Appendix A

Scattering for time independent potential

This appendix is devoted to the construction of wave operators for a Schrödinger operator of the form

$$A = -\frac{1}{2}\frac{d^2}{dx^2} + V(x),$$

where V is a real-valued potential in $\mathcal{S}(\mathbb{R})$. If W_+ stands for the wave operator defined by (A.5) below, one knows that $W_+W_+^* = P_{\mathrm{ac}}$, $W_+^*W_+ = \mathrm{Id}_{L^2}$, where P_{ac} is the spectral projector associated to the absolutely continuous spectrum of A. Moreover, one has the intertwining property

$$W_+^* A W_+ = -\frac{1}{2}\frac{d^2}{dx^2}.$$

Our main result below is that, under convenient assumptions on V, operator W_+ acting on odd functions may be represented from pseudo-differential operators (see Proposition A.1.1). Let us mention that, even if we give quite complete proofs, our approach here is not original, and that we strongly rely on the classical paper of Deift and Trubowitz [17] and on the work of Weder [85].

A.1 Statement of main proposition

We consider $V : \mathbb{R} \to \mathbb{R}$ a potential belonging to $\mathcal{S}(\mathbb{R})$. Then the operator

$$-\frac{1}{2}\Delta + V = -\frac{1}{2}\frac{d^2}{dx^2} + V$$

is a self-adjoint operator whose spectrum is made of an absolutely continuous part, equal to $[0, +\infty[$, and of finitely many negative eigenvalues (see [17]). For ξ in $\mathbb{R}$, we define the Jost function $f_1(x,\xi)$ (resp. $f_2(x,\xi)$) as the unique solution to

$$-\frac{d^2}{dx^2} f + 2V(x) f = \xi^2 f \tag{A.1}$$

that satisfies $f_1(x,\xi) \sim e^{ix\xi}$ when x goes to $+\infty$ (resp. $f_2(x,\xi) \sim e^{-ix\xi}$ when x goes to $-\infty$). We set

$$\begin{aligned} m_1(x,\xi) &= e^{-ix\xi} f_1(x,\xi), \\ m_2(x,\xi) &= e^{ix\xi} f_2(x,\xi). \end{aligned} \tag{A.2}$$

We shall say that the potential V is generic if

$$\int_{-\infty}^{+\infty} V(x) m_1(x,0)\, dx \neq 0. \tag{A.3}$$

Notice that the above integral is convergent as $m_1(x,\xi)$ is bounded when x goes to $+\infty$ and has at most polynomial growth as x goes to $-\infty$ (see [17, Lemma 1] and Lemma A.1.1 below). We say that V is very exceptional if

$$\int_{-\infty}^{+\infty} V(x)m_1(x,0)\,dx = 0 \quad \text{and} \quad \int_{-\infty}^{+\infty} V(x)xm_1(x,0)\,dx = 0. \tag{A.4}$$

If one sets $V(x) = -\frac{3}{4}\cosh^{-2}(\frac{x}{2})$, as for the potential of interest in this paper (see equation (2.5)), it is proved in [13, Lemma 2.1] that the transmission coefficient of this potential satisfies $T(0) = 1$ (see [17] or below for the definition of the transmission coefficient). This implies on the one hand that (A.3) does not hold (as (A.3) is equivalent to $T(0) = 0$ – see [17, 85] or (A.32) below) and that moreover

$$\int xV(x)m_1(x,0)\,dx = 0,$$

i.e. that (A.4) holds, as follows from (A.26) and (A.31).

We denote by W_+ the wave operator associated to $A = -\frac{1}{2}\Delta + V$, defined as the strong limit

$$W_+ = s - \lim_{t\to+\infty} e^{itA}e^{-itA_0}, \tag{A.5}$$

where $A_0 = -\frac{1}{2}\Delta$. One knows (see Weder [85] and references therein) that

$$W_+W_+^* = P_{\mathrm{ac}}, \quad W_+^*W_+ = \mathrm{Id}_{L^2}, \tag{A.6}$$

where P_{ac} is the orthogonal projector on the absolutely continuous spectrum and, more generally, that if $\mathfrak{b}$ is any Borel function on $\mathbb{R}$,

$$\mathfrak{b}(A)P_{\mathrm{ac}} = W_+\mathfrak{b}(A_0)W_+^*, \quad \mathfrak{b}(A_0) = W_+^*\mathfrak{b}(A)W_+. \tag{A.7}$$

Notice that since A and A_0 preserve the space of odd functions, so do W_+, W_+^*. For odd w, we shall obtain an expression for W_+w given by the following proposition.

Proposition A.1.1. *Assume that V is an even potential that is either generic or very exceptional. Let $\chi_\pm$ be smooth functions, supported for $\pm x \geq -1$, with values in the interval $[0,1]$, with $\chi_-(x) = \chi_+(-x)$, $\chi_+(x) + \chi_-(x) \equiv 1$. There are an odd smooth real-valued function θ, and a smooth function $(x,\xi) \mapsto b(x,\xi)$ satisfying*

$$\begin{aligned} \left|\partial_\xi^\beta b(x,\xi)\right| &\leq C_\beta && \text{for all } \beta \in \mathbb{N}, \\ \left|\partial_x^\alpha\partial_\xi^\beta b(x,\xi)\right| &\leq C_{\alpha\beta N}\langle x\rangle^{-N} && \text{for all } \alpha \in \mathbb{N}^*,\ \beta \in \mathbb{N},\ N \in \mathbb{N}, \end{aligned} \tag{A.8}$$

and

$$\overline{b(x,-\xi)} = b(x,\xi), b(-x,-\xi) = b(x,\xi) \tag{A.9}$$

such that if we set $c(\xi) = e^{i\theta(\xi)}\mathbb{1}_{\xi>0} + e^{-i\theta(\xi)}\mathbb{1}_{\xi<0}$, then for any odd function w,

$$W_+w = b(x,D_x)\circ c(D_x)w \tag{A.10}$$

with

$$b(x,D)v = \frac{1}{2\pi}\int e^{ix\xi}b(x,\xi)\hat{w}(\xi)\,d\xi.$$

A.2 Proof of main proposition

We shall give here the proof of Proposition A.1.1, relying on the results of Deift and Trubowitz [17] and Weder [85].

If V is a real-valued even potential, the Jost functions satisfy by uniqueness $f_1(-x,\xi) = f_2(x,\xi)$ so that (A.2) implies that

$$m_1(-x,\xi) = m_2(x,\xi). \tag{A.11}$$

By [17, Lemma 1], m_1 solves the Volterra equation

$$m_1(x,\xi) = 1 + \int_x^{+\infty} D_\xi(x'-x)2V(x')m_1(x',\xi)\,dx' \tag{A.12}$$

where

$$D_\xi(x) = \int_0^x e^{2ix'\xi}\,dx' = \frac{e^{2ix\xi}-1}{2i\xi}. \tag{A.13}$$

If V is in $\mathcal{S}(\mathbb{R})$, then [17, Lemma 1 (ii)] shows that

$$\begin{aligned} \left|\partial_x^\alpha\partial_\xi^\beta(m_1(x,\xi)-1)\right| &\le C_{\alpha\beta N}\langle x\rangle^{-N}\langle\xi\rangle^{-1-\beta} \quad \text{for all } x > -M,\ \xi\in\mathbb{R},\\ \left|\partial_x^\alpha\partial_\xi^\beta(m_2(x,\xi)-1)\right| &\le C_{\alpha\beta N}\langle x\rangle^{-N}\langle\xi\rangle^{-1-\beta} \quad \text{for all } x < M,\ \xi\in\mathbb{R}, \end{aligned} \tag{A.14}$$

holds for m_1 (and thus also for m_2) when $\alpha = \beta = 0$. To get also estimates for the derivatives, we need to establish the following lemma, whose proof relies on the same ideas as in [17]:

Lemma A.2.1. *Denote for any β, N in $\mathbb{N}$ by $\Omega_N^\beta(x)$ a smooth positive function such that $\Omega_N^\beta(x) = \langle x\rangle^{-N}$ for $x \ge 1$ and $\Omega_N^\beta(x) = \langle x\rangle^\beta$ for $x \le -1$. Then for any N, α, β in $\mathbb{N}$, there is $C > 0$ such that for any ξ with $\operatorname{Im}\xi \ge 0$, any x,*

$$\left|\partial_x^\alpha\partial_\xi^\beta(m_1(x,\xi)-1)\right| \le C\Omega_N^{\beta+1}(x)\langle\xi\rangle^{-1-\beta}. \tag{A.15}$$

Proof. Following the proof of [17, Lemma 1], we write

$$m_1(x,\xi) = 1 + \sum_{n=1}^{+\infty} g_n(x,\xi) \tag{A.16}$$

with

$$g_n(x,\xi) = \int_{x\le x_1\le\cdots\le x_n}\prod_{j=1}^n D_\xi(x_j - x_{j-1})2V(x_j)\,dx_1\cdots dx_n, \tag{A.17}$$

using the convention $x_0 = x$. Set $\Omega(x) = \Omega_0^1(x)$ and

$$K_\xi(y,y') = D_\xi(y-y')\Omega(y')^{-1}2V(y)\Omega(y).$$

Then we may rewrite g_n as

$$g_n(x,\xi) = \Omega(x) \int_{x\le x_1\le\cdots\le x_n} \prod_{j=1}^{n} K_\xi(x_j, x_{j-1})\Omega(x_n)^{-1}\, dx_1 \cdots dx_n,$$

or equivalently

$$g_n(x,\xi) = \Omega(x) \int_{y_1\ge 0,\dots,y_n\ge 0} \prod_{j=1}^{n} K_\xi(x + y_1 + \cdots + y_j, x + y_1 + \cdots + y_{j-1}) \times \Omega(x + y_1 + \cdots + y_n)^{-1}\, dy_1 \cdots dy_n. \tag{A.18}$$

By (A.13), we have

$$\left|\partial_\xi^\beta D_\xi(y)\right| \le C_\beta \langle\xi\rangle^{-1}\langle y\rangle^{1+\beta}.$$

Fix some integer m. The definition of K_ξ implies that for $\alpha + \beta \le m$

$$\begin{aligned} &\left|\partial_x^\alpha \partial_\xi^\beta K_\xi(x + y_1 + \cdots + y_j, x + y_1 + \cdots + y_{j-1})\right| \\ &\quad \le C\langle\xi\rangle^{-1}\Omega(x + y_1 + \cdots + y_{j-1})^{-1}\langle x + y_1 + \cdots + y_j\rangle^{-1-\beta} \\ &\qquad \times W(x + y_1 + \cdots + y_j)\langle y_j\rangle^{1+\beta}, \end{aligned} \tag{A.19}$$

where W is some smooth rapidly decaying function. When $y_1 \ge 0, \dots, y_j \ge 0$, we may bound

$$\langle y_j\rangle^{1+\beta}\Omega(x + y_1 + \cdots + y_{j-1})^{-1}\langle x + y_1 + \cdots + y_j\rangle^{-1-\beta} \le C\Omega(x)^\beta.$$

Consequently, (A.18) implies that

$$|\partial_x^\alpha \partial_\xi^\beta g_n(x,\xi)| \le C\Omega(x)^{\beta+1}\langle\xi\rangle^{-n} \times \int_{y_1\ge 0,\dots,y_n\ge 0} \prod_{j=1}^{n} W(x + y_1 + \cdots + y_j)\, dy_1 \cdots dy_n. \tag{A.20}$$

Define $G(x) = \int_x^{+\infty} W(z)\, dz$, so that the last integral above may be written

$$(-1)^{n-1} \int_{y_1\ge 0,\dots,y_{n-1}\ge 0} \prod_{j=1}^{n-1} G'(x + y_1 + \cdots + y_j) \times G(x + y_1 + \cdots + y_{n-1})\, dy_1 \cdots dy_{n-1} = \frac{1}{n!}G(x)^n.$$

As $|G(x)| \le C_N \Omega_N^0(x)$ for any N, it follows from (A.20) that, for any N,

$$|\partial_x^\alpha \partial_\xi^\beta g_n(x,\xi)| \le \frac{C_N^{n+1}}{n!}\langle\xi\rangle^{-n}\Omega_N^{\beta+1}(x). \tag{A.21}$$

If we sum for $n \ge \beta + 1$, we get a bound by the right-hand side of (A.15).

We are thus left with studying

$$\sum_{n=1}^{\beta} \partial_x^{\alpha} \partial_{\xi}^{\beta} g_n(x,\xi). \tag{A.22}$$

Notice that (A.21) summed for $n = 1, \ldots, \beta$ gives, when $|\xi| \le 1$, estimate (A.15) for (A.22) as well. Assume from now on that $|\xi| \ge 1$ and let us prove by induction on $n = 1, \ldots, \beta$ that $|\partial_x^{\alpha}\partial_{\xi}^{\beta} g_n(x,\xi)|$ is bounded by the right-hand side of (A.15). We may write from (A.17)

$$\begin{aligned} g_n(x,\xi) &= \int_{x \le x_1} D_{\xi}(x_1 - x) 2V(x_1) g_{n-1}(x_1,\xi)\, dx_1 \\ &= \int_{y_1 \ge 0} D_{\xi}(y_1) 2V(y_1 + x) g_{n-1}(y_1 + x, \xi)\, dy_1 \end{aligned} \tag{A.23}$$

with $g_0 \equiv 1$. We use in (A.23) the last expression (A.13) for D_{ξ}. We have then to consider two kind of terms. The first one is

$$\begin{aligned} &\int_{y_1 \ge 0} \frac{e^{2iy_1\xi}}{\xi} 2V(y_1 + x) g_{n-1}(y_1 + x, \xi)\, dy_1 \\ &\quad = -\frac{1}{2i\xi^2} 2V(x) g_{n-1}(x,\xi) \\ &\qquad - \int_{y_1 \ge 0} \frac{e^{2iy_1\xi}}{2i\xi^2} \partial_{y_1}\big(2V(y_1 + x) g_{n-1}(y_1 + x, \xi)\big)\, dy_1. \end{aligned}$$

Repeating the integrations by parts, we end up with contributions that, according to the induction hypothesis (and the fact that $g_0 \equiv 1$), satisfy estimates of the form (A.15) (with $\Omega_N^{\beta}(x)$ replaced by $\langle x \rangle^{-N}$), and an integral term of the form

$$\int_{y_1 \ge 0} \frac{e^{2iy_1\xi}}{\xi^{M+1}} \partial_{y_1}^{M}\big(2V(y_1 + x) g_{n-1}(y_1 + x, \xi)\big)\, dy_1 \tag{A.24}$$

for M as large as we want. If $M = \beta$, we see that (A.24) satisfies (A.15). The second type of terms coming from (A.23) to consider is

$$\frac{1}{\xi} \int_{y_1 \ge 0} 2V(y_1 + x) g_{n-1}(y_1 + x, \xi)\, dy_1$$

which trivially satisfies (A.15) by the induction hypothesis applied to g_{n-1}. This concludes the proof. ■

In order to obtain the representation (A.10) for $W_+ w$, when w is odd, we recall first the definition of the transmission and reflection coefficients. The Wronskian of $(f_1(x,\xi), f_1(x,-\xi))$ (resp. $(f_2(x,\xi), f_2(x,-\xi))$) is non-zero for any ξ in $\mathbb{R}^*$ (see [17, p. 144]), so that, for real $\xi \neq 0$, we may find unique coefficients $T_1(\xi), T_2(\xi)$

non-zero, $R_1(\xi)$, $R_2(\xi)$ such that

$$\begin{aligned} f_2(x,\xi) &= \frac{R_1(\xi)}{T_1(\xi)} f_1(x,\xi) + \frac{1}{T_1(\xi)} f_1(x,-\xi) \\ f_1(x,\xi) &= \frac{R_2(\xi)}{T_2(\xi)} f_2(x,\xi) + \frac{1}{T_2(\xi)} f_2(x,-\xi). \end{aligned} \tag{A.25}$$

By [17, Theorem I], these functions extend as smooth functions on $\mathbb{R}$, and they satisfy the following properties:

$$\begin{gathered} T_1(\xi) = T_2(\xi) \stackrel{\text{def}}{=} T(\xi), \\ T(\xi)\overline{R_2(\xi)} + R_1(\xi)\overline{T(\xi)} = 0, \\ |T(\xi)|^2 + |R_j(\xi)|^2 = 1, \quad j = 1,2, \\ \overline{T(\xi)} = T(-\xi), \quad \overline{R_j(\xi)} = R_j(-\xi). \end{gathered} \tag{A.26}$$

If the potential V is even, we have seen that

$$f_1(-x,\xi) = f_2(x,\xi),$$

so that, plugging this equality in the first relation of (A.25), comparing to the second one, and using that $T_1 = T_2$, we conclude that

$$R_1(\xi) = R_2(\xi). \tag{A.27}$$

We denote by $R(\xi)$ this common value. The integral representations of the scattering coefficients (see [17, p. 145])

$$\begin{aligned} \frac{R(\xi)}{T(\xi)} &= \frac{1}{2i\xi} \int e^{2ix\xi} 2V(x) m_1(x,\xi)\, dx, \\ \frac{1}{T(\xi)} &= 1 - \frac{1}{2i\xi} \int 2V(x) m_1(x,\xi)\, dx \end{aligned} \tag{A.28}$$

together with (A.15) and the fact that $V \in \mathcal{S}(\mathbb{R})$, show that for any N, β,

$$\partial_\xi^\beta R(\xi) = O(\langle\xi\rangle^{-N}), \quad \partial_\xi^\beta (T(\xi) - 1) = O(\langle\xi\rangle^{-1-\beta}). \tag{A.29}$$

We need the following lemma:

Lemma A.2.2. *The functions T, R satisfy*

$$T(0) = 1 + R(0) \tag{A.30}$$

in the following two cases:

- *The generic case $\int V(x) m_1(x,0)\, dx \neq 0$.*
- *The very exceptional case $\int V(x) m_1(x,0)\, dx = 0$ and $\int V(x) x m_1(x,0)\, dx = 0$.*

Proof. Summing the two equalities (A.28) and making an expansion at $\xi = 0$ using (A.15), we get

$$\begin{aligned} R(\xi) + 1 &= T(\xi)\Big(1 - \frac{1}{i\xi}\int_{-\infty}^{+\infty} V(x)m_1(x,\xi)\,dx \\ &\qquad + \frac{1}{i\xi}\int_{-\infty}^{+\infty} e^{2ix\xi}V(x)m_1(x,\xi)\,dx\Big) \\ &= T(\xi)\Big(1 + 2\int_{-\infty}^{+\infty} xV(x)m_1(x,0)\,dx + O(\xi)\Big), \quad \xi \to 0, \end{aligned}$$

so that

$$R(0) + 1 - T(0) = 2T(0)\int_{-\infty}^{+\infty} xV(x)m_1(x,0)\,dx. \tag{A.31}$$

In the generic case, by (A.28),

$$T(\xi) = i\xi\Big(-\int_{-\infty}^{+\infty} V(x)m_1(x,0)\,dx + O(\xi)\Big)^{-1}, \quad \xi \to 0, \tag{A.32}$$

so that $T(0) = 0$. This shows that (A.31) vanishes in the two considered cases. ■

Proof of Proposition A.1.1. We have to prove that W_+ acting on odd functions is given by (A.10). Recall (see for instance Weder [85] formula (2.20), Schechter [74]) that $W_+ w$ is given by

$$W_+ w = F_+^* \hat{w}, \tag{A.33}$$

where F_+^* is the adjoint of the distorted Fourier transform, given by

$$F_+^* \Phi = \frac{1}{2\pi}\int \psi_+(x,\xi)\Phi(\xi)\,d\xi, \tag{A.34}$$

where

$$\psi_+(x,\xi) = \mathbb{1}_{\xi>0}T(\xi)f_1(x,\xi) + \mathbb{1}_{\xi<0}T(-\xi)f_2(x,-\xi). \tag{A.35}$$

Let $\chi_\pm$ be the functions defined in the statement of Proposition A.1.1 and write

$$\psi_+(x,\xi) = \chi_+(x)\psi_+(x,\xi) + \chi_-(x)\psi_+(x,\xi).$$

Replace in $\chi_+\psi_+$ (resp. $\chi_-\psi_+$) ψ_+ by (A.35), where we express f_2 from f_1 (resp. f_1 for f_2) using the first (resp. second) formula (A.25). We get, using notation (A.2),

$$\begin{aligned} \psi_+(x,\xi) = \chi_+(x)\Big(&e^{ix\xi}\big(T(\xi)m_1(x,\xi)\mathbb{1}_{\xi>0} + m_1(x,\xi)\mathbb{1}_{\xi<0}\big) \\ &+ e^{-ix\xi}R(-\xi)m_1(x,-\xi)\mathbb{1}_{\xi<0}\Big) \\ + \chi_-(x)\Big(&e^{ix\xi}\big(m_2(x,-\xi)\mathbb{1}_{\xi>0} + T(-\xi)m_2(x,-\xi)\mathbb{1}_{\xi<0}\big) \\ &+ e^{-ix\xi}R(\xi)m_2(x,\xi)\mathbb{1}_{\xi>0}\Big). \end{aligned} \tag{A.36}$$

Using (A.11), we deduce from (A.33), (A.34) and (A.36) that

$$W_+ w = \frac{1}{2\pi}\int e^{ix\xi} e_1(x,\xi)\hat{w}(\xi)\,d\xi + \frac{1}{2\pi}\int e^{-ix\xi} e_2(x,\xi)\hat{w}(\xi)\,d\xi \tag{A.37}$$

with

$$\begin{aligned} e_1(x,\xi) &= \chi_+(x) m_1(x,\xi)\big(T(\xi)\mathbb{1}_{\xi>0} + \mathbb{1}_{\xi<0}\big) \\ &\qquad\qquad + \chi_-(x) m_1(-x,-\xi)\big(\mathbb{1}_{\xi>0} + T(-\xi)1_{\xi<0}\big), \\ e_2(x,\xi) &= \chi_+(x) R(-\xi) m_1(x,-\xi)\mathbb{1}_{\xi<0} + \chi_-(x) R(\xi) m_1(-x,\xi)\mathbb{1}_{\xi>0}. \end{aligned} \tag{A.38}$$

If w is odd, we may rewrite (A.37) as

$$W_+ w = \frac{1}{2\pi}\int e^{ix\xi} a(x,\xi)\hat{w}(\xi)\,d\xi$$

with

$$\begin{aligned} a(x,\xi) &= e_1(x,\xi) - e_2(x,-\xi) \\ &= \chi_+(x) m_1(x,\xi)\big((T(\xi) - R(\xi))\mathbb{1}_{\xi>0} + \mathbb{1}_{\xi<0}\big) \\ &\qquad\qquad + \chi_-(x) m_1(-x,-\xi)\big(\mathbb{1}_{\xi>0} + (T(-\xi) - R(-\xi))\mathbb{1}_{\xi<0}\big). \end{aligned} \tag{A.39}$$

By properties (A.26), $|T(\xi) - R(\xi)|^2 = 1$ and by (A.30), $T(0) - R(0) = 1$. We may thus find a unique smooth real-valued function $\theta(\xi)$, satisfying $\theta(0) = 0$, such that $T(\xi) - R(\xi) = e^{2i\theta(\xi)}$. Moreover, using (A.26), one gets that θ is odd, and by (A.29) it satisfies $\partial^\beta \theta(\xi) = O(\langle\xi\rangle^{-1-\beta})$. We define

$$c(\xi) = e^{i\theta(\xi)}\mathbb{1}_{\xi>0} + e^{-i\theta(\xi)}\mathbb{1}_{\xi<0} \tag{A.40}$$

so that in (A.39)

$$\begin{aligned} (T(\xi) - R(\xi))\mathbb{1}_{\xi>0} + \mathbb{1}_{\xi<0} &= e^{i\theta(\xi)} c(\xi), \\ \mathbb{1}_{\xi>0} + (T(-\xi) - R(-\xi))\mathbb{1}_{\xi<0} &= e^{-i\theta(\xi)} c(\xi) \end{aligned}$$

and $a(x,\xi) = b(x,\xi)c(\xi)$, where b is a smooth function satisfying (A.8) given by

$$b(x,\xi) = \chi_+(x) m_1(x,\xi) e^{i\theta(\xi)} + \chi_-(x) m_1(-x,-\xi) e^{-i\theta(\xi)}.$$

We thus got $W_+ w = b(x, D_x) \circ c(D_x) w$ for odd w. Moreover, the definition of f_1 and m_1 shows that $\overline{f_1(x,\xi)} = f_1(x,-\xi)$, $\overline{m_1(x,\xi)} = m_1(x,-\xi)$, so that it follows from the expression of b that equalities (A.9) hold. ■

Remarks. We make the following observations.

- The proof of the last result shows that b satisfies better estimates than those written in (A.8): Actually, on the right-hand side of these inequalities, one could insert a factor $\langle\xi\rangle^{-\beta}$. We wrote the estimates without this factor because we shall have in any case to consider also more general classes of symbols, for which only (A.8) holds.

- The difference between generic or very exceptional potentials versus exceptional ones appears, as is well known, when considering the action of the Fourier multiplier $c(\xi)$ on L^∞ based spaces. Since $\partial^\beta \theta(\xi) = O(\langle \xi \rangle^{-1-\beta})$ when $|\xi| \to +\infty$, $c(\xi) - 1$ coincides with a symbol of order -1 outside a neighborhood of zero. Consequently, if $\chi_0 \in C_0^\infty(\mathbb{R})$ is equal to one close to zero, $(1-\chi_0)(D_x)c(D_x)$ is bounded on L^∞. On the other hand, $\chi_0(\xi)c(\xi)$ is Lipschitz at zero if the potential is generic or very exceptional, since $\theta(0) = 0$, so that $\chi_0(D_x)c(D_x)$ is also bounded on L^∞. In the exceptional potential case, $c(\xi)$ has a jump at $\xi = 0$, and L^∞ bounds for $c(D_x)$ do not hold.