H^s} \tag{7.83}$$

with operators $K_{H,i_1,i_2}^{\ell_1,\ell_2}$ in the class $\mathcal{K}'_{1,\frac{1}{2}}(1, i_1, i_2)$ introduced in Definition F.4.1.

Consider first the case $v_1 = (D_t - P_0)u'^{\text{app},1}$. We apply Corollary F.5.4 when ℓ_1 or ℓ_2 is non-zero and (F.46) if $\ell_1 = \ell_2 = 0$. We obtain for $\sigma > 0$ small and $s\sigma$ large enough a bound of (7.83) by

$$\begin{aligned} Ct^{-\frac{3}{4}}\Big(&t^{\sigma}\|L(D_t - P_0)u'^{\text{app},1}\|_{L^2}\big(\|\tilde{u}\|_{H^s} + \|u'^{\text{app},1}\|_{H^s}\big) \\ &+ t^{\sigma}\big(\|L\tilde{u}\|_{L^2} + \|Lu'^{\text{app},1}\|_{L^2}\big)\|(D_t - P_0)u'^{\text{app},1}\|_{H^s} \\ &+ \|(D_t - P_0)u'^{\text{app},1}\|_{H^s}\big(\|\tilde{u}\|_{H^s} + \|u'^{\text{app},1}\|_{H^s}\big)\Big). \end{aligned} \tag{7.84}$$

By the end of the statement of Proposition 4.1.2, $u'^{\text{app},1}_+$ satisfies estimates of the form (4.46)–(4.47) and also (4.39)–(4.41). Moreover, $\tilde{u}$ satisfies (7.3). Plugging these estimates in (7.84), we get a better upper bound than (7.71).

Consider next the case $v_1 = (D_t - P_0)\tilde{u}$, $\ell_1 = 1$ in (7.83). Decompose

$$K_{H,i_1,i_2}^{\ell_1,\ell_2} = K_< + K_>,$$

where $K_<$ (resp. $K_>$) is defined by the same formula (F.25) as $K_{H,i_1,i_2}^{\ell_1,\ell_2}$, but with the function k cut-off for $|\xi_1| \le 2\langle\xi_2\rangle$ (resp. $|\xi_2| \le 2\langle\xi_1\rangle$). We need to bound

$$t^{-\frac{3}{2}}\|K_<(L_{i_1}(D_t - i_1 p(D_x))\tilde{u}_{i_1}, L_{i_2}^{\ell_2}v_{2,i_2})\|_{H^s}, \tag{7.85}$$

$$t^{-\frac{3}{2}}\|K_>(L_{i_1}(D_t - i_1 p(D_x))\tilde{u}_{i_1}, L_{i_2}^{\ell_2}v_{2,i_2})\|_{H^s}, \tag{7.86}$$

where $\ell_2 = 0$ or 1 and $v_2 = \tilde{u}$ or $u'^{\text{app},1}$. Consider first expression (7.85). We decompose the first argument in $K_<$ under the form $g_1 + g_2$, where, for $\chi \in C_0^\infty(\mathbb{R})$, equal to one close to zero,

$$g_1 = (1-\chi)(t^{-\beta}D_x)\big(L_{i_1}(D_t - i_1 p(D_x))\tilde{u}_{i_1}\big), \tag{7.87}$$

$$g_2 = \chi(t^{-\beta}D_x)(f_{1,i_1} + xf_{2,i_1}), \tag{7.88}$$

where we used decomposition (7.74). Using the definition of L_{i_1} and (7.73), we may rewrite g_1 as a sum $g_1 = tg_1' + xg_1''$ with according to (7.73), for any $\sigma_0 \le s$,

$$\|g_1'\|_{H^{\sigma_0}} + \|g_1''\|_{H^{\sigma_0}} \le t^{-\beta(s-\sigma_0)} \varepsilon t^{\delta-\frac{1}{2}}. \tag{7.89}$$

Applying (F.38)–(F.40) (with the roles of f_1, f_2 interchanged), we see that (7.85) with the first argument of $K_<$ replaced by g_1 has Sobolev norm bounded from above by

$$C t^{\frac{1}{4}-\beta(s-\sigma_0)} \varepsilon t^{\delta-\frac{1}{2}} \big(\|\tilde{u}\|_{H^s} + \|u'^{\mathrm{app},1}\|_{H^s}\big).$$

If $s\beta$ is large enough, we get an estimate by the right-hand side of (7.71). On the other hand, if we replace the first argument of $K_<$ in (7.85) by g_2, we reduce ourselves to

$$t^{-\frac{3}{2}} \|K_<(\tilde{\chi}(t^{-\beta} D_x) \tilde{f}_{1,i_1}, L_{i_2}^{\ell_2} v_2)\|_{H^s}, \tag{7.90}$$

$$t^{-\frac{3}{2}} \|K_<(x \tilde{\chi}(t^{-\beta} D_x) \tilde{f}_{2,i_1}, L_{i_2}^{\ell_2} v_2)\|_{H^s} \tag{7.91}$$

for new functions $\tilde{f}_1, \tilde{f}_2$ satisfying the same estimates (7.75)–(7.76) as f_1, f_2 and $\tilde{\chi}$ in $C_0^\infty(\mathbb{R})$. Decomposing $L_{i_2} = x + i_2 t p'(D_x)$ and using (F.38)–(F.39) with the roles of f_1, f_2 interchanged, we bound (7.90) by

$$t^{-\frac{3}{4}} \|\tilde{\chi}(t^{-\beta} D_x) \tilde{f}_{1,i_1}\|_{H^{\sigma_0}} \|v_2\|_{H^s}.$$

By (7.75) and (7.3)–(7.4), this is smaller than

$$t^{-\frac{3}{4}+\beta\sigma_0} t^{-\frac{1}{2}} \big(t^{\frac{1}{4}} (\varepsilon^2 \sqrt{t})^\theta\big) \varepsilon t^\delta$$

so than the right-hand side of (7.71) if $t \le \varepsilon^{-4+c}$ and β is small enough. To study (7.91), we decompose again L_{i_2} as above and use (F.39) and (F.40), to obtain a bound in

$$t^{-\frac{1}{4}} \|\tilde{\chi}(t^{-\beta} D_x) \tilde{f}_2\|_{H^{\sigma_0}} \|v_2\|_{H^s}.$$

By (7.76) for $\tilde{f}_2$ and (7.3), (7.4), we obtain a bound by the right-hand side of (7.71).

Let us study next (7.86). If $\ell_2 = 1$, we use (F.52) (with f_1 and f_2 interchanged) and if $\ell_2 = 0$ we use (F.58). We bound thus (7.86) by

$$C t^{-\frac{3}{4}} \|(D_t - P_0)\tilde{u}\|_{H^s} \big(t^{\beta\sigma_0} \big(\|L\tilde{u}\|_{L^2} + \|Lu'^{\mathrm{app},1}\|_{L^2}\big) + \|\tilde{u}\|_{H^s} + \|u'^{\mathrm{app},1}\|_{H^s}\big).$$

If we use (7.73), (7.3), (7.4), we bound this by the right-hand side of (7.71), using again $t \le \varepsilon^{-4+c}$, and taking β small enough.

To conclude Step 1, we still have to consider (7.83) with $v_1 = (D_t - P_0)\tilde{u}$ and $\ell_1 = 0$, i.e. to bound

$$t^{-\frac{3}{2}} \|K_{H,i_1,i_2}^{0,\ell_2}(D_t - i_1 p(D_x))\tilde{u}_{i_1}, L_{i_2}^{\ell_2} v_{2,i_2})\|_{H^s}.$$

Expressing L_{i_2} and using (F.54) and (F.46), we obtain a bound in

$$t^{-\frac{3}{4}} \|(D_t - P_0)\tilde{u}\|_{H^s} \big(\|\tilde{u}\|_{H^s} + \|u'^{\mathrm{app},1}\|_{H^s}\big).$$

Using (7.73), (7.3), (7.4), we obtain a bound of the form (7.71). This concludes the proof of Step 1.

Step 2: Proof of (7.72). Again, properties (E.20)–(E.22) of operator $C(t)$ allow us to ignore it in the proof of the estimates. We shall have thus to bound $\|L\hat{\mathcal{M}}'_2(v_1, v_2)\|_{L^2}$ where $\hat{\mathcal{M}}'_2$ has structure (6.47) and v_1, v_2 are given by equation (7.82). If we express $L_\pm = x \pm tp'(D_x)$, we are reduced to studying

$$t^{-\frac{1}{2}}\|K^{\ell_1,\ell_2}_{H,i_1,i_2}\big(L^{\ell_1}_{i_1}v_{1,i_1}, L^{\ell_2}_{i_2}v_{2,i_2}\big)\|_{L^2}, \tag{7.92}$$

$$t^{-\frac{3}{2}}\|xK^{\ell_1,\ell_2}_{H,i_1,i_2}\big(L^{\ell_1}_{i_1}v_{1,i_1}, L^{\ell_2}_{i_2}v_{2,i_2}\big)\|_{L^2}. \tag{7.93}$$

By Definition F.4.1 of the class $\mathcal{K}'_{1,1/2}(i)$, $xK^{\ell_1,\ell_2}_{H,i_1,i_2}$ may be written as $t^{\frac{1}{2}}\tilde{K}^{\ell_1,\ell_2}_{H,i_1,i_2}$ for another operator in $\mathcal{K}'_{1,1/2}(i)$. It is thus enough to bound (7.92).

We consider first the case $v_1 = (D_t - P_0)u'^{\mathrm{app},1}$. By (F.50), (F.47), we bound (7.92) by

$$Ct^{-\frac{3}{4}}\big(\|(D_t - P_0)u'^{\mathrm{app},1}\|_{H^s} + t^\sigma\|L(D_t - P_0)u'^{\mathrm{app},1}\|_{L^2}\big)$$
$$\times\big(\|Lu'^{\mathrm{app},1}\|_{L^2} + \|L\tilde{u}\|_{L^2} + \|u'^{\mathrm{app},1}\|_{L^2} + \|\tilde{u}\|_{L^2}\big)$$

for any $\sigma > 0$ (if $s\sigma$ is large enough). Since by Proposition 4.1.2, $u'^{\mathrm{app},1}$ satisfies (4.46)–(4.47), we deduce from (7.3)–(7.4) an estimate better than (7.72).

Consider next the case $v_1 = (D_t - P_0)\tilde{u}$, $\ell_1 = 1$ in (7.92). We replace $L(D_t - P_0)\tilde{u}$ by the right-hand side of (7.74). By (F.47) and (F.51), the f_1 contribution to (7.92) is bounded from above by

$$Ct^{-\frac{3}{4}}\|f_1\|_{L^2}\big(t^\sigma\big(\|Lu'^{\mathrm{app},1}\|_{L^2} + \|L\tilde{u}\|_{L^2}\big) + \|u'^{\mathrm{app}}\|_{H^s} + \|\tilde{u}\|_{H^s}\big).$$

Using (7.75), (7.3), (7.4), we get an estimate in

$$Ct^{-1}\big(t^{\frac{1}{4}}(\varepsilon^2\sqrt{t})^\theta\big)\big((\varepsilon^2\sqrt{t})^\theta t^\sigma + \varepsilon t^{\delta-\frac{1}{4}}\big).$$

If σ is small enough, and since $t \le \varepsilon^{-4+c}$, we get a bound of the form (7.72).

On the other hand, if we replace $(D_t - P_0)\tilde{u}$ by xf_2, (7.92) is reduced to

$$t^{-\frac{1}{2}}\|K^{\ell_1,\ell_2}_{H,i_1,i_2}(xf_{2,i_1}, L^{\ell_2}_{i_2}v_{2,i_2})\|_{L^2}. \tag{7.94}$$

A ∂_{ξ_1}-integration by parts in (F.25) using (F.27) shows that (7.94) is reduced to

$$\|\tilde{K}^{\ell_1,\ell_2}_{H,i_1,i_2}(f_{2,i_1}, L^{\ell_2}_{i_2}v_{2,i_2})\|_{L^2}$$

for a new operator in the same class. Using (F.47) and (F.51), we get a bound in

$$Ct^{-\frac{1}{4}}\|f_2\|_{L^2}\big(\big(\|Lu'^{\mathrm{app},1}\|_{L^2} + \|L\tilde{u}\|_{L^2}\big)t^\sigma + \|u'^{\mathrm{app},1}\|_{H^s} + \|\tilde{u}\|_{H^s}\big).$$

Using (7.76), (7.3), (7.4), we obtain a bound of the form (7.72).

Consider finally the case $v_1 = (D_t - P_0)\tilde{u}$, $\ell_1 = 0$ in (7.92). By (F.47), we get a bound of (7.92) by

$$Ct^{-\frac{3}{4}}\|(D_t - P_0)\tilde{u}\|_{H^s}\big(\|L\tilde{u}\|_{L^2} + \|Lu'^{\mathrm{app},1}\|_{L^2} + \|\tilde{u}\|_{L^2} + \|u'^{\mathrm{app},1}\|_{L^2}\big).$$

If we plug there (7.73) and (7.3)–(7.4), we get an estimate of the form (7.72). This concludes the proof. ∎

This concludes the study of terms of the form (6.65). It remains to study (6.66), (6.67) and (6.68).

Proposition 7.3.6. *The following statements hold.*

(i) *Denote*

$$F(t) = C(t)R_j(\underbrace{\tilde{u},\dots,\tilde{u}}_{\ell}, u^{\mathrm{app}},\dots,u^{\mathrm{app}}), \quad j = 3,4,\ 0 \le \ell \le j, \tag{7.95}$$

with R_j of the form (6.34)–(6.35). Then there is a function e satisfying (5.41) such that

$$\|F(t)\|_{H^s} \le t^{-1}\varepsilon t^{\delta} e(t,\varepsilon), \tag{7.96}$$

$$\|L_\pm F(t)\|_{L^2} \le t^{-1}\big(t^{\frac{1}{4}}(\varepsilon^2\sqrt{t})^{\theta}\big)e(t,\varepsilon). \tag{7.97}$$

(ii) *Denote*

$$F(t) = C(t)R_2(\underbrace{\tilde{u},\dots,\tilde{u}}_{\ell}, u'^{\mathrm{app},1},\dots,u'^{\mathrm{app},1})$$

with $0 \le \ell \le 2$ and $R_2 = \left[\begin{smallmatrix} R_{2,+} \\ R_{2,-}\end{smallmatrix}\right]$ given by (6.49). Then (7.96) and (7.97) hold.

(iii) *Let $F(t) = C(t)(\mathcal{R}(t,\cdot) + \tilde{\mathcal{R}}_3(t,\cdot) + \tilde{\mathcal{R}}_4(t,\cdot)) + \tilde{\mathcal{R}}_2(t,\cdot)$ with $\mathcal{R}, \tilde{\mathcal{R}}_j$ as in (6.68). Then (7.96) and (7.97) hold.*

Proof. (i) By (6.35) and (D.32) (and the boundedness of $C(t)$ on H^s), we bound $\|F(t)\|_{H^s}$ by

$$C\big(\|\tilde{u}\|_{W^{\rho_0,\infty}} + \|u^{\mathrm{app}}\|_{W^{\rho_0,\infty}}\big)^{j-1}\big(\|\tilde{u}\|_{H^s} + \|u^{\mathrm{app}}\|_{H^s}\big).$$

As $j \ge 3$, (7.1) and (7.3) imply (7.96).

To prove (7.97), we use once again that by (E.20)–(E.22), we may ignore the factor $C(t)$, and have to estimate LR_j in L^2. This expression is a sum of quantities of the form (6.36)–(6.38), so of the form (7.77)–(7.79) with $v_\ell = \tilde{u}_\pm$ or $v_\ell = u'^{\mathrm{app}}_\pm + u''^{\mathrm{app}}_\pm$.

When v_1 in (7.77) is replaced by $\tilde{u}_\pm$ or $u'^{\mathrm{app}}_\pm$, we use (D.33) to estimate the L^2 norm of these terms by

$$C\big(\|\tilde{u}\|_{W^{\rho_0,\infty}} + \|u^{\mathrm{app}}\|_{W^{\rho_0,\infty}}\big)^{j-1}\big(\|L\tilde{u}\|_{L^2} + \|Lu'^{\mathrm{app}}\|_{L^2}\big)$$

so by the right-hand side of (7.97) by (7.1)–(7.3), since $j \geq 3$. If $v_1 = u''^{\text{app}}$, we have a bound by (7.81) so by

$$\frac{1}{t} t^{\frac{1}{4}} (\varepsilon^2 \sqrt{t})^{\theta} \big((\varepsilon^2 \sqrt{t})^{\frac{1}{2}+\theta'-\theta} t^{\delta} \log(1+t) \log(1+t\varepsilon^2) \big) \tag{7.98}$$

which is bounded by the right-hand side of (7.97) for $\delta > 0$ small, θ, θ' close to $\frac{1}{2}$ if $t \leq \varepsilon^{-4+c}$.

Expression (7.78) is controlled as (7.77). For (7.79), we use (D.71) if at least one of the functions v_j is equal to $\tilde{u}_{\pm}$ or $u'^{\text{app}}_{\pm}$, which brings the wanted estimate (7.97) by (7.1)–(7.3). If all arguments v_j are equal to $u''^{\text{app}}_{\pm}$, we use (D.72), that brings again an estimate of the form (7.98). This concludes the proof of (i).

(ii) Again, we may forget operator $C(t)$. We have to study

$$t^{-2} \| K^{\ell_1,\ell_2}_{L,i_1,i_2}(L^{\ell_1}_{i_1} v_{1,i_1}, L^{\ell_2}_{i_2} v_{2,i_2}) \|_{H^s}, \tag{7.99}$$

$$t^{-2} \| L_{\pm} K^{\ell_1,\ell_2}_{L,i_1,i_2}(L^{\ell_1}_{i_1} v_{1,i_1}, L^{\ell_2}_{i_2} v_{2,i_2}) \|_{L^2} \tag{7.100}$$

with $K^{\ell_1,\ell_2}_{L,i_1,i_2}$ in $\mathcal{K}'_{1/2,1}(i)$, and v_1, v_2 equal to $\tilde{u}$ or $u'^{\text{app},1}$. Since estimates (7.4) are better than (7.3), we may argue just in the case $v_1 = v_2 = \tilde{u}$. Then (7.99) is just (7.31) multiplied by $t^{-\frac{1}{2}}$. It is then estimated by (7.32)–(7.34) multiplied by $t^{-\frac{1}{2}}$ and thus by (7.35) multiplied by $t^{-\frac{1}{2}}$, so by $\varepsilon t^{\delta-1} t^{\sigma} (\varepsilon^2 \sqrt{t})^{\theta}$. For $t \leq \varepsilon^{-4+c}$, this is of the form of the right-hand side of (7.96) if σ is small enough. Let us bound next (7.100). Using the expression $L_{\pm} = x \pm t p'(D_x)$, we have to estimate

$$t^{-1} \| K^{\ell_1,\ell_2}_{L,i_1,i_2}(L^{\ell_1}_{i_1} v_{1,i_1}, L^{\ell_2}_{i_2} v_{2,i_2}) \|_{L^2}, \tag{7.101}$$

$$t^{-2} \| x K^{\ell_1,\ell_2}_{L,i_1,i_2}(L^{\ell_1}_{i_1} v_{1,i_1}, L^{\ell_2}_{i_2} v_{2,i_2}) \|_{L^2}. \tag{7.102}$$

By (F.47), (F.50), (F.51), we bound (7.101) by

$$C t^{-\frac{5}{4}} \big(\|L\tilde{u}\|_{L^2} t^{\sigma} + \|\tilde{u}\|_{H^s} \big)^2.$$

Using (7.3), we obtain

$$C t^{-1} ((\varepsilon^2 \sqrt{t})^{\theta} t^{\frac{1}{4}}) t^{2\sigma} (\varepsilon^2 \sqrt{t})^{\theta}$$

which is smaller than the right-hand side of (7.97) for $t \leq \varepsilon^{-4+c}$ if σ is small enough.

Finally, to study (7.102), we notice, as after (7.38), that this expression may be bounded by $t^{-\frac{1}{2}}$ times (7.101), so has the wanted bounds.

(iii) The contributions $C(t)\tilde{\mathcal{R}}_3$, $C(t)\tilde{\mathcal{R}}_4$, $\tilde{\mathcal{R}}_2$ are estimated by (6.59), (6.54), (6.55), so largely by the right-hand side of (7.96)–(7.97), using (7.1)–(7.3). The fact that $C(t)\mathcal{R}$ satisfies these estimates follows from inequalities (5.39)–(5.40) satisfied by $\mathcal{R}$ (or (6.26)–(6.27)). This concludes the proof. ∎

We conclude this chapter summarizing the estimates we have obtained.

Proposition 7.3.7. *Let $c > 0$ (small) be given, $0 < \theta' < \theta < \frac{1}{2}$ with θ' close to $\frac{1}{2}$. Let $T \in [1, \varepsilon^{-4+c}]$ and assume that we are given on $[1,T] \times \mathbb{R}$ functions $\tilde{u}_+$, u'^{app}_+, u''^{app}_+, $u'^{\text{app},1}_+$ that satisfy estimates (7.1)–(7.4), for some small $\delta > 0$, some constants $C(A, A')$, D, any ε in an interval $]0, \varepsilon_0]$, and such that $\tilde{u}$ solves (6.61). Then there are $D_0 > 0$, $\varepsilon_0' \in]0, \varepsilon_0]$ such that if $D \geq D_0$ and $\varepsilon \in]0, \varepsilon_0']$, for any $t \in [1, T]$, the L^2 estimates in (7.3) may be improved to*

$$\|\tilde{u}_+(t,\cdot)\|_{H^s} \leq \frac{D}{2}\varepsilon t^{\delta}, \tag{7.103}$$

$$\|L_+\tilde{u}_+(t,\cdot)\|_{L^2} \leq \frac{D}{2} t^{\frac{1}{4}}(\varepsilon^2\sqrt{t})^{\theta}. \tag{7.104}$$

Proof. By Corollary 6.2.5, we know that

$$(D_t - P_0)\mathring{u} = \hat{\mathcal{R}} \tag{7.105}$$

if we define

$$\mathring{u} = C(t)\Big(\tilde{u} - \sum_{j=3}^{4} \hat{\mathcal{M}}_j(\tilde{u}, u^{\text{app}})\Big) - \hat{\mathcal{M}}_2'(\tilde{u}, u'^{\text{app},1}). \tag{7.106}$$

By Proposition 7.1.1, Proposition 7.2.1 and the boundedness properties (E.19)–(E.22) of $C(t)$, we have

$$\|\mathring{u} - C(t)\tilde{u}\|_{H^s} \leq \varepsilon t^{\delta} e(t,\varepsilon), \tag{7.107}$$

$$\|L(\mathring{u} - C(t)\tilde{u})\|_{L^2} \leq t^{\frac{1}{4}}(\varepsilon^2\sqrt{t})^{\theta} e(t,\varepsilon), \tag{7.108}$$

where e satisfies (5.41).

The right-hand side $\hat{\mathcal{R}}$ of (7.105) is the sum of terms (6.62)–(6.68). These terms have been estimated in Proposition 7.1.2, Proposition 7.2.3, Proposition 7.3.1, Proposition 7.3.3, Proposition 7.3.4, Proposition 7.3.6, which imply that

$$\begin{aligned} \|\hat{R}(t,\cdot)\|_{H^s} &\leq \varepsilon t^{\delta-1} e(t,\varepsilon), \\ \|L\hat{R}(t,\cdot)\|_{L^2} &\leq t^{-1} t^{\frac{1}{4}}(\varepsilon^2\sqrt{t})^{\theta} e(t,\varepsilon). \end{aligned} \tag{7.109}$$

By the fact that L commutes to $(D_t - P_0)$, it follows from the energy inequality applied to (7.105) that

$$\|\mathring{u}(t,\cdot)\|_{H^s} \leq \|\mathring{u}(1,\cdot)\|_{H^s} + \varepsilon t^{\delta} e(t,\varepsilon), \tag{7.110}$$

$$\|L\mathring{u}(t,\cdot)\|_{L^2} \leq \|L\mathring{u}(1,\cdot)\|_{L^2} + t^{\frac{1}{4}}(\varepsilon^2\sqrt{t})^{\theta} e(t,\varepsilon) \tag{7.111}$$

and then, by (7.107)–(7.108) and (E.14), (E.19)–(E.22) that

$$\|\tilde{u}(t,\cdot)\|_{H^s} \leq C\|\tilde{u}(1,\cdot)\|_{H^s} + \varepsilon t^{\delta} e(t,\varepsilon), \tag{7.112}$$

$$\|L\tilde{u}(t,\cdot)\|_{L^2} \leq C\big(\|L\tilde{u}(1,\cdot)\|_{L^2} + \|\tilde{u}(1,\cdot)\|_{L^2}\big) + t^{\frac{1}{4}}(\varepsilon^2\sqrt{t})^{\theta} e(t,\varepsilon) \tag{7.113}$$

for some constant C, some new factors $e(t,\varepsilon)$. Recall that $\tilde{u}_+$ has been defined from u_+ in (5.34), and that since this function is $O(\varepsilon)$ at time $t = 1$ in the space $\{f \in H^s : xf \in L^2\}$ by (2.24) and (2.22), we may take D so large that the first term on the right-hand side of (7.112)–(7.113) is smaller than $\frac{D}{4}\varepsilon$. If ε is small enough, we thus get (7.103)–(7.104) using (5.41). ∎