Chapter 8

L^∞ estimates and end of bootstrap

The goal of this chapter is to conclude the bootstrap argument that gives our main theorem. At the end of the preceding chapter, we have seen that assuming a priori estimates (7.3), we could prove that the first and last ones hold with a better constant. Here, we shall bootstrap the $W^{\rho,\infty}$ bound in (7.3). Once this is done, we still have to go back to the original unknowns of the statement of our main Theorem 2.1.1 and to deduce from estimates of $\tilde{u}$ and from the study made in Section 4.2 the bounds of the quantities that appear in that theorem.

8.1 L^∞ estimates

One cannot deduce an L^∞ estimate of the form of the second inequality in (7.3) from the Sobolev estimates satisfied by $\tilde{u}_+, L_+\tilde{u}_+$ through Klainerman–Sobolev inequalities: the fact that $\|L_+\tilde{u}_+\|_{L^2}$ admits only an $O(t^{\frac{1}{4}})$ bound would be too rough in order to do so. Instead, we deduce from the equation satisfied by $\tilde{u}$ an ODE, that will allow us to get the wanted L^∞ bound.

We shall reduce ourselves to the semiclassical framework, defining from the solution $\tilde{u} = \begin{bmatrix} \tilde{u}_+ \\ \tilde{u}_- \end{bmatrix}$ of (6.61) a function $\underline{\tilde{u}} = \begin{bmatrix} \underline{\tilde{u}}_+ \\ \underline{\tilde{u}}_- \end{bmatrix}$ by

$$\tilde{u}_\pm = \frac{1}{\sqrt{t}}\underline{\tilde{u}}_\pm\Big(t, \frac{x}{t}\Big) = (\Theta_t\underline{\tilde{u}})(t,x) \tag{8.1}$$

using notation (B.15). We set $h = t^{-1}$ and decompose for a given $\rho \geq 0$,

$$\langle hD_x\rangle^\rho \underline{\tilde{u}}_\pm = \underline{\tilde{u}}^\rho_{\pm,\Lambda} + \underline{\tilde{u}}^\rho_{\pm,\Lambda^c} \tag{8.2}$$

with according to notation (D.91)

$$\underline{\tilde{u}}^\rho_{\pm,\Lambda} = \mathrm{Op}_h^{\mathrm{W}}\Big(\gamma\Big(\frac{x \pm p'(\xi)}{\sqrt{h}}\Big)\Big)\mathrm{Op}_h^{\mathrm{W}}(\langle\xi\rangle^\rho)\underline{\tilde{u}}_\pm, \tag{8.3}$$

where $\gamma \in C_0^\infty(\mathbb{R})$ has small enough support and is equal to 1 close to zero. We denote by $\tilde{u}^\rho_{\pm,\Lambda}, \tilde{u}^\rho_{\pm,\Lambda^c}$ the functions corresponding to $\underline{\tilde{u}}^\rho_{\pm,\Lambda}, \underline{\tilde{u}}^\rho_{\pm,\Lambda^c}$ by a change of variables of the form (8.1).

The contribution $\underline{\tilde{u}}^\rho_{\pm,\Lambda^c}$ has nice L^∞ bounds by Klainerman–Sobolev estimates:

Proposition 8.1.1. *For any $\sigma > 0$, any s with $s\sigma$ large enough, one has the following estimate:*

$$\|\tilde{u}^\rho_{\pm,\Lambda^c}\|_{L^\infty} \leq Ct^{-\frac{3}{4}+\sigma}\big(\|L_\pm\tilde{u}_\pm\|_{L^2} + \|\tilde{u}_\pm\|_{H^s}\big). \tag{8.4}$$

Proof. Translating that on $\underline{\tilde{u}}^{\rho}_{\pm,\Lambda^c}$, this means

$$\|\underline{\tilde{u}}^{\rho}_{\pm,\Lambda^c}\|_{L^\infty} \leq C h^{\frac{1}{4}-\sigma}\big(\|\mathcal{L}_{\pm}\underline{\tilde{u}}_{\pm}\|_{L^2} + \|\underline{\tilde{u}}_{\pm}\|_{H^s_h}\big).$$

This is just statement (D.87) in Proposition D.3.4. ∎

We study from now on the function $\underline{\tilde{u}}^{\rho}_{\pm,\Lambda}$. We first prove some bounds for expressions (5.43)–(5.49), whose sum is equal to $(D_t - p(D_x))\tilde{u}_+$. If $W(t,x)$ is some function and $\underline{W}$ is defined from W by (8.1), i.e. $W(t,\cdot) = \Theta_t \underline{W}(t,\cdot)$, we denote by $\underline{W}^{\rho}_{\Lambda}$ the function defined by (8.3) with sign $+$ and $\underline{\tilde{u}}_{\pm}$ replaced by $\underline{W}$, and we shall call W^{ρ}_{Λ} the function $W^{\rho}_{\Lambda} = \Theta_t \underline{W}^{\rho}_{\Lambda}$.

Lemma 8.1.2. *Let*

$$a(t) = \frac{\sqrt{3}}{3}(a_+(t) - a_-(t)), \quad a^{\text{app}}(t) = \frac{\sqrt{3}}{3}(a^{\text{app}}_+(t) - a^{\text{app}}_-(t)),$$

where $a_- = -\bar{a}_+$, $a^{\text{app}}_- = -\overline{a^{\text{app}}_+}$, and where a_+, a^{app}_+ satisfy by (4.96)–(4.100)

$$|a^{\text{app}}_+(t)| \leq C t_\varepsilon^{-\frac{1}{2}}, \quad |a_+(t) - a^{\text{app}}_+(t)| \leq C t_\varepsilon^{-\frac{3}{2}} \tag{8.5}$$

for t in the interval $[1,T]$, $T \leq \varepsilon^{-4+c}$, where these functions are defined. Assume moreover that on that interval, the functions $\tilde{u}_+, u'^{\text{app}}_+, u''^{\text{app}}_+$ satisfy (7.1)–(7.3). Then the quantities (5.43)–(5.49) satisfy the following estimates, with a constant C depending on the constants A, A', D in (7.1)–(7.3):

$$\|(5.43)\|_{W^{\rho,\infty}} \leq C t^{-\frac{3}{2}}(\varepsilon^2\sqrt{t})^{\theta}, \tag{8.6}$$

$$\|(5.44)\|_{W^{\rho,\infty}} \leq C t^{-\frac{3}{2}}(\varepsilon^2\sqrt{t})^{\theta}, \tag{8.7}$$

$$\|(5.45)\|_{W^{\rho,\infty}} \leq C t^{-\frac{3}{2}}(\varepsilon^2\sqrt{t})^{\theta}, \tag{8.8}$$

$$\|(5.46)^{\rho}_{\Lambda}\|_{L^\infty} \leq C t^{-\frac{3}{2}+\sigma}(\varepsilon^2\sqrt{t})^{\theta}, \tag{8.9}$$

$$\|(5.47)\|_{W^{\rho,\infty}} \leq C t^{-\frac{3}{2}}(\varepsilon^2\sqrt{t})^{\theta}, \tag{8.10}$$

$$\|(5.48)\|_{W^{\rho,\infty}} \leq C t^{-\frac{3}{2}}(\varepsilon^2\sqrt{t})^{\theta}, \tag{8.11}$$

$$\|(5.49)\|_{W^{\rho,\infty}} \leq C t^{-\frac{3}{2}+\sigma}(\varepsilon^2\sqrt{t})^{\theta}, \tag{8.12}$$

where $\sigma > 0$ may be taken as small as one wants if $s\sigma$ is large enough (s being the index of Sobolev estimates (7.1)–(7.3)) relatively to ρ, and where in (8.9) one uses the notation W^{ρ}_{Λ} defined before the statement of the lemma.

Proof. We prove the inequalities separately.

Inequality (8.6). This inequality follows from (5.58) and the fact that $t_\varepsilon^{-\frac{1}{2}} \leq \varepsilon$.

Inequality (8.7). We have seen in the proof of Proposition 5.2.1 that (5.44) is a sum of terms of the form (5.60) or (5.61), with conditions (5.62) or (5.63), i.e. may be written from

$$\mathrm{Op}(m)(v_1,\dots,v_n), \tag{8.13}$$

where m is in $\tilde{S}_{1,0}(\prod_{j=1}^n \langle\xi_j\rangle^{-1} M_0^\nu, n)$, with $n \geq 3$ and v_j equal to $\tilde{u}_\pm$ or $u'^{\mathrm{app}}\pm$ or $u''^{\mathrm{app}}_\pm$ or R (with R satisfying (5.25)–(5.26)). In particular, by Sobolev estimates, one has

$$\|R(t,\cdot)\|_{W^{\rho,\infty}} \leq C\left(\frac{(\varepsilon^2\sqrt{t})^{\theta'} t^\sigma}{\sqrt{t}}\right)^4 \varepsilon t^\delta. \tag{8.14}$$

If we apply (D.39), we obtain for the $W^{\rho,\infty}$ norm of (8.13) a bound in

$$\begin{aligned}&\big(\|\tilde{u}_+\|_{W^{\rho,\infty}} + \|u'^{\mathrm{app}}_+\|_{W^{\rho,\infty}} + \|u''^{\mathrm{app}}_+\|_{W^{\rho,\infty}} + \|R\|_{W^{\rho,\infty}}\big)^2\\ &\quad\times\Big(t^\sigma\big(\|\tilde{u}_+\|_{W^{\rho,\infty}} + \|u'^{\mathrm{app}}_+\|_{W^{\rho,\infty}} + \|u''^{\mathrm{app}}_+\|_{W^{\rho,\infty}} + \|R\|_{W^{\rho,\infty}}\big)\\ &\qquad + t^{-1}\big(\|\tilde{u}_+\|_{H^s} + \|u'^{\mathrm{app}}_+\|_{H^s} + \|u''^{\mathrm{app}}_+\|_{H^s} + \|R\|_{H^s}\big)\Big).\end{aligned}$$

By (7.1)–(7.3) and (5.25), (8.14), this is smaller than the right-hand side of (8.7) (if we use that $(\varepsilon^2\sqrt{t})^{3\theta'-\theta} t^\sigma \leq C$ for $t \leq \varepsilon^{-4+c}$).

Inequality (8.8). Expression (5.45) to estimate has been seen to be of the form (5.71) or (5.72), with either (5.73) or (5.74). Terms corresponding to (5.73) are of the form (8.13) and, as we have just seen, satisfy the wanted bound. We have just to consider expressions (5.71) or (5.72) under (5.74), i.e. quantities of the form

$$\mathrm{Op}(m')(v_1,v_2), \tag{8.15}$$

where m' is in $\tilde{S}'_{1,0}(\prod_{j=1}^2 \langle\xi_j\rangle^{-1} M_0^\nu, 2)$, and v_1, v_2 taken among $\tilde{u}_\pm$, $u'^{\mathrm{app}}_\pm$, $u''^{\mathrm{app}}_\pm$, R. If both v_1, v_2 are different from $u''^{\mathrm{app}}_\pm$, we use (D.77) with $r = 2$, $n = 2$, $\ell = 0$. We get a bound in

$$\begin{aligned}&t^{-2+\sigma}\big(\|u'^{\mathrm{app}}_+\|_{H^s} + \|\tilde{u}_+\|_{H^s} + \|R\|_{H^s}\\ &\qquad + \|L_+ u'^{\mathrm{app}}_+\|_{L^2} + \|L_+\tilde{u}_+\|_{L^2} + \|L_+ R\|_{L^2}\big)^2\end{aligned} \tag{8.16}$$

(estimating the $W^{\rho_0,\infty}$ norm from the H^s one). It follows from (5.25) and (5.26) that $\|L_+R\|_{L^2} \leq C(t^{\frac14}(\varepsilon^2\sqrt{t})^\theta)$. Using also (7.1) and (7.3), we estimate (8.16) by the right-hand side of (8.8), when $t \leq \varepsilon^{-4+c}$ if σ is small enough. Consider next the case when v_1 or v_2 is equal to $u''^{\mathrm{app}}_\pm$. If for instance $v_1 = u''^{\mathrm{app}}_\pm$ and $v_2 = \tilde{u}_\pm$ or $u'^{\mathrm{app}}_\pm$ or R, we apply (D.77) with $n = 2$, $\ell = 1$. The first term on the right-hand side of this expression is largely estimated by (8.8) if r is taken large enough. The second one is smaller than

$$\begin{aligned}&Ct^{-2+\sigma}\big(\|u''^{\mathrm{app}}_+\|_{W^{\rho,\infty}} + \|L_+ u''^{\mathrm{app}}_+\|_{W^{\rho,\infty}}\big)\\ &\quad\times\big(\|u'^{\mathrm{app}}_+\|_{H^s} + \|\tilde{u}_+\|_{H^s} + \|R\|_{H^s}\\ &\qquad + \|L_+u'^{\mathrm{app}}_+\|_{L^2} + \|L_+\tilde{u}_+\|_{L^2} + \|L_+R\|_{L^2}\big).\end{aligned}$$

By (7.1)–(7.3) and (5.25)–(5.26), this is largely bounded by the right-hand side of inequality (8.8).

If v_1 and v_2 are both equal to $u''^{\text{app}}_{\pm}$, we use (D.77) with $\ell = n = 2$. We obtain a bound in $t^{-2+\sigma}(\log(1+t))^2(\log(1+t\varepsilon^2))^2$ for the second contribution to the right-hand side of (D.77). If σ is small enough, this is better than (8.8) since $\theta \leq \frac{1}{2}$.

Inequality (8.9). It follows from (D.82) (with a large enough r) translated in the non-semiclassical framework, that for any function W

$$\|W^{\rho}_{\Lambda}\|_{L^\infty} \leq C\big(t^{-\frac{1}{4}+\sigma}\|W\|_{L^2} + t^{-2}\|W\|_{H^s}\big). \tag{8.17}$$

To estimate (8.9), we decompose expression (5.46) as the sum of (5.80)–(5.83). Consider first the nonlinear quantity (5.82), that may be written as (5.85). By (D.88) and the fact that $a(t) = O(t_\varepsilon^{-1/2})$, its contribution to (8.9) is bounded from above by

$$t^{\sigma} t_\varepsilon^{-\frac{1}{2}}\big(\|\mathrm{Op}(m')(v_1,\dots,v_n)\|_{W^{\rho,\infty}} + t^{-r}\|\mathrm{Op}(m')(v_1,\dots,v_n)\|_{H^s}\big) \tag{8.18}$$

for any r, if $\sigma > 0$ and $s\sigma$ is large enough, m' being in $\tilde{S}'_{1,0}(\prod_{j=1}^n \langle\xi_j\rangle^{-1} M_0^{\nu}, n)$, $2 \leq n \leq 4$, v_j being equal to $\tilde{u}_\pm$ or $u'^{\text{app}}_{\pm}$ or $u''^{\text{app}}_{\pm}$. Since (8.18) involves expressions of the form (8.13) or (8.15), we already know that the first term is estimated by the right-hand side of (8.9). The second term is easily bounded, as r is arbitrary.

We have thus just to consider the linear expressions (5.80), (5.81), (5.83). As $a(t) = O(t_\varepsilon^{-1/2})$, $a(t) - a^{\text{app}}(t) = O(t_\varepsilon^{-3/2})$ by (8.5), the expressions to study are of the form

$$\begin{aligned} &t_\varepsilon^{-\frac{1}{2}}\mathrm{Op}(m')\tilde{u}_\pm,\\ &t_\varepsilon^{-\frac{1}{2}}\mathrm{Op}(m')R, \end{aligned} \tag{8.19}$$

$$\begin{aligned} &t_\varepsilon^{-\frac{3}{2}}\mathrm{Op}(m')u'^{\text{app}}_{\pm},\\ &t_\varepsilon^{-\frac{3}{2}}\mathrm{Op}(m')u''^{\text{app}}_{\pm}, \end{aligned} \tag{8.20}$$

where m' is in $\tilde{S}'_{1,0}(\langle\xi\rangle^{-1}, 1)$. We replace in (8.17) W by (8.19) or (8.20). It follows from (D.71) and (D.32) with $n = 1$ that the contribution of (8.19) to the right-hand side of (8.17) is bounded from above by

$$t^{-\frac{5}{4}+\sigma} t_\varepsilon^{-\frac{1}{2}}\big(\|\tilde{u}_\pm\|_{H^s} + \|R\|_{H^s} + \|L_\pm \tilde{u}_\pm\|_{L^2} + \|L_\pm R\|_{L^2}\big).$$

Combined with (7.1), (7.3) and (5.25)–(5.26), this gives an estimate in $t^{-\frac{3}{2}+\sigma}(\varepsilon^2\sqrt{t})^{\theta}$ as wanted.

To study the contribution of (8.20) to the right-hand side of (8.17), we just apply the Sobolev boundedness of $\mathrm{Op}(m')$ to get

$$t_\varepsilon^{-\frac{3}{2}} t^{-\frac{1}{4}+\sigma}\big(\|u'^{\text{app}}_{+}\|_{H^s} + \|u''^{\text{app}}_{+}\|_{H^s}\big).$$

Combining with (7.1) and (7.2), we get again the wanted bound. This concludes the study of (8.9).

Inequality (8.10). Expression (5.47) is made of terms of the form (5.45) or (5.44) multiplied by the decaying factor $a(t)$. It is thus estimated by better quantities than the right-hand side of (8.7)–(8.8).

Inequality (8.11). To estimate (5.48), we notice first that terms in that expression corresponding to $|I| \geq 2$ have already been treated in the proof of (8.7) and (8.8). It remains thus to study the linear terms, that are of the form

$$a(t)^j \operatorname{Op}(m')u_\pm, \quad j \geq 2,$$

with m' in $\tilde{S}'_{1,0}(\langle\xi\rangle^{-1}, 1)$. By expression (5.59) of u_+, we shall get terms of the form (5.82) with $a(t)$ replaced by $a(t)^2$. These terms have already been considered in the study of (8.7) and (8.8) (see (8.13) and (8.15)). We obtain also linear terms in

$$\begin{aligned} &a(t)^j \operatorname{Op}(m')\tilde{u}_\pm, && a(t)^j \operatorname{Op}(m')u'^{\text{app}}_\pm, \\ &a(t)^j \operatorname{Op}(m')u''^{\text{app}}_\pm, && a(t)^j \operatorname{Op}(m')R \end{aligned} \tag{8.21}$$

with $j \geq 2$. To study those terms in (8.21) of the form $a(t)^j \operatorname{Op}(m')w$ with $w = \tilde{u}_\pm$ or $u'^{\text{app}}_\pm$ or R, we use (D.77) with $n = 1$, $\ell = 0$. We obtain an estimate of the $W^{\rho,\infty}$ norm in

$$\begin{aligned} Ct_\varepsilon^{-1}t^{-1+\sigma}\big(&\|u'^{\text{app}}_+\|_{H^s} + \|\tilde{u}_+\|_{H^s} + \|R_+\|_{H^s} \\ &+ \|L_+\tilde{u}_+\|_{L^2} + \|L_+u'^{\text{app}}_+\|_{L^2} + \|L_+R\|_{L^2}\big). \end{aligned}$$

Combined with (7.1)–(7.2) and (5.25)–(5.26), this largely implies a bound by the right-hand side of (8.11). Finally, the $W^{\rho,\infty}$ norm of the terms in (8.21) involving $u''^{\text{app}}_\pm$ is estimated using (D.77) when $n = 1, \ell = 1$. One obtains

$$Ct_\varepsilon^{-1}t^{-1+\sigma}\big(\|u''^{\text{app}}_+\|_{H^s} + \|u''^{\text{app}}_+\|_{W^{\rho,\infty}} + \|L_+u''^{\text{app}}_+\|_{W^{\rho,\infty}}\big)$$

which by (7.2) is also largely estimated by (8.11).

Inequality (8.12). Finally, (8.12) follows from the fact that (5.49) satisfies bounds (4.38), that largely imply (8.12). ■

We may deduce from the above lemma an L^∞ bound for $(D_t - p(D_x))\tilde{u}_+$.

Proposition 8.1.3. *Denote $f_+ = (D_t - p(D_x))\tilde{u}_+$ and define $\underline{f}_+$ by*

$$f_+(t,x) = \frac{1}{\sqrt{t}}\underline{f}_+\Big(t, \frac{x}{t}\Big) = \Theta_t \underline{f}_+(t,x) \tag{8.22}$$

using notation (B.15). *According to* (D.91), *define*

$$\underline{f}^{\rho}_{+,\Lambda} = \operatorname{Op}_h^{\mathrm{W}}\Big(\gamma\Big(\frac{x + p'(\xi)}{\sqrt{h}}\Big)\Big)\operatorname{Op}_h^{\mathrm{W}}(\langle\xi\rangle^\rho)\underline{f}_+. \tag{8.23}$$

Then, under a priori assumption (7.3) *on $\tilde{u}_+$, for any $\sigma > 0$, any s such that $s\sigma$ is large enough, one has*

$$\|\underline{f}^{\rho}_{+,\Lambda}(t,\cdot)\|_{L^\infty} \leq Ch^{1-\sigma}(\varepsilon^2\sqrt{t})^\theta. \tag{8.24}$$

Proof. Recall that

$$f_+ = (D_t - p(D_x))\tilde{u}_+$$

is given by the sum of expressions (5.43)–(5.49). Call $f_{+,2}$ contribution (5.46) and $f_{+,1}$ the sum of all other contributions. Define $\underline{f}^{\rho}_{+,j,\Lambda}$, $j = 1, 2$, from $\underline{f}_{+,j}$ as in (8.23). Then (8.9) shows that $\underline{f}^{\rho}_{+,2,\Lambda}$ satisfies (8.24). To obtain the same estimates for $\underline{f}^{\rho}_{+,1,\Lambda}$, we apply (D.88) in order to bound the different contributions to $\underline{f}^{\rho}_{+,1,\Lambda}$ in L^∞ from (8.6)–(8.8) and (8.10)–(8.12), using moreover (7.73) in order to estimate the H^s norm in (D.88) (taking the power N in the pre-factor h^N large enough). This concludes the proof. ∎

We shall now write an ODE satisfied by function (8.3).

Proposition 8.1.4. *Assume a priori assumptions (7.1)–(7.3). There is a real-valued function θ_h, supported in $]-1,1[$ such that $\underline{\tilde{u}}^{\rho}_{+,\Lambda}$ defined by (8.3) satisfies*

$$\big(D_t - \theta_h(x)\sqrt{1-x^2}\big)\underline{\tilde{u}}^{\rho}_{+,\Lambda} = O_{L^\infty}\big(t^{-1+\sigma}(\varepsilon^2\sqrt{t})^{\theta}\big), \tag{8.25}$$

where $\sigma > 0$ is as small as one wants (if s in estimate (7.3) is large enough relatively to $\frac{1}{\sigma}$).

Proof. Denote as in the preceding proposition $f_+ = (D_t - p(D_x))\tilde{u}_+$, so that

$$(D_t - p(D_x))\big(\langle D_x\rangle^{\rho}\tilde{u}_+\big) = \langle D_x\rangle^{\rho} f_+.$$

If $\underline{f}_+$ is given by (8.22) and $\underline{\tilde{u}}_+$ by (8.1), this is equivalent to

$$\Big(D_t - \mathrm{Op}_h^{\mathrm{W}}\big(x\xi + \sqrt{1+\xi^2}\big)\Big)\mathrm{Op}_h^{\mathrm{W}}(\langle\xi\rangle^{\rho})\underline{\tilde{u}}_+ = \mathrm{Op}_h^{\mathrm{W}}(\langle\xi\rangle^{\rho})\underline{f}_+. \tag{8.26}$$

We make act $\mathrm{Op}_h^{\mathrm{W}}(\gamma(\frac{x+p'(\xi)}{\sqrt{h}}))$ on (8.26). By (D.94) and the definition (8.3) of $\underline{\tilde{u}}^{\rho}_{+,\Lambda}$, we obtain

$$\Big(D_t - \mathrm{Op}_h^{\mathrm{W}}\big(x\xi + \sqrt{1+|\xi|}\big)\Big)\underline{\tilde{u}}^{\rho}_{+,\Lambda} = \underline{f}^{\rho}_{+,\Lambda} + R_1 + R_2 \tag{8.27}$$

with

$$R_1 = h\mathrm{Op}_h^{\mathrm{W}}\Big(\gamma_{-1}\Big(\frac{x+p'(\xi)}{\sqrt{h}}\Big)\Big(\frac{x+p'(\xi)}{\sqrt{h}}\Big)\Big)\mathrm{Op}_h^{\mathrm{W}}(\langle\xi\rangle^{\rho})\underline{\tilde{u}}_+, \tag{8.28}$$

$$R_2 = h^{\frac{3}{2}}\mathrm{Op}_h^{\mathrm{W}}(r)\mathrm{Op}_h^{\mathrm{W}}(\langle\xi\rangle^{\rho})\underline{\tilde{u}}_+, \tag{8.29}$$

where $|\partial_z^{\alpha}\gamma_{-1}(z)| \le C_\alpha\langle z\rangle^{-1-\alpha}$ and r satisfies

$$|\partial_x^{\alpha_1}\partial_\xi^{\alpha_2}(h\partial_h)^k r(x,\xi,h)| \le Ch^{-\frac{\alpha_1+\alpha_2}{2}}\Big\langle\frac{x+p'(\xi)}{\sqrt{h}}\Big\rangle^{-1}. \tag{8.30}$$

By [82, Lemma 4.2], R_1 may be replaced by

$$h^{\frac{1}{2}}\mathrm{Op}_h^{\mathrm{W}}\Big(\gamma_{-1}\Big(\frac{x+p'(\xi)}{\sqrt{h}}\Big)(x+p'(\xi))\langle\xi\rangle^{\rho}\chi(h^{\beta}\xi)\Big)\underline{\tilde{u}}_+ \tag{8.31}$$

modulo a quantity estimated in L^∞ by

$$Ch^{\frac{5}{4}-\sigma}\big(\|\mathcal{L}_+\underline{\tilde{u}}_+\|_{L^2} + \|\tilde{u}_+\|_{H^s}\big) \tag{8.32}$$

for some $\sigma > 0$, σ going to zero with β. By a priori assumption (7.3) (translated on $\underline{\tilde{u}}_+$) this is estimated by the right-hand side of (8.25). By [82, estimate (4.25) of Lemma 4.3], the L^∞ norm of (8.31) is also controlled by (8.32), so by the right-hand side of (8.25).

Let us check that R_2 given by (8.29) is also bounded by the same quantity. This follows from semiclassical Sobolev injection together with the a priori Sobolev estimate in (7.3). Moreover, by (8.24), the $\underline{f}^\rho_{+,\Lambda}$ contribution in (8.27) is also bounded by the right-hand side of (8.25).

It remains to write the left-hand side of (8.27) as the left-hand side of (8.25), up to some new contributions to the right-hand side of the latter. This follows from Proposition D.3.6, where the right-hand side of the second inequality of (D.93) is again estimated using (7.3). This concludes the proof. ∎

8.2 Bootstrap of L^∞ estimates

We have shown in Proposition 7.3.7 that under a priori assumptions (7.1)–(7.4), we could improve the Sobolev estimates in (7.3) to (7.103)–(7.104). Our first goal here will be to improve also the L^∞ estimate.

Proposition 8.2.1. *Assume that* (7.1)–(7.3) *hold true on an interval* $[1, T]$. *Let* $c > 0$ *be given. Then if* D *in* (7.3) *has been taken large enough, there is* $\varepsilon_0 \in]0, 1]$ *such that, for all* $\varepsilon \in]0, \varepsilon_0]$, *all* $1 \le t \le T \le \varepsilon^{-4+c}$, *one has the bound*

$$\|\tilde{u}_+\|_{W^{\rho,\infty}} \le \frac{D}{2}\frac{(\varepsilon^2\sqrt{t})^{\theta'}}{\sqrt{t}}. \tag{8.33}$$

Proof. We have to bound $\langle D_x\rangle^\rho \tilde{u}_+$ in L^∞. By (8.1) and the notation introduced after (8.3) for $\tilde{u}^\rho_{+,\Lambda}$, $\tilde{u}^\rho_{+,\Lambda^c}$, it suffices to show

$$\|\tilde{u}^\rho_{+,\Lambda}\|_{L^\infty} \le \frac{D}{4}t^{-\frac{1}{2}}(\varepsilon^2\sqrt{t})^{\theta'}, \tag{8.34}$$

$$\|\tilde{u}^\rho_{+,\Lambda^c}\|_{L^\infty} \le \frac{D}{4}t^{-\frac{1}{2}}(\varepsilon^2\sqrt{t})^{\theta'}. \tag{8.35}$$

By (8.4) and a priori estimate (7.3), one may bound (8.35) by $Ct^{-\frac{1}{2}+\sigma}(\varepsilon^2\sqrt{t})^{\theta}$. Since $\theta' < \theta$ and $t \le \varepsilon^{-4+c}$, we bound this by the quantity $Ct^{-\frac{1}{2}}(\varepsilon^2\sqrt{t})^{\theta'}e(t,\varepsilon)$, where e satisfies (5.41), if σ has been taken small enough relatively to $c(\theta-\theta')$.

We are left with estimating (8.34). It is equivalent to show that

$$\|\underline{\tilde{u}}^\rho_{+,\Lambda}\|_{L^\infty} \le \frac{D}{4}(\varepsilon^2\sqrt{t})^{\theta'}$$

if ε is small enough. Computing $\partial_t |\underline{\tilde{u}}^\rho_{+,\Lambda}(t,x)|^2$ from (8.25) and integrating in time, we get

$$|\underline{\tilde{u}}^\rho_{+,\Lambda}(t,x)| \le |\underline{\tilde{u}}^\rho_{+,\Lambda}(1,x)| + C\int_1^t \tau^{-1+\sigma}(\varepsilon^2\sqrt{\tau})^\theta \, d\tau.$$

If D has been taken large enough so that $\|\underline{\tilde{u}}^\rho_{+,\Lambda}(1,\cdot)\|_{L^\infty} \le \frac{D}{8}\varepsilon$, we get the wanted estimate, using again that $t \le \varepsilon^{-4+c}$ and that σ may be taken small relatively to $c(\theta-\theta')$. This concludes the proof. ∎

Propositions 7.3.7 and 8.2.1 allowed us to bootstrap estimates (7.3). To be able to finish the proof of the main theorem, we shall have to bootstrap as well the inequalities satisfied by g. We prove first some technical lemmas.

Proposition 8.2.2. *Let Z be a function in $\mathcal{S}(\mathbb{R})$. Assume that the function $\tilde{u}_+$ satisfies estimate (7.3). For any neighborhood $\mathcal{W}$ of $\{-1,1\}$ in $\mathbb{R}$, there is $\varepsilon_0 > 0$ (depending only on $\mathcal{W}$ and on the constants in (7.3)) such that for any λ in $\mathbb{R}-\mathcal{W}$, there are functions $\varphi_\pm(\lambda,t)$, $\psi_\pm(\lambda,t)$ defined for $t \in [1,\varepsilon^{-4+c}]$, $\varepsilon \in]0,\varepsilon_0]$, satisfying the estimates*

$$|\varphi_\pm(\lambda,t)| \le t^{-\frac{1}{2}}(\varepsilon^2\sqrt{t})^{\theta'}, \tag{8.36}$$

$$|\psi_\pm(\lambda,t)| \le t^{-1}(\varepsilon^2\sqrt{t})^{\theta'} \tag{8.37}$$

and solving the equation

$$(D_t-\lambda)\varphi_\pm(\lambda,t) = \langle Z,\tilde{u}_\pm\rangle + \psi_\pm(\lambda,t). \tag{8.38}$$

Moreover, denoting $\langle Z,\tilde{u}\rangle$ for the vector $\left[\begin{smallmatrix}\langle Z,\tilde{u}_+\rangle\\ \langle Z,\tilde{u}_-\rangle\end{smallmatrix}\right]$, one has the bound

$$|\langle Z,\tilde{u}\rangle| \le t^{-\frac{3}{4}}(\varepsilon^2\sqrt{t})^{\theta'}. \tag{8.39}$$

Proof. We shall use the following notation: we set $f = o(g)$ when we may write $|f| \le |g|e(t,\varepsilon)$ for some $e(t,\varepsilon)$ satisfying (5.41). In particular, for any given N, taking ε small enough, we may bound $|f|$ by $\frac{1}{N}|g|$.

We prove the proposition in the case of sign $+$. Let us show first that on the right-hand side of (8.38), we may replace $\langle Z,\tilde{u}_+\rangle$ by $\langle Z(C(t)\tilde{u})_+\rangle$, up to a contribution to ψ_+. Since $((\mathrm{Id}-C(t))\tilde{u})_+$ is odd, and Z is in $\mathcal{S}$, we may use (4.79) to write

$$\begin{aligned}\langle Z,((\mathrm{Id}-C(t))\tilde{u})_+\rangle &= \frac{1}{t}\int_{-1}^1 \langle Z^1,(L(\mathrm{Id}-C(t))\tilde{u})_+(\mu x)\rangle\, d\mu\\ &\quad - \frac{1}{t}\int_{-1}^1 \langle Z^2,((\mathrm{Id}-C(t))\tilde{u})_+(\mu x)\rangle \mu\, d\mu\end{aligned} \tag{8.40}$$

for new functions Z^1, Z^2 in $\mathcal{S}(\mathbb{R})$. By (7.3) and L^2 boundedness of $C(t)$, the last term is $O(\varepsilon t^{\delta-1}) = o((\varepsilon^2\sqrt{t})^{\theta'}t^{-1})$. It may thus be integrated to $\psi_+(\lambda,t)$. In the first term on the right-hand side of (8.40) we write using (E.20)

$$L(\mathrm{Id}-C(t))\tilde{u} = (\mathrm{Id}-\tilde{C}(t))L\tilde{u} + \tilde{C}_1(t)\tilde{u}.$$

By (E.21), (E.22) and (7.3), we get

$$\|L(\mathrm{Id}-C(t))\tilde{u}\|_{L^2} \le C(\varepsilon^2\sqrt{t})^{\theta'}\big[\varepsilon^{\iota}t^{-m+\frac{1}{2}+\delta'}(\varepsilon^2\sqrt{t})^{\theta-\theta'} + \varepsilon^{1+\iota-2\theta'}t^{\frac{1}{2}-m+\delta-\frac{\theta'}{2}}\big]. \tag{8.41}$$

As θ,θ' are fixed with $\theta'<\theta<\frac{1}{2}$ and θ' close to $\frac{1}{2}$, and as $\delta',\frac{1}{2}-m$ may be taken as small as we want, the bracket above is $o(1)$ when $t\le\varepsilon^{-4+c}$ and ε goes to zero. Thus (8.41) plugged in the first term on the right-hand side of (8.40) shows that this term is $o(t^{-1}(\varepsilon^2\sqrt{t})^{\theta'})$, so satisfies (8.37). We are thus reduced to studying equation

$$(D_t-\lambda)\varphi_+(\lambda,t) = \langle Z,(C(t)\tilde{u})_+\rangle + \psi_+(\lambda,t). \tag{8.42}$$

Recall the function $\mathring{u}$ defined in (7.106). We may write

$$\begin{aligned} &\langle Z,(C(t)\tilde{u})_+\rangle = \langle Z,\mathring{u}_+\rangle + \psi_1(t),\\ &\quad\psi_1(t) = \langle Z,(\hat{\mathcal{M}}_2'(\tilde{u},u'^{\mathrm{app},1}))_+\rangle + \sum_{j=3}^{4}\langle Z,(C(t)\hat{\mathcal{M}}_j(\tilde{u},u'^{\mathrm{app},1}))_+\rangle. \end{aligned} \tag{8.43}$$

By (7.5), we may bound the last sum by

$$Ct^{-1}(\varepsilon^2\sqrt{t})^{\theta'}\big(t^{\delta}(\varepsilon^2\sqrt{t})^{\theta'}\varepsilon + \varepsilon^{5-2\theta'}t^{1-\frac{\theta'}{2}+\delta+\sigma}\big).$$

As $t\le\varepsilon^{-4+c}$, this is smaller than the right-hand side of (8.37) (for δ,σ small).

Let us show that the first term on the right-hand side of the expression of ψ_1 satisfies also (8.37). It suffices to show that $\|\hat{\mathcal{M}}_2'(\tilde{u},u'^{\mathrm{app},1})\|_{L^2} = o(t^{-1}(\varepsilon^2\sqrt{t})^{\theta'})$. Recall that $\hat{\mathcal{M}}_2'(\tilde{u},u'^{\mathrm{app},1})$ is given by (6.60) in terms of expressions $\hat{\mathcal{M}}_2'^{\ell}$, that have structure (6.47), i.e. that may be written from expressions

$$t^{-\frac{3}{2}}K^{\ell_1,\ell_2}\big(L_{\pm}^{\ell_1}f_{1,\pm},L_{\pm}^{\ell_2}f_{2,\pm}\big), \tag{8.44}$$

where $0\le\ell_1,\ell_2\le 1$, K^{ℓ_1,ℓ_2} is in $\mathcal{K}'_{1,1/2}(1,\pm,\pm)$ and f_1,f_2 equal to $\tilde{u}$ or $u'^{\mathrm{app},1}$ (see (F.35)). If we apply (F.47), (F.50), (F.51), we obtain a bound for the L^2 norm of (8.44) in

$$Ct^{-\frac{3}{2}-\frac{1}{4}+\sigma}\big(\|L_+\tilde{u}_+\|_{L^2} + \|L_+u'^{\mathrm{app},1}_+\|_{L^2} + \|\tilde{u}_+\|_{H^s} + \|u'^{\mathrm{app},1}_+\|_{H^s}\big)^2$$

so according to (7.3) and (7.4) by

$$Ct^{-\frac{3}{2}+\sigma}(\varepsilon^2\sqrt{t})^{\theta}t^{\frac{1}{4}}(\varepsilon^2\sqrt{t})^{\theta}$$

which is better than (8.37). On the right-hand side of (8.42), up to incorporating ψ_1 to ψ_+, we thus may replace $\langle Z,(C(t)\tilde{u})_+\rangle$ by $\langle Z,\mathring{u}_+\rangle$, i.e. we reduced equation (8.42) to

$$(D_t-\lambda)\varphi_+(\lambda,t) = \langle Z,\mathring{u}_+\rangle + \psi_+ \tag{8.45}$$

for a new ψ_+. Since $\mathring{u}_+$ is odd and Z in $\mathcal{S}(\mathbb{R})$, we may write using (4.79) again

$$\langle Z, \mathring{u}_+\rangle = \frac{1}{t}\int_{-1}^{1}\langle Z^1, (L_+\mathring{u}_+)(\mu\cdot)\rangle\, d\mu - \frac{1}{t}\int_{-1}^{1}\langle Z^2, \mathring{u}_+(\mu\cdot)\rangle\mu\, d\mu \tag{8.46}$$

for new functions Z^1, Z^2 in the space $\mathcal{S}(\mathbb{R})$. By inequality (7.110), the last term is $O(\varepsilon t^{\delta-1}) = o((\varepsilon^2\sqrt{t})^{\theta'}t^{-1})$. It may thus be incorporated to $\psi_+(\lambda, t)$. We decompose the first integral on the right-hand side of (8.46) as $I_1 + I_2$, with

$$\begin{aligned} I_2 &= \int_{-1}^{1}\Big\langle Z^1, \Big(\chi\Big(\sqrt{t}(\lambda - \sqrt{1+D_x^2}\Big)\Big)(L_+\mathring{u}_+)\Big)(\mu\cdot)\Big\rangle\, d\mu \\ &= \int_{-1}^{1}\Big\langle \chi\Big(\sqrt{t}(\lambda - \sqrt{1+D_x^2}\Big)\Big)\Big(Z^1\Big(\frac{\cdot}{\mu}\Big)\Big), L_+\mathring{u}_+\Big\rangle\frac{d\mu}{\mu}, \end{aligned} \tag{8.47}$$

where $\chi \in C_0^\infty(\mathbb{R})$ is real valued, equal to one close to zero. By Cauchy–Schwarz,

$$|I_2| \le \int_{-1}^{1}\Big\|\chi\Big(\sqrt{t}(\lambda - \sqrt{1+D_x^2}\Big)\Big)\Big(Z^1\Big(\frac{\cdot}{\mu}\Big)\Big)\Big\|_{L^2}\frac{d\mu}{\mu}\|L_+\mathring{u}_+\|_{L^2}. \tag{8.48}$$

Since $\lambda \notin \mathcal{W}$, $\|\chi(\sqrt{t}(\lambda - \sqrt{1+\xi^2}))\|_{L^2(d\xi)} = O(t^{-\frac{1}{4}})$, so that the L^2 norm inside the above integral is bounded by

$$Ct^{-\frac{1}{4}}\Big\|Z^1\Big(\frac{\cdot}{\mu}\Big)\Big\|_{L^1} = O(\mu C t^{-\frac{1}{4}}).$$

By (7.111), it follows that the contribution of I_2 to the first term in (8.46) satisfies (8.37), so may be incorporated to ψ_+. We have thus written by (8.40) and (8.46)

$$\langle Z, \mathring{u}_+\rangle = \frac{1}{t}I_1 + \psi_+^1, \tag{8.49}$$

where ψ_+^1 satisfies the same estimates as ψ_+ (with an arbitrary small multiplicative constant on the right-hand side) and

$$I_1 = \int_{-1}^{1}\Big\langle Z^1, \Big((1-\chi)\Big(\sqrt{t}(\lambda - \sqrt{1+D_x^2}\Big)\Big)(L_+\mathring{u}_+)\Big)(\mu\cdot)\Big\rangle\, d\mu. \tag{8.50}$$

We thus reduced (8.45) to

$$(D_t - \lambda)\varphi_+(\lambda, t) = \frac{1}{t}I_1 + \psi_+(\lambda, t) \tag{8.51}$$

for a new ψ_+. We define

$$\begin{aligned} \varphi_+(\lambda, t) &= \frac{1}{t}\int_{-1}^{1}\Big\langle Z^1, \Big(\frac{(1-\chi)\big(\sqrt{t}(\lambda - \sqrt{1+D_x^2}\big)\big)}{\sqrt{1+D_x^2}-\lambda}L_+\mathring{u}_+\Big)(\mu\cdot)\Big\rangle\, d\mu \\ &+ = \frac{1}{\sqrt{t}}\int_{-1}^{1}\Big\langle \chi_1\Big(\sqrt{t}(\lambda - \sqrt{1+D_x^2}\Big)\Big)\Big(Z^1\Big(\frac{\cdot}{\mu}\Big)\Big), L_+\mathring{u}_+\Big\rangle\frac{d\mu}{\mu}, \end{aligned} \tag{8.52}$$

where $\chi_1(z) = \frac{\chi(z)-1}{z}$. Arguing as in (8.48) and using inequality (7.111), we obtain that $\varphi_+(\lambda, t)$ satisfies (8.36). If we compute $(D_t - \lambda)\varphi_+(\lambda, t)$, we get the following terms:

$$\frac{i}{t}\varphi_+(\lambda, t), \tag{8.53}$$

$$\frac{1}{t}\int_{-1}^{1}\Big\langle Z^1, \Big(\frac{(1-\chi)\big(\sqrt{t}(\lambda - \sqrt{1+D_x^2})\big)}{\sqrt{1+D_x^2}-\lambda}(D_t - p(D_x))L_+\mathring{u}_+\Big)(\mu\cdot)\Big\rangle\, d\mu, \tag{8.54}$$

$$\frac{1}{t}I_1(t), \tag{8.55}$$

$$-\frac{i}{2t^{\frac{3}{2}}}\int_{-1}^{1}\Big\langle Z^1, \Big(\chi'(\sqrt{t}(\lambda - \sqrt{1+D_x^2}))L_+\mathring{u}_+\Big)(\mu\cdot)\Big\rangle\, d\mu. \tag{8.56}$$

According to (8.51), we shall have proved (8.38) (in the case of sign $+$) if we show that (8.53), (8.54), (8.56) satisfy estimates (8.37), with a small constant in front of the right-hand side of this inequality. For (8.53), this follows from (8.52) and (8.36). We may rewrite (8.54) as

$$\frac{1}{\sqrt{t}}\int_{-1}^{1}\Big\langle \chi_1\Big(\sqrt{t}(\lambda - \sqrt{1+D_x^2})\Big)\Big(Z^1\Big(\frac{\cdot}{\mu}\Big)\Big), (D_t - \sqrt{1+D_x^2})L_+\mathring{u}_+\Big\rangle\frac{d\mu}{\mu}.$$

Arguing as in (8.48), we estimate that by

$$Ct^{-\frac{3}{4}}\|(D_t - \sqrt{1+D_x^2})L_+\mathring{u}_+\|_{L^2}.$$

Since L_+ commutes to $(D_t - \sqrt{1+D_x^2})$, it follows from (7.105) and (7.109) that this is bounded by

$$t^{-\frac{3}{2}}(\varepsilon^2\sqrt{t})^{\theta}e(t,\varepsilon) = o(t^{-1}(\varepsilon^2\sqrt{t})^{\theta'})$$

which implies an estimate of the form (8.37). Finally, (8.56) is bounded by

$$Ct^{-\frac{3}{2}}\int_{-1}^{1}\Big\|\chi'\Big(\sqrt{t}(\lambda - \sqrt{1+D_x^2})\Big)\Big(Z^1\Big(\frac{\cdot}{\mu}\Big)\Big)\Big\|_{L^2}\|L_+\mathring{u}_+\|_{L^2}\frac{d\mu}{\mu} \le Ct^{-\frac{3}{2}}(\varepsilon^2\sqrt{t})^{\theta}$$

according to (7.111). This is again better than needed.

Finally, estimate (8.39) follows from (8.40) (that is bounded by (8.37)), (8.43), the fact that ψ_1 is $o(t^{-1}(\varepsilon^2\sqrt{t})^{\theta'})$, (8.46) were we plug (7.110) and (7.111). This concludes the proof. ■

Our next task will be to show that a priori assumptions (7.1)–(7.3) imply that inequalities (4.92)–(4.93) that we assume in Section 4.2 in order to get estimates for the solution of the ODE (4.94), hold.

Lemma 8.2.3. *Assume that estimates (7.1)–(7.3) hold. Then inequality (4.92) is true, with a constant B' depending only on the constants A, A', D in (7.1)–(7.3).*

Proof. We divide the proof into two steps.

Step 1. Consider first the contribution Φ_2 on the left-hand side of (4.92). Recall that Φ_2 is given by (2.36), (2.38) so may be written as a sum of terms

$$\iint e^{ix(\xi_1+\xi_2)} m'(x,\xi_1,\xi_2)\hat{u}_\pm(\xi_1)\hat{u}_\pm(\xi_2)\,d\xi_1\,d\xi_2\,dx \tag{8.57}$$

with

$$m'(x,\xi_1,\xi_2) = \kappa(x)Y(x)b(x,\xi_1)b(x,\xi_2)p(\xi_1)^{-1}p(\xi_2)^{-1}.$$

By estimates (A.8) satisfied by b, and the fact that Y is in $\mathcal{S}(\mathbb{R})$, we have that m' belongs to $\tilde{S}'_{0,0}(\prod_{j=1}^2\langle\xi_j\rangle^{-1},2)$ and Φ_2 is thus a sum of expressions

$$\int \mathrm{Op}(m')(u_\pm,u_\pm)\,dx.$$

On the other hand, recall that u_+ is related to $\tilde{u}_+$ by (5.59), with a remainder R satisfying (5.25) and (5.26). By Corollary B.2.6, we get that (8.57) may be written as a sum of expressions

$$\int \mathrm{Op}(\tilde{m}')(v_1,\dots,v_n)\,dx \tag{8.58}$$

where $n \geq 2$ and v_j is equal to $u'^{\mathrm{app}}_\pm$ or $u''^{\mathrm{app}}_\pm$, or $\tilde{u}_\pm$ or R, with a symbol $\tilde{m}'$ in $\tilde{S}'_{1,0}(\prod_{j=1}^2\langle\xi_j\rangle^{-1}M_0^\nu,2)$ for some ν.

Consider first the case when at least one of the arguments v_j, say the last one, is not equal to $u''^{\mathrm{app}}_\pm$. Since $\tilde{m}'$ is rapidly decaying as $\langle M_0(\xi)^{-1}|y|\rangle^{-N}$, we may estimate (8.58) from the L^2 norm of the integrand. If $n=2$, we use (D.76) when v_1 is different from $u''^{\mathrm{app}}_\pm$ and (D.75) if $v_1 = u''^{\mathrm{app}}_\pm$. We obtain for (8.58) a bound in

$$\begin{aligned} Ct^{-2+\sigma}\big(&\|L_+\tilde{u}_+\|_{L^2} + \|L_+u'^{\mathrm{app}}_+\|_{L^2} + \|L_+R\|_{L^2} + \|\tilde{u}_+\|_{H^s} + \|u'^{\mathrm{app}}_+\|_{H^s} \\ &+ \|R\|_{H^s} + \|L_+u''^{\mathrm{app}}_+\|_{W^{\rho_0,\infty}} + \|u''^{\mathrm{app}}_+\|_{W^{\rho_0,\infty}}\big) \\ \times\big(&\|L_+\tilde{u}_+\|_{L^2} + \|L_+u'^{\mathrm{app}}_+\|_{L^2} + \|L_+R\|_{L^2} + \|\tilde{u}_+\|_{H^s} \\ &+ \|u'^{\mathrm{app}}_+\|_{H^s} + \|R\|_{H^s}\big). \end{aligned} \tag{8.59}$$

We plug there (7.1)–(7.3) and (5.25)–(5.26). We obtain a bound in $t^{-\frac{3}{2}+\sigma}(\varepsilon^2\sqrt{t})^{2\theta}$. As $\theta > \theta'$ and $t \leq \varepsilon^{-4+c}$, we see that if σ is small enough, this is smaller than the right-hand side of (4.92).

If $n \geq 3$ in (8.58), and again at least one v_j, say the last one, is different from $u''^{\mathrm{app}}_\pm$, we use Corollary D.2.8. By (D.71), we estimate then (8.58) by

$$\begin{aligned} Ct^{-1}\big(&\|u'^{\mathrm{app}}_+\|_{W^{\rho_0,\infty}} + \|u''^{\mathrm{app}}_+\|_{W^{\rho_0,\infty}} + \|\tilde{u}_+\|_{W^{\rho_0,\infty}} + \|R_+\|_{W^{\rho_0,\infty}}\big)^{n-1} \\ \times\big(&\|L_+\tilde{u}_+\|_{L^2} + \|L_+u'^{\mathrm{app}}_+\|_{L^2} + \|L_+R\|_{L^2} + \|\tilde{u}_+\|_{L^2} \\ &+ \|u'^{\mathrm{app}}_+\|_{L^2} + \|R\|_{L^2}\big). \end{aligned}$$

Using (7.1)–(7.3) and (5.25) (together with Sobolev injection), (5.26), we get a bound in $t^{-2}(\varepsilon^2\sqrt{t})^{2\theta'}(\varepsilon^2\sqrt{t})^\theta t^{\frac{1}{4}}$, which is better than what we want.

It remains to study (8.58) when all arguments v_j are equal to $u''^{\text{app}}_{\pm}$. Again by the rapid decay in x of the symbol $\tilde{m}'$, it is enough to control the L^∞ norm of the integrand (up to changing the definition of $\tilde{m}'$). We may use then (D.77) with $n = \ell \geq 2$. We obtain a bound in

$$t^{-2+\sigma}\big(\|u''^{\text{app}}_{+}\|_{W^{\rho_0,\infty}} + \|L_+u''^{\text{app}}_{+}\|_{W^{\rho_0,\infty}} + t^{-\frac{1}{2}}\|u''^{\text{app}}_{+}\|_{H^s}\big)^2. \tag{8.60}$$

Using (7.2) and the fact that $\theta' < \frac{1}{2}$, $\sigma \ll 1$, one controls that by $t^{-\frac{3}{2}}(\varepsilon^2\sqrt{t})^{2\theta'}$ for $t \leq \varepsilon^{-4+c}$. This concludes the proof of (4.92) for contribution Φ_2.

Step 2. We study next the term $t_\varepsilon^{-\frac{3}{2}+\frac{j}{2}}\Gamma_j(u_+, u_-)$ in (4.92), for $1 \leq j \leq 3$. Recall that Γ_j is given by (2.36)–(2.39). It has thus again the structure (8.58) with $n = j$, as it follows from the expression (5.59) of u_+ in terms of u^{app}_{+}, $\tilde{u}_+$, R and the composition results of Appendix B. If $j \geq 2$, our preceding reasoning implies the wanted bound. We thus just have to consider

$$t_\varepsilon^{-1}\int \operatorname{Op}(\tilde{m}')(v)\,dv \tag{8.61}$$

with $\tilde{m}'$ in $\tilde{S}'_{1,0}(\langle\xi\rangle^{-1}, 1)$ and $v = u'^{\text{app}}_{\pm}, u''^{\text{app}}_{\pm}, \tilde{u}_\pm, R$. When v is not equal to $u''^{\text{app}}_{\pm}$, we use (D.71) in order to bound (8.61) by

$$\begin{aligned} Ct_\varepsilon^{-1}t^{-1}\big(&\|L_+u'^{\text{app}}_{+}\|_{L^2} + \|L_+\tilde{u}_+\|_{L^2} + \|L_+R\|_{L^2}\\ &+ \|u'^{\text{app}}_{+}\|_{L^2} + \|\tilde{u}_+\|_{L^2} + \|R\|_{L^2}\big) \end{aligned}$$

which by (7.1)–(7.3) and (5.25)–(5.26) is bounded from above by $t_\varepsilon^{-1}t^{-1}(\varepsilon^2\sqrt{t})^{\theta}t^{\frac{1}{4}}$. One checks that this quantity is $O(t^{-\frac{3}{2}}(\varepsilon^2\sqrt{t})^{2\theta'})$ using $\theta' < \theta < \frac{1}{2}$.

If v in (8.61) is equal to $u''^{\text{app}}_{\pm}$, we bound (8.61) by

$$Ct_\varepsilon^{-1}\|\operatorname{Op}(\tilde{m}')v\|_{L^\infty}$$

(for a new symbol $\tilde{m}'$). We use (D.77) to get a bound in

$$t_\varepsilon^{-1}t^{-1+\sigma}\big[\|u''^{\text{app}}_{+}\|_{W^{\rho_0,\infty}} + \|L_+u''^{\text{app}}_{+}\|_{W^{\rho_0,\infty}} + t^{-\frac{1}{2}}\|u''^{\text{app}}_{+}\|_{H^s}\big]. \tag{8.62}$$

Using (7.2), one bounds the bracket by $t^{\sigma'}t^{\frac{1}{4}}(\varepsilon^2\sqrt{t})^{\frac{1}{2}}$ for any $\sigma' > 0$. As $t \leq \varepsilon^{-4+c}$, one concludes that if σ, σ' are small enough, (8.62) is $O(t^{-\frac{3}{2}}(\varepsilon^2\sqrt{t})^{2\theta'})$. This concludes the proof of the lemma. ∎

We show next that a priori assumptions (7.1)–(7.3) imply as well estimates (4.93).

Lemma 8.2.4. *Assume that estimates* (7.1)–(7.3) *hold true. Then inequality* (4.93) *holds true with a constant B' depending only on A, A', D in* (7.1)–(7.3).

Proof. Recall that $\Phi_1(u_+, u_-)$ is given by (2.36), i.e. taking (2.37) into account, by

$$\frac{\sqrt{3}}{3}\langle Y, Y(x)\kappa(x)b(x, D_x)p(D_x)^{-1}(u_+ - u_-)\rangle. \tag{8.63}$$

Expressing u_+ using (5.59), we get that, if we define

$$Z = \frac{\sqrt{3}}{3} p(D_x)^{-1} b(x, D_x)^*(\kappa(x)Y(x)^2),$$

the term inside the modulus on the left-hand side of (4.93) may be written as the sum of an expression $\langle Z, R\rangle$ with R satisfying (5.25) and of expressions of the form (8.58) with $n \geq 2$. We have seen that these last quantities may be bounded by (8.59) or (8.60), and thus by the right-hand side of (4.93). On the other hand, by (5.25) $\langle Z, R\rangle$ is also $O(t^{-\frac{3}{2}}(\varepsilon^2\sqrt{t})^{2\theta'})$. This concludes the proof. ■

Corollary 8.2.5. *Assume that estimates (7.1)–(7.3) hold true. Then Assumption (H_1') of Section 4.2 holds.*

Proof. We have seen that by Lemmas 8.2.3 and 8.2.4, inequalities (4.92) and (4.93) hold. It remains to check that for any $\lambda \in \mathbb{R} - \{-1, 1\}$, there are functions $\varphi_\pm(\lambda, t)$, $\psi_\pm(\lambda, t)$ as at the end of the statement of condition (H_1'). But this is exactly the statement of Proposition 8.2.2. ■

8.3 End of bootstrap argument

We give here the proof of Theorem 2.1.1. We shall have to gather all estimates we proved in the preceding chapters. We first restate the main estimates in Theorem 2.1.1.

Proposition 8.3.1. *There is ρ_0 in $\mathbb{N}$ and for any $\rho \geq \rho_0$, any $c \in]0, 1[$, any $\theta' \in]0, \frac{1}{2}[$ close to $\frac{1}{2}$, any large enough $N \in \mathbb{N}$, there are $\varepsilon_0 > 0$, $C > 0$ such that if $0 < \varepsilon < \varepsilon_0$, the solution φ of equation (2.11) with odd initial conditions with bounds (2.10) satisfies for $t \in [1, \varepsilon^{-4+c}]$ the following estimates (using notation (2.7) and (2.8)):*

$$\begin{aligned} \|P_{\mathrm{ac}}\varphi(t,\cdot)\|_{W^{\rho,\infty}} &\leq Ct^{-\frac{1}{2}}(\varepsilon^2\sqrt{t})^{\theta'}, \\ \|\langle x\rangle^{-2N} P_{\mathrm{ac}}\varphi(t,\cdot)\|_{W^{\rho,\infty}} &\leq Ct^{-\frac{3}{4}}(\varepsilon^2\sqrt{t})^{\theta'}, \\ \|\langle x\rangle^{-2N} D_t P_{\mathrm{ac}}\varphi(t,\cdot)\|_{W^{\rho-1,\infty}} &\leq Ct^{-\frac{3}{4}}(\varepsilon^2\sqrt{t})^{\theta'} \end{aligned} \tag{8.64}$$

and $a(t)$ may be written as $a(t) = e^{it\frac{\sqrt{3}}{2}} g_+(t) - e^{-it\frac{\sqrt{3}}{2}} g_-(t)$ with

$$\begin{aligned} |g_\pm(t)| &\leq C\varepsilon(1 + t\varepsilon^2)^{-\frac{1}{2}}, \\ |\partial_t g_\pm(t)| &\leq C\varepsilon t^{-\frac{1}{2}}(1 + t\varepsilon^2)^{-\frac{1}{2}}. \end{aligned} \tag{8.65}$$

Proof. Recall that we have defined in (2.18) and (2.19)

$$w = b(x, D_x)^* P_{\mathrm{ac}}\varphi, \quad P_{\mathrm{ac}}\varphi = b(x, D_x)w. \tag{8.66}$$

We have introduced in (2.24)

$$u_+ = (D_t + p(D_x))w. \tag{8.67}$$

We shall prove the following inequalities, where the last two ones are just the restatement of (8.65):

$$\begin{aligned}\|u_+(t,\cdot)\|_{W^{\rho,\infty}} &\leq Ct^{-\frac{1}{2}}(\varepsilon^2\sqrt{t})^{\theta'},\\ \|u_+(t,\cdot)\|_{H^s} &\leq C\varepsilon t^{\delta}\end{aligned} \tag{8.68}$$

and

$$\begin{aligned}|g_\pm(t)| &\leq C\varepsilon(1+t\varepsilon^2)^{-\frac{1}{2}},\\ |\partial_t g_\pm(t)| &\leq C\varepsilon t^{-\frac{1}{2}}(1+t\varepsilon^2)^{-\frac{1}{2}}.\end{aligned} \tag{8.69}$$

We shall deduce these estimates from bounds on $\tilde{u}_+$ that we establish by bootstrap of (7.3). Actually, let us show that if (7.3) holds on some interval $[1,T]$ with $T \leq \varepsilon^{-4+c}$ with a constant D, then it still holds with D replaced by $\frac{D}{2}$, as soon as D has been fixed large enough, and ε smaller than some ε_0 (depending on D). Proposition 7.3.7 shows that this statement holds for the Sobolev and L^2 estimate as soon as bounds (7.1), (7.2), (7.4) hold true (with constants A, A' that may depend on D). By Proposition 8.2.1, the $W^{\rho,\infty}$ estimate of $\tilde{u}_+$ may also be bootstrapped.

Let us next show that we may bootstrap as well estimate (4.99) on g. According to Proposition 4.2.1, we may do so as soon as Assumption (H_1') holds true. By Corollary 8.2.5, this follows under a priori conditions (7.1)–(7.3). Property (7.3) is the bootstrap assumption. On the other hand, (7.1), (7.2), (7.4) hold, for convenient constants $C(A,A')$ by Proposition 4.1.2 as soon as (4.3)–(4.7) hold. The first of these inequalities is the bootstrap assumption (4.99) on g. The other ones are (8.36)–(8.39), that, according to Proposition 8.2.2, hold under the bootstrap assumption (7.3).

Let us now deduce (8.68) from estimates (7.1)–(7.3) and (4.3), that hold on $[1,\varepsilon^{-4+c}]$ for ε small, according to our bootstrap assumption. Recall that u_+ is given by (5.59) (or (5.24)) by

$$u_+ = u'^{\text{app}}_+ + u''^{\text{app}}_+ + \tilde{u}_+ + \sum_{\substack{2\leq|I|\leq 4\\ I=(I',I'')}} \mathrm{Op}(\tilde{m}_I)(\tilde{u}_{I'}, u^{\text{app}}_{I''}) + R, \tag{8.70}$$

where R satisfies (5.25). This (and Sobolev injection) shows that R satisfies better bounds than those given by (8.68). By (7.1)–(7.3), the first three terms in (8.70) satisfy also the wanted bounds. Finally, the terms in the sum are also estimated by these bounds using (7.1)–(7.3) and (D.32), (D.39).

Let us check inequalities (8.69). Recall that $a(t) = \frac{\sqrt{3}}{3}(a_+(t) - a_-(t))$, where $a_- = -\overline{a_+}$ and a_+ is given by (4.96). We set then, using notation (4.97) and (4.98),

$$g_+(t) = \frac{\sqrt{3}}{3}e^{-it\frac{\sqrt{3}}{2}}(a^{\text{app}}_+(t) + S(t)) \tag{8.71}$$

and $g_-(t) = -\overline{g_+(t)}$. It follows from the expressions of a^{app}_+, S and (4.97)–(4.101) that

$$g_+(t) = O\big(t_\varepsilon^{-\frac{1}{2}}\big), \quad \partial_t g_+(t) = O\big(t_\varepsilon^{-\frac{1}{2}}t^{-\frac{1}{2}}\big).$$

It remains to prove (8.64). By (2.19) and (2.24),

$$P_{\mathrm{ac}}\varphi = b(x, D_x)w = \frac{1}{2} b(x, D_x) p(D_x)^{-1}(u_+ - u_-). \tag{8.72}$$

By Proposition D.1.5, the operator $b(x, D_x)p(D_x)^{-1}\langle D_x\rangle^{-\alpha}$ is bounded on $W^{\rho',\infty}$ if $\alpha > 0$. It follows that the first estimate (8.64) follows from (8.68) if we modify the value of ρ on the left-hand side of (8.64).

To obtain the weighted estimates in (8.64), let us write from (8.72) and (2.24)

$$\langle x\rangle^{-2N} P_{\mathrm{ac}}\varphi = \frac{1}{2}\langle x\rangle^{-2N} b(x, D_x) p(D_x)^{-1}(u_+ - u_-), \tag{8.73}$$

$$\langle x\rangle^{-2N} D_t P_{\mathrm{ac}}\varphi = \frac{1}{2}\langle x\rangle^{-2N} b(x, D_x)(u_+ + u_-). \tag{8.74}$$

On the right-hand side of (8.73), we replace u_+ by its expression (8.70). We have to bound the following quantities:

$$\begin{aligned} &\|\langle x\rangle^{-2N} b(x, D_x)p(D_x)^{-1} u'^{\mathrm{app}}_+\|_{W^{\rho,\infty}},\\ &\|\langle x\rangle^{-2N} b(x, D_x)p(D_x)^{-1} \tilde{u}_+\|_{W^{\rho,\infty}}, \end{aligned} \tag{8.75}$$

$$\|\langle x\rangle^{-2N} b(x, D_x)p(D_x)^{-1} u''^{\mathrm{app}}_+\|_{W^{\rho,\infty}}, \tag{8.76}$$

$$\sum_{\substack{2\le |I|\le 4\\ I=(I',I'')}} \|\langle x\rangle^{-2N} b(x, D_x)p(D_x)^{-1} \mathrm{Op}(m_I)(\tilde{u}_{I'}, u^{\mathrm{app}}_{I''})\|_{W^{\rho,\infty}}, \tag{8.77}$$

$$\|\langle x\rangle^{-2N} b(x, D_x)p(D_x)^{-1} R\|_{W^{\rho,\infty}}. \tag{8.78}$$

If $N = 2$, the assumptions of Proposition D.2.5 with $n = 1$ are satisfied. We may thus apply Corollary D.2.11 with $\ell = 0$. Taking into account (7.1) and (7.3), we obtain for (8.75) a bound in

$$t^{-\frac{3}{4}+\sigma}(\varepsilon^2\sqrt{t})^{\theta} + t^{-1}\frac{(\varepsilon^2\sqrt{t})^{\theta'}}{\sqrt{t}}.$$

For (8.76), we apply also Corollary D.2.11, but with $\ell = 1$. We obtain by (7.2) a bound in

$$t^{-1+\sigma}\log(1+t)\log(1+t\varepsilon^2) = O\big(t^{-\frac{3}{4}+2\sigma}(\varepsilon^2\sqrt{t})^{\frac{1}{2}}\big).$$

modulo a bound in $t^{-1}\frac{(\varepsilon^2\sqrt{t})^{\theta'}}{\sqrt{t}}$. To estimate (8.77), we use again Corollary D.2.11, with $n = |I|$ and ℓ equal to the number of arguments equal to $u''^{\mathrm{app}}_\pm$, $n - \ell$ equal to the number of arguments equal to $\tilde{u}_\pm$ or $u'^{\mathrm{app}}_\pm$. If N is taken large enough, we get better estimates than those holding for (8.75) and (8.76). Finally, Sobolev injection and (5.25) provide for (8.78) a better upper bound than the one in (8.64). We thus got estimates of $\|\langle x\rangle^{-N} P_{\mathrm{ac}}\varphi(t,\cdot)\|_{W^{\rho,\infty}}$ in $t^{-\frac{3}{4}}(\varepsilon^2\sqrt{t})^{\theta'}$ since σ is as small as we want, $t \le \varepsilon^{-4+c}$, and $\theta < \frac{1}{2}$. This implies the second inequality of (8.64).

The proof of the last inequality (8.64) is similar, starting from (8.74). ■