Appendix B

(Semiclassical) pseudo-differential operators

This appendix is devoted to the definition and main properties of classes of multilinear pseudo-differential operators and their semiclassical counterparts. Recall that the symbol of a pseudo-differential operator of order $m \in \mathbb{R}$ is in general a smooth function $(x, \xi) \mapsto a(x, \xi)$ defined on $\mathbb{R}^d \times \mathbb{R}^d$, satisfying for any multi-indices α, β estimates of the form

$$|\partial_x^\alpha \partial_\xi^\beta a(x, \xi)| \leq C_{\alpha,\beta} \langle \xi \rangle^{m-\rho|\beta|+\delta|\alpha|}, \tag{B.1}$$

where $0 \leq \delta \leq \rho \leq 1$ (see Hörmander [42, 43]). One associates to such a symbol an operator acting on test functions in $\mathcal{S}(\mathbb{R})$ by a quantization rule, that may be given for instance by the usual quantization

$$\mathrm{Op}(a)u = \frac{1}{(2\pi)^d} \int e^{ix\cdot\xi} a(x, \xi)\hat{u}(\xi)\, d\xi = \frac{1}{(2\pi)^d} \int e^{i(x-y)\cdot\xi} a(x, \xi)u(y)\, dy\, d\xi$$

or by the Weyl quantization

$$\mathrm{Op}^{\mathrm{W}}(a)u = \frac{1}{(2\pi)^d} \int e^{i(x-y)\cdot\xi} a\Big(\frac{x+y}{2}, \xi\Big)u(y)\, dy\, d\xi.$$

We shall be here more interested in the semiclassical version of this calculus, namely smooth symbols $(x, \xi, h) \mapsto a(x, \xi, h)$ that depend on a parameter $h \in]0, 1]$, and that satisfy bounds of the form

$$|\partial_x^\alpha \partial_\xi^\beta (h\partial_h)^k a(x, \xi, h)| \leq C_{\alpha,\beta,k} M(x, \xi) \tag{B.2}$$

with a fixed "weight function" $M(x, \xi)$ (see Dimassi and Sjöstrand [24]). For instance, a function satisfying (B.1) with $\rho = \delta = 0$ obeys these inequalities with $M \equiv 1$. One defines then the semiclassical quantization of a by the formulas

$$\begin{aligned} \mathrm{Op}_h(a)u = a(x, hD_x, h)u &= \frac{1}{(2\pi)^d} \int e^{ix\cdot\xi} a(x, h\xi, h)\hat{u}(\xi)\, d\xi \\ &= \frac{1}{(2\pi h)^d} \int e^{i(x-y)\cdot\frac{\xi}{h}} a(x, \xi, h)u(y)\, dy\, d\xi \end{aligned} \tag{B.3}$$

or for the Weyl quantization by

$$\mathrm{Op}_h^{\mathrm{W}}(a)u = \frac{1}{(2\pi h)^d} \int e^{i(x-y)\cdot\frac{\xi}{h}} a\Big(\frac{x+y}{2}, \xi, h\Big)u(y)\, dy d\xi. \tag{B.4}$$

One has then a symbolic calculus. Assume for instance that we are given two symbols a, b satisfying (B.2) with $M \equiv 1$. Then there is a symbol c in the same class such that

$\mathrm{Op}_h(a) \circ \mathrm{Op}_h(b) = \mathrm{Op}_h(c)$. Moreover, one may get an asymptotic expansion of c in terms of powers of the semiclassical parameter h, whose first terms are given by

$$c(x,\xi,h) = a(x,\xi,h)b(x,\xi,h) + \frac{h}{i}\sum_{j=1}^{d} \partial_{\xi_j} a(x,\xi,h)\partial_{x_j} b(x,\xi,h) + \cdots . \tag{B.5}$$

It turns out that we shall be interested only in the case of one variable $d = 1$, but with more general classes of symbols. In Appendix A, we have used symbols $b(x,\xi)$ satisfying inequalities (A.8). It turns out that, if one translates in the semiclassical framework the operators $b(x, D_x)$ (see (B.15) and (B.16) below), one is led to consider instead of (B.3) the more general operator

$$b\Big(\frac{x}{h}, hD_x\Big)u = \frac{1}{2\pi}\int e^{ix\xi} b\Big(\frac{x}{h}, h\xi\Big)\hat{u}(\xi)\,d\xi. \tag{B.6}$$

Of course, the function $(x,\xi) \mapsto b(\frac{x}{h},\xi)$ does not satisfy the estimates in (B.2), since ∂_x-derivatives make lose powers of h^{-1}. On the other hand, because of (A.8), taking a ∂_x-derivatives makes gain a weight in $\langle \frac{x}{h}\rangle^{-N}$ for any N. We shall thus consider symbols depending on two space variables, $(y,x,\xi) \mapsto a(y,x,\xi,h)$, such that at fixed y, $(x,\xi,h) \mapsto a(y,x,\xi,h)$ satisfies the estimates in (B.2), and that for any $\ell > 0$, $(x,\xi,h) \mapsto \partial_y^\ell a(y,x,\xi,h)$ satisfies (B.2) with on the right-hand side of these inequalities an arbitrarily decaying factor in $\langle \frac{x}{h}\rangle^{-N}$. We shall quantify such symbols as

$$\mathrm{Op}_h(a)u = a\Big(\frac{x}{h}, x, hD_x, h\Big)u = \frac{1}{2\pi}\int e^{ix\xi} a\Big(\frac{x}{h}, x, h\xi, h\Big)\hat{u}(\xi)\,d\xi. \tag{B.7}$$

In that way, instead of getting for the composition of two such symbols an expansion of the form (B.5), we shall obtain

$$c(y,x,\xi,h) = a(y,x,\xi,h)b(y,x,\xi,h) + hr_1 + r_1', \tag{B.8}$$

where r_1 is in the same class as a, b and where r_1' is rapidly decaying in $\frac{x}{h}$, i.e. satisfies (B.2) with on the right-hand side an extra arbitrary factor in $\langle \frac{x}{h}\rangle^{-N}$.

It turns out that we shall not just need linear, but also multilinear operators, defined instead of (B.7) by formula (B.14) below. The goal of this chapter is thus to define such operators and study their composition properties, establishing the generalization of formulas of the form (B.8) to this multilinear framework.

B.1 Classes of symbols and their quantization

We shall use classes of semiclassical multilinear pseudo-differential operators, analogous to those introduced in [20]. We shall use also the non-semiclassical counterparts of these operators that are deduced from the former by conjugation through dilations. We refer to Dimassi and Sjöstrand [24] for a reference text on semiclassical calculus. Recall first the following definition.

Definition B.1.1. An order function on $\mathbb{R} \times \mathbb{R}^p$ is a function M from $\mathbb{R} \times \mathbb{R}^p$ to $\mathbb{R}_+$, $(x, \xi_1, \dots, \xi_p) \mapsto M(x, \xi_1, \dots, \xi_p)$, such that there is N_0 in $\mathbb{N}$, $C > 0$ and for any $(x, \xi_1, \dots, \xi_p)$, $(x', \xi_1', \dots, \xi_p')$ in $\mathbb{R} \times \mathbb{R}^p$,

$$M(x', \xi_1', \dots, \xi_p') \leq C \langle x - x' \rangle^{N_0} \prod_{j=1}^{p} \langle \xi_j - \xi_{j'} \rangle^{N_0} M(x, \xi_1, \dots, \xi_p). \tag{B.9}$$

An example of an order function that we use several times is

$$M_0(\xi_1, \dots, \xi_p) = \Big(\sum_{1 \leq i < j \leq p} \langle \xi_i \rangle^2 \langle \xi_j \rangle^2 \Big)^{\frac{1}{2}} \Big(\sum_{i=1}^{p} \langle \xi_i \rangle^2 \Big)^{-\frac{1}{2}}. \tag{B.10}$$

Actually, this function is smooth and is equivalent to $1 + \max_2(|\xi_1|, \dots, |\xi_p|)$, where $\max_2(|\xi_1|, \dots, |\xi_p|)$ is the second largest among $|\xi_1|, \dots, |\xi_p|$.

We shall introduce several classes of semiclassical symbols, depending on a semi-classical parameter $h \in]0, 1]$:

Definition B.1.2. Let p be in $\mathbb{N}^*$, M an order function on $\mathbb{R} \times \mathbb{R}^p$, M_0 the function defined in (B.10). Let (β, κ) be in $[0, +\infty[\times \mathbb{N}$. We denote by $S_{\kappa,\beta}(M, p)$ the space of smooth functions

$$\begin{gathered} (y, x, \xi_1, \dots, \xi_p, h) \mapsto a(y, x, \xi_1, \dots, \xi_p, h), \\ \mathbb{R} \times \mathbb{R} \times \mathbb{R}^p \times]0, 1] \to \mathbb{C} \end{gathered} \tag{B.11}$$

satisfying for any $\alpha_0 \in \mathbb{N}$, $\alpha \in \mathbb{N}^p$, $k \in \mathbb{N}$, $N \in \mathbb{N}$, $\alpha_0' \in \mathbb{N}^*$ the bounds

$$\big| \partial_x^{\alpha_0} \partial_\xi^\alpha (h\partial_h)^k a(y, x, \xi, h) \big| \leq C M(x, \xi) M_0(\xi)^{\kappa(\alpha_0 + |\alpha|)} \big(1 + \beta h^\beta M_0(\xi)\big)^{-N} \tag{B.12}$$

and

$$\begin{aligned} &\big| \partial_y^{\alpha_0'} \partial_x^{\alpha_0} \partial_\xi^\alpha (h\partial_h)^k a(y, x, \xi, h) \big| \\ &\quad \leq C M(x, \xi) M_0(\xi)^{\kappa(\alpha_0 + |\alpha|)} \big(1 + \beta h^\beta M_0(\xi)\big)^{-N} \big(1 + M_0(\xi)^{-\kappa} |y|\big)^{-N}, \end{aligned} \tag{B.13}$$

where ξ stands for $(\xi_1, \dots, \xi_p)$.

We denote by $S'_{\kappa,\beta}(M, p)$ the subspace of $S_{\kappa,\beta}(M, p)$ of those symbols that satisfy (B.13) including for $\alpha_0' = 0$.

We shall set $S'^{N'}_{\kappa,\beta}(M, p)$ for the space of functions satisfying the bound in (B.13) including for the case $\alpha_0' = 0$, but with the last factor $(1 + M_0(\xi)^{-\kappa}|y|)^{-N}$ replaced by $(1 + M_0(\xi)^{-\kappa}|y|)^{-N'}$, for a fixed power N' instead of for all N.

Remarks. We make the following observations.

- If $p = 1$, then $M_0(\xi) = 1$ and symbols of the class $S_{\kappa,\beta}(M, 1)$ that do not depend on y are just usual symbols of pseudo-differential operators as defined in [24] for instance. For symbols depending on y, we impose that if we take at least one

∂_y-derivative, we get a rapid decay in $|y|$ in the case of the class $S_{\kappa,\beta}(M,1)$. For elements of $S'_{\kappa,\beta}(M,1)$, this rapid decay has to hold including without taking any ∂_y-derivative. Notice also that when $p=1$, the classes we define do not depend on the parameters κ,β.

- The parameter κ in the definition of the classes of symbols measures the power of $M_0(\xi)$ that we lose when taking ∂_x- or ∂_ξ-derivatives. As these losses involve only "small frequencies", they will be affordable.
- When $\beta>0$, we have an extra gain in $\langle h^\beta M_0(\xi)\rangle^{-N}$ for any N, that allows to trade off the loss $M_0(\xi)^\kappa$ for $h^{-\beta\kappa}$. If β is small, this reduces these losses to those ones used usually in definitions of semiclassical symbols as in [24]. Moreover, an element of $S_{\kappa,0}(M,p)$ may be always reduced to an element of $S_{\kappa,\beta}(M,p)$ multiplying it by $\chi(h^\beta M_0(\xi))$ for some χ in $C_0^\infty(\mathbb{R})$.

We shall quantize symbols in $S_{\kappa,\beta}(M,p)$ as p-linear operators acting a p-tuple of functions by

$$\begin{aligned}\operatorname{Op}_h(a)(\underline{v}_1,\dots,\underline{v}_p)\\ &= \frac{1}{(2\pi)^p}\int e^{ix(\xi_1+\cdots+\xi_p)}a\Big(\frac{x}{h},x,h\xi_1,\dots,h\xi_p\Big)\prod_{j=1}^{p}\hat{\underline{v}}_j(\xi_j)\,d\xi_1\cdots d\xi_p\\ &= \frac{1}{(2\pi h)^p}\int e^{i\sum_{j=1}^{p}(x-x_j')\frac{\xi_j}{h}}a\Big(\frac{x}{h},x,\xi_1,\dots,\xi_p\Big)\prod_{j=1}^{p}\underline{v}_j(x_j')\,dx'd\xi.\end{aligned} \tag{B.14}$$

We shall call (B.14) the semiclassical quantization of a. We shall also use a classical quantization, depending on the parameter $t=\frac{1}{h}\geq 1$, related to (B.14) through conjugation by dilations: If $t\geq 1$, and $\underline{v}$ is a test function on $\mathbb{R}$, define the L^2 isometry Θ_t by

$$\Theta_t\underline{v}(x)=\frac{1}{\sqrt{t}}\underline{v}\Big(\frac{x}{t}\Big). \tag{B.15}$$

We shall set for a an element of $S_{\kappa,\beta}(M,p)$,

$$\operatorname{Op}^t(a)(v_1,\dots,v_p)=h^{\frac{p-1}{2}}\Theta_t\circ\operatorname{Op}_h(a)\big(\Theta_{t^{-1}}v_1,\dots,\Theta_{t^{-1}}v_p\big) \tag{B.16}$$

with $h=t^{-1}$. Explicitly, we get from (B.14)

$$\begin{aligned}\operatorname{Op}^t(a)(v_1,\dots,v_p)\\ &= \frac{1}{(2\pi)^p}\int e^{ix(\xi_1+\cdots+\xi_p)}a\Big(x,\frac{x}{t},\xi_1,\dots,\xi_p\Big)\prod_{j=1}^{p}\hat{v}_j(\xi_j)\,d\xi_1\cdots d\xi_p.\end{aligned} \tag{B.17}$$

Remark that if $a(y,x,\xi)$ is independent of x, then $\operatorname{Op}^t(a)$ is independent of t, and if $p=1$, $\operatorname{Op}^t(a)$ is just the usual pseudo-differential operator of symbol $a(y,\xi)$. In this case, we shall just write $\operatorname{Op}(a)$ for $\operatorname{Op}^t(a)$.

B.2 Symbolic calculus

We prove first a proposition generalizing [20, Proposition 1.5].

Proposition B.2.1. *Let n', n'' be in $\mathbb{N}^*$, $n = n' + n'' - 1$. Let $M'(x, \xi_1, \dots, \xi_{n'})$ and $M''(x, \xi_{n'}, \dots, \xi_n)$ be two order functions on $\mathbb{R} \times \mathbb{R}^{n'}$ and $\mathbb{R} \times \mathbb{R}^{n''}$, respectively. In particular, they satisfy (B.9) and we shall denote by N_0'' an integer such that*

$$M''(x', \xi_{n'}, \dots, \xi_n) \le C \langle x - x' \rangle^{N_0''} M''(x, \xi_{n'}, \dots, \xi_n). \tag{B.18}$$

Let $(\kappa, \beta) \in \mathbb{N} \times [0, 1]$, a in $S_{\kappa,\beta}(M', n')$, b in $S_{\kappa,\beta}(M'', n'')$. Assume either that $(\kappa, \beta) = (0, 0)$ or $0 < \beta\kappa \le 1$ or that symbol b is independent of x. Define

$$M(x, \xi_1, \dots, \xi_n) = M'(x, \xi_1, \dots, \xi_{n'-1}, \xi_{n'} + \dots + \xi_n) M''(x, \xi_{n'}, \dots, \xi_n). \tag{B.19}$$

Then there is ν in $\mathbb{N}$, that depends only on N_0'' in (B.18), and symbols

$$c_1 \in S_{\kappa,\beta}(M M_0^{\nu\kappa}, n), c_1' \in S'_{\kappa,\beta}(M M_0^{\nu\kappa}, n) \tag{B.20}$$

such that one may write

$$\mathrm{Op}_h(a)[v_1, \dots, v_{n'-1}, \mathrm{Op}_h(b)(v_{n'}, \dots, v_n)] = \mathrm{Op}_h(c)[v_1, \dots, v_n], \tag{B.21}$$

where

$$\begin{aligned} c(y, x, \xi_1, \dots, \xi_n) = a(y, x, \xi_1, &\dots, \xi_{n'-1}, \xi_{n'} + \dots + \xi_n) \\ &\times b(y, x, \xi_{n'}, \dots, \xi_n) \\ &+ h c_1(y, x, \xi_1, \dots, \xi_n) + c_1'(y, x, \xi_1, \dots, \xi_n). \end{aligned} \tag{B.22}$$

Moreover, if b is independent of y, c_1' in (B.22) vanishes and if b is independent of x, then c_1 vanishes. In addition, if a is in $S'_{\kappa,\beta}(M', n')$ or b is in $S'_{\kappa,\beta}(M'', n'')$, then c and c_1 are in $S'_{\kappa,\beta}(M M_0^{\nu\kappa}, n)$.

Let us prove first a lemma:

Lemma B.2.2. *Let $\xi' = (\xi_1, \dots, \xi_{n'-1})$ and $\xi'' = (\xi_{n'}, \dots, \xi_n)$, $\xi = (\xi', \xi'')$. Then*

$$M_0(\xi', \xi_{n'} + \dots + \xi_n) \le C M_0(\xi), \quad M_0(\xi'') \le C M_0(\xi). \tag{B.23}$$

Moreover, if ζ is a real number and $|\zeta| / M_0(\xi)$ is small enough,

$$\max\big(M_0(\xi', \xi_{n'} + \dots + \xi_n - \zeta), M_0(\xi'')\big) \ge c M_0(\xi) \tag{B.24}$$

for some $c > 0$.

Proof. Estimate (B.23) follows from the fact that $M_0(\xi_1, \dots, \xi_n)$ is equivalent to $1 + \max_2(|\xi_1|, \dots, |\xi_n|)$.

To prove estimate (B.24), we may assume that $|\xi_n| \ge |\xi_{n-1}| \ge \dots \ge |\xi_{n'}|$ and $|\xi_1| \ge |\xi_2| \ge \dots \ge |\xi_{n'-1}|$. Moreover, if $n = n'$, then (B.24) is trivial, so that we may assume $n' < n$.

Case 1. Assume $|\xi_n| \geq |\xi_1|$. If $|\xi_n| \sim |\xi_{n-1}|$, then both $M_0(\xi'')$ and $M_0(\xi)$ are of the magnitude of $\langle \xi_{n-1} \rangle$, so (B.24) is trivial.

Let us assume that $|\xi_{n-1}| \ll |\xi_n|$.

- If in addition $|\xi_n| \sim |\xi_1|$, then $M_0(\xi) \sim \langle \xi_n \rangle \sim \langle \xi_1 \rangle$ and
$$\langle \xi_{n'} + \cdots + \xi_n - \zeta \rangle \sim \langle \xi_n \rangle,$$
so that
$$M_0(\xi', \xi_{n'} + \cdots + \xi_n - \zeta) \sim M_0(\xi', \xi_n) \sim \langle \xi_n \rangle \sim \langle \xi_1 \rangle$$
and (B.24) holds.
- If $|\xi_1| \ll |\xi_n|$, then $M_0(\xi) \sim \max(\langle \xi_1 \rangle, \langle \xi_{n-1} \rangle)$ and $M_0(\xi'') \sim \langle \xi_{n-1} \rangle$, so that $M_0(\xi', \xi_{n'} + \cdots + \xi_n - \zeta) \sim M_0(\xi', \xi_n) \sim \langle \xi_1 \rangle$ and (B.24) holds again.

Case 2. Assume $|\xi_1| \geq |\xi_n|$. Then $M_0(\xi) \sim \max(\langle \xi_2 \rangle, \langle \xi_n \rangle)$.

- If $|\xi_n| \geq |\xi_2|$ and $|\xi_n| \sim |\xi_{n-1}|$, then $M_0(\xi'') \sim \langle \xi_n \rangle$, so that (B.24) holds.
- If $|\xi_n| \geq |\xi_2|$ and $|\xi_n| \gg |\xi_{n-1}|$, then we have $|\xi_{n'} + \cdots + \xi_n - \zeta| \sim |\xi_n|$, so that $M_0(\xi', \xi_{n'} + \cdots + \xi_n - \zeta) \sim \langle \xi_n \rangle$ and (B.24) holds.
- If $|\xi_2| \geq |\xi_n|$, then $M_0(\xi', \xi_{n'} + \cdots + \xi_n - \zeta) \sim \langle \xi_2 \rangle$, so that (B.24) holds as well.

This concludes the proof. ∎

Proof of Proposition B.2.1. Going back to the definition (B.14) of quantization, we may write the composition (B.21) as the right-hand side of this expression, with a symbol c given by the oscillatory integral

$$c(y, x, \xi) = \frac{1}{2\pi} \int e^{-iz\zeta} a(y, x, \xi', \xi_{n'} + \cdots + \xi_n - \zeta) \times b(y - z, x - hz, \xi'')\, dz\, d\zeta. \tag{B.25}$$

We decompose

$$a(y, x, \xi', \xi_{n'} + \cdots + \xi_n - \zeta) = a(y, x, \xi', \xi_{n'} + \cdots + \xi_n) - \zeta \tilde{a}(y, x, \xi', \xi_{n'} + \cdots + \xi_n, \zeta) \tag{B.26}$$

with

$$\tilde{a}(y, x, \xi', \tilde{\xi}, \zeta) = \int_0^1 \Big(\frac{\partial a}{\partial \tilde{\xi}}\Big)(y, x, \xi', \tilde{\xi} - \lambda\zeta)\, d\lambda. \tag{B.27}$$

It follows from (B.23) that

$$M_0(\xi', \xi_{n'} + \cdots + \xi_n - \lambda\zeta) \leq C(M_0(\xi) + \langle \zeta \rangle). \tag{B.28}$$

Using (B.12) and the definition of order functions, we get that $\tilde{a}$ satisfies

$$\begin{aligned} &|\partial_x^{\alpha_0} \partial_\xi^\alpha \partial_\zeta^\gamma (h\partial_h)^k \tilde{a}(y, x, \xi', \xi_{n'} + \cdots + \xi_n, \zeta)| \\ &\quad \leq C(M_0(\xi) + \langle \zeta \rangle)^{\kappa(1+|\alpha|+|\gamma|+\alpha_0)} \langle \zeta \rangle^{N_0} M'(x, \xi', \xi_{n'} + \cdots + \xi_n) \\ &\qquad \times \int_0^1 \big(1 + \beta h^\beta M_0(\xi', \xi_{n'} + \cdots + \xi_n - \lambda\zeta)\big)^{-N}\, d\lambda \end{aligned} \tag{B.29}$$

for any $\alpha, \alpha_0, \gamma, k, N$. If one takes at least one ∂_y-derivative, the same estimate holds, with an extra factor

$$\left(1+(M_0(\xi)+\langle\zeta\rangle)^{-\kappa}|y|\right)^{-N} \tag{B.30}$$

using (B.13) and (B.28). If we plug (B.26) in (B.25), we get the first term on the right-hand side of (B.22) and, by integration by parts, the following two contributions:

$$-\frac{i}{2\pi}\int e^{-iz\zeta}\tilde{a}(y,x,\xi',\xi_{n'}+\cdots+\xi_n,\zeta)\frac{\partial b}{\partial y}(y-z,x-hz,\xi'')\,dz\,d\zeta, \tag{B.31}$$

$$-\frac{ih}{2\pi}\int e^{-iz\zeta}\tilde{a}(y,x,\xi',\xi_{n'}+\cdots+\xi_n,\zeta)\frac{\partial b}{\partial x}(y-z,x-hz,\xi'')\,dz\,d\zeta. \tag{B.32}$$

Let us show that (B.31) (resp. (B.32)) provides the contribution c_1' (resp. hc_1) in equation (B.22).

Study of (B.31). If we insert under integral (B.31) a cut-off $(1-\chi_0)(\zeta)$ for some C_0^∞ function χ_0 equal to one close to zero and make N_1 integrations by parts in z, we gain a factor ζ^{-N_1}, up to making act on $\frac{\partial b}{\partial y}(y-z,x-hz,\xi'')$ at most N_1 ∂_z-derivatives. By (B.12) and (B.13), each of these ∂_z-derivatives makes lose $\langle hM_0(\xi'')^\kappa\rangle$ if it falls on the x argument of $\frac{\partial b}{\partial y}$, and does not make lose anything if it falls on the y argument. Consequently, if $\beta=\kappa=0$, or if b is independent of x, we get no loss, while if $\kappa\beta>0$, we get a loss that may be compensated since, in this case, we get by (B.12) and (B.13) a factor $\langle h^\beta M_0(\xi'')\rangle^{-N}$ in the estimates, with an arbitrary N. Since we assume $\beta\kappa\le 1$, $\langle h^\beta M_0(\xi'')\rangle^{-N}\langle hM_0(\xi'')^\kappa\rangle^{N_1}=O(\langle h^\beta M_0(\xi'')\rangle^{-N/2})$ if N is large enough relatively to N_1. In other words, up to changing the definition of b, we may insert under (B.31) an extra factor decaying like $\langle\zeta\rangle^{-N_1}$ as well as its derivatives, for a given N_1.

We perform next N_2 integrations by parts using the operator

$$\left\langle z(\langle\zeta\rangle+M_0(\xi))^{-\kappa}\right\rangle^{-2}\left(1-(\langle\zeta\rangle+M_0(\xi))^{-2\kappa}zD_\zeta\right). \tag{B.33}$$

By estimates (B.28) and (B.29), each of these integrations by parts makes gain a factor $\langle z(\langle\zeta\rangle+M_0(\xi))^{-\kappa}\rangle^{-1}$. Using (B.29), (B.13), the definition (B.19) of M and (B.18), we bound the modulus of (B.31) by

$$\begin{aligned}
&CM(x,\xi)\int\langle\zeta\rangle^{-N_1+N_0}\left\langle z(\langle\zeta\rangle+M_0(\xi))^{-\kappa}\right\rangle^{-N_2}(\langle\zeta\rangle+M_0(\xi))^\kappa\\
&\quad\times\langle hz\rangle^{N_0''}\left(1+M_0(\xi)^{-\kappa}|y-z|\right)^{-N}\\
&\quad\times\int_0^1\left(1+\beta h^\beta M_0(\xi',\xi_{n'}+\cdots+\xi_n-\lambda\zeta)\right)^{-N}d\lambda\\
&\quad\times(1+\beta h^\beta M_0(\xi''))^{-N}\,dz\,d\zeta
\end{aligned} \tag{B.34}$$

for arbitrary N_1, N_2, N and given N_0, N_0'' (coming from (B.9) and (B.18)), the factor in $(1+M_0(\xi)^{-\kappa}|y-z|)^{-N}$ coming from the last factor in (B.13) of $\frac{\partial b}{\partial y}$. If N_1-N_0 is large enough, and if we integrate for $|\zeta|\ge cM_0(\xi)$, then the factor $\langle\zeta\rangle^{-N_1+N_0}$

provides a decay in $M_0(\xi)^{-N'}$ for any given N'. On the other hand, if we integrate for $|\zeta| \leq cM_0(\xi)$, we may use (B.24) that shows that the product of the last two factors in (B.34) is smaller than $C(1+\beta h^\beta M_0(\xi))^{-N}$. We thus get a bound in

$$\begin{aligned} &CM(x,\xi)(1+\beta h^\beta M_0(\xi))^{-N} \\ &\qquad \times \int \langle\zeta\rangle^{-N_1+N_0+N}\big\langle z(\langle\zeta\rangle + M_0(\xi))^{-\kappa}\big\rangle^{-N_2}(\langle\zeta\rangle + M_0(\xi))^{\kappa} \\ &\qquad \times \langle hz\rangle^{N_0''}\big(1+M_0(\xi)^{-\kappa}|y-z|\big)^{-N}\,dz\,d\zeta \\ &\quad \leq CM(x,\xi)\big(1+\beta h^\beta M_0(\xi)\big)^{-N} M_0(\xi)^{(2+N_0'')\kappa}\big(1+M_0(\xi)^{-\kappa}|y|\big)^{-N} \end{aligned} \tag{B.35}$$

if $N_1 \gg N_2 \gg N + N_0 + N_0''$. We thus get an estimate of the form (B.13), with $\alpha_0 = 0$, $\alpha = 0$, and the order function M replaced by $M(x,\xi)M_0(\xi)^{\kappa(2+N_0'')}$.

If we make the same computation after taking a $\partial_x^{\alpha_0}$ and a ∂_ξ^α-derivative of (B.31), we replace, according to estimate (B.29), the factor $(M_0(\xi)+\langle\zeta\rangle)^\kappa$ in (B.34) by $(M_0(\xi)+\langle\zeta\rangle)^{\kappa(1+\alpha_0+|\alpha|)}$, so that we obtain again a bound of the form (B.13), with still M replaced by $M(x,\xi)M_0(\xi)^{\nu\kappa}$ with $\nu = 2+N_0''$.

Study of (B.32). The difference with the preceding case is that the ∂_x-derivative acting on b makes lose an extra factor $M_0(\xi)^\kappa$, and that we do not have in (B.34) the factor in $(1+M_0(\xi)^{-\kappa}|y-z|)^{-N}$. Instead of (B.35), we thus get a bound in

$$CM(x,\xi)M_0(\xi)^{\nu\kappa}\big(1+\beta h^\beta M_0(\xi)\big)^{-N}$$

for some ν depending only on N_0''. On the other hand, if one takes a ∂_y-derivative of (B.32), either it falls on b, which reduces one to an expression of the form (B.31), or on $\tilde{a}$, so that one gains a factor (B.30) in the estimates. In both cases, it shows that a bound of form (B.13) holds. One studies in the same way the derivatives, and shows that (B.32) provides the hc_1 contribution in (B.22).

If b does not depend on y, then (B.31) vanishes identically so that there is no c_1' contribution in (B.33). If it is independent of x, the term hc_1 given by (B.32) vanishes.

Finally, if one assumes that b is in $S'_{\kappa,\beta}(M'',n'')$, then estimates of the form (B.35), i.e. with the factor $(1+M_0(\xi)^{-\kappa}|y-z|)^{-N}$, hold also for the study of term (B.32), so that we get that c_1 in (B.22) is also in $S'_{\kappa,\beta}(MM_0^\nu,n)$. In the same way, if a is in $S'_{\kappa,\beta}(M',n')$, one gets in (B.29) an extra factor of the form (B.30) on the right-hand side, so that (B.32) is again in $S'_{\kappa,\beta}(M,n)$. This concludes the proof. ■

Let us write a special case of Proposition B.2.1.

Corollary B.2.3. *Let $p(\xi) = \langle\xi\rangle$ and let $b(y,\xi_1,\dots,\xi_n)$ be a function satisfying estimates*

$$\begin{aligned} |\partial_\xi^\alpha b(y,\xi)| &\leq C\prod_{j=1}^n \langle\xi_j\rangle^{-1} M_0(\xi)^{1+|\alpha|}, \\ |\partial_y^{\alpha_0'}\partial_\xi^\alpha b(y,\xi)| &\leq C_N\prod_{j=1}^n \langle\xi_j\rangle^{-1} M_0(\xi)^{1+|\alpha|}\langle y\rangle^{-N} \end{aligned} \tag{B.36}$$

for all $\alpha'_0 \in \mathbb{N}^*$, $\alpha \in \mathbb{N}^n$, $N \in \mathbb{N}$. *Then*

$$\begin{aligned}&\mathrm{Op}_h(p(\xi))\big[\mathrm{Op}_h(b)(v_1,\dots,v_n)\big]\\&\quad = \mathrm{Op}_h\big(p(\xi)b(y,\xi)\big)(v_1,\dots,v_n) + \mathrm{Op}_h(c'_1)(v_1,\dots,v_n),\end{aligned} \tag{B.37}$$

where c'_1 *satisfies*

$$|\partial_y^{\alpha'_0}\partial_\xi^\alpha c'_1(y,\xi)| \le C_N \prod_{j=1}^n \langle \xi_j\rangle^{-1} M_0(\xi)^{1+|\alpha|}\langle y\rangle^{-N} \tag{B.38}$$

for all α'_0, α, N.

Proof. We may not directly apply the proposition, as the order function it would provide on the right-hand side of (B.38) would not be the right one. Though, we may apply its proof that shows that the composed operator (B.37) is given by (B.31) with $\tilde{a}$ given by (B.27), i.e.

$$-\frac{i}{2\pi}\int_0^1\int e^{-iz\zeta} p'(\xi_1+\dots+\xi_n-\lambda\zeta)\frac{\partial b}{\partial y}(y-z,\xi_1,\dots,\xi_n)\,dz\,d\zeta\,d\lambda. \tag{B.39}$$

Performing integrations by parts in z, ζ, we may bound the modulus of (B.39) by

$$C\int \langle z\rangle^{-N}\langle\zeta\rangle^{-N}\langle y-z\rangle^{-N}\,dz\,d\zeta \prod_{j=1}^n \langle\xi_j\rangle^{-1}M_0(\xi)$$

which gives (B.38) performing the same computations for the derivatives. ∎

We shall use also the following corollary.

Corollary B.2.4. *Let* b *be a symbol in* $S_{\kappa,\beta}(M,n)$ *for some order function* M*, some* n *in* $\mathbb{N}^*$*, with* (κ,β) *satisfying the assumptions of Proposition* B.2.1*. Assume moreover that* $b(y,x,\xi_1,\dots,\xi_n)$ *is supported inside* $|\xi_1|+\dots+|\xi_{n-1}| \le C\langle\xi_n\rangle$*. There is* $\nu \ge 0$ *such that for any* $s \ge 0$*, one may write*

$$\langle hD\rangle^s \mathrm{Op}_h\big(b\langle\xi_n\rangle^{-s}\big) = \mathrm{Op}_h(c) \tag{B.40}$$

with a symbol c *in* $S_{\kappa,\beta}(MM_0^\nu, n)$*. The result holds also if* b *(and then* c*) satisfy* (B.13) *with the last exponent* N *replaced by 2, i.e. if* b *is in* $S'^2_{\kappa,\beta}(M,n)$*, then* c *lies in* $S'^2_{\kappa,\beta}(MM_0^\nu,n)$*.*

Proof. We apply Proposition B.2.1 with $a(\xi) = \langle\xi\rangle^s \in S_{\kappa,\beta}(\langle\xi\rangle^s,1)$ (for any (κ,β)) and for second symbol $b(y,x,\xi_1,\dots,\xi_n)\langle\xi_n\rangle^{-s}$. Notice that, because of the support assumption on b, this symbol belongs to the class $S_{\kappa,\beta}(M(x,\xi)(\sum_{j=1}^n\langle\xi_j\rangle)^{-s},n)$. Then by (B.20), c in (B.40) belongs to $S_{\kappa,\beta}(\tilde{M}(x,\xi)M_0^{\nu\kappa},n)$, where ν depends only on the exponent N''_0 in (B.18), which is independent of s, and where $\tilde{M}$ is given, according to (B.19), by

$$\tilde{M}(x,\xi_1,\dots,\xi_n) = \langle\xi_1+\dots+\xi_n\rangle^s M(x,\xi)\Big(\sum_{j=1}^n\langle\xi_j\rangle\Big)^{-s} \le CM(x,\xi).$$

The conclusion follows, as the last statement of the corollary comes from the fact that when taking a ∂_y-derivative of c given by (B.25), it falls on the b factor as $a(\xi) = \langle \xi \rangle^s$ and makes appear a gain $(1 + M_0(\xi)^{-\kappa}|y - z|)^{-2}$ if we assume that (B.13) holds with last exponent equal to 2. ∎

Let us state a result on the adjoint. Since we shall need it only for linear operators, we limit ourselves to that case.

Proposition B.2.5. *Let $M(x, \xi)$ be an order function on $\mathbb{R} \times \mathbb{R}$ and let a be an element of $S_{0,0}(M, 1)$. Define*

$$a^*(y, x, \xi) = \frac{1}{2\pi} \int e^{-iz\zeta} \bar{a}(y - z, x - hz, \xi - \zeta)\, dz\, d\zeta. \tag{B.41}$$

Then a^* belongs to $S_{0,0}(M, 1)$ and $(\mathrm{Op}_h(a))^* = \mathrm{Op}_h(a^*)$.

Proof. By a direct computation $(\mathrm{Op}_h(a))^*$ is given by $\mathrm{Op}_h(a^*)$ if a^* is defined by (B.41). Making ∂_z and ∂_ζ integrations by parts, one checks that a^* belongs to the wanted class. ∎

Remark. It follows from (B.25), (B.31), (B.32), that if a, b in the statement of Proposition B.2.1 satisfy

$$\begin{aligned} a(-y, -x, -\xi_1, \dots, -\xi_{n'}) &= (-1)^{n'-1} a(y, x, \xi_1, \dots, \xi_{n'}),\\ b(-y, -x, -\xi_1, \dots, -\xi_{n''}) &= (-1)^{n''-1} b(y, x, \xi_1, \dots, \xi_{n''}), \end{aligned} \tag{B.42}$$

then symbol c in (B.22) satisfies

$$c(-y, -x, -\xi_1, \dots, -\xi_n) = (-1)^{n-1} a(y, x, \xi_1, \dots, \xi_n) \tag{B.43}$$

and a similar statement for c_1, c_1'. One has an analogous property for a^*.

To conclude this appendix, let us translate Propositions B.2.1 and B.2.5 in the framework of the non-semiclassical quantization introduced in (B.16) and (B.17).

Corollary B.2.6. *The following statements hold.*

(i) *Let n', n'' be in $\mathbb{N}^*$, $n = n' + n'' - 1$, M', M'' two order functions on $\mathbb{R} \times \mathbb{R}^{n'}$ and $\mathbb{R} \times \mathbb{R}^{n''}$, respectively. Let (κ, β) be in $\mathbb{N} \times [0, 1]$, a in $S_{\kappa,\beta}(M', n')$, b in $S_{\kappa,\beta}(M'', n'')$. Assume that either $(\kappa, \beta) = (0, 0)$ or $0 < \kappa\beta \le 1$ or that b is independent of x. Then if M is defined in (B.19), there are ν in $\mathbb{N}$, symbols c_1 in $S_{\kappa,\beta}(MM_0^{\nu\kappa}, n)$, c_1' in $S'_{\kappa,\beta}(MM_0^{\nu\kappa}, n)$ such that if*

$$\begin{aligned} c(y, x, \xi_1, \dots, \xi_n) = a(y, x, \xi_1, &\dots, \xi_{n'-1}, \xi_{n'} + \dots + \xi_n)\\ &\times b(y, x, \xi_{n'}, \dots, \xi_n)\\ &+ t^{-1} c_1(y, x, \xi_1, \dots, \xi_n)\\ &+ c_1'(y, x, \xi_1, \dots, \xi_n), \end{aligned} \tag{B.44}$$

then for any functions $v_1, \dots, v_n$,

$$\mathrm{Op}^t(a)[v_1, \dots, v_{n'-1}, \mathrm{Op}^t(b)(v_{n'}, \dots, v_n)] = \mathrm{Op}^t(c)[v_1, \dots, v_n]. \quad \text{(B.45)}$$

Moreover, if b *is independent of* x, *then* c_1 *vanishes in* (B.44). *Finally, if* a *is in* $S'_{\kappa,\beta}(M', n')$ *or* b *is in* $S'_{\kappa,\beta}(M'', n'')$, *then* c *is in* $S'_{\kappa,\beta}(MM_0^{\nu\kappa}, n)$.

(ii) *In the same way, if* a *is in* $S_{0,0}(M, 1)$, *then* $\mathrm{Op}^t(a)^* = \mathrm{Op}^t(a^*)$, *for a symbol* a^* *in the same class. Moreover, if* a *satisfies* (B.42), *so does* a^*.

Proof. Statement (i) is just the translation of Proposition B.2.1. Statement (ii) follows from Proposition B.2.5. ■

We get also translating Corollary B.2.3:

Corollary B.2.7. *Under the assumptions and notation of Corollary* B.2.3, *one has*

$$\begin{aligned} &\mathrm{Op}\big(p(\xi)\big)\mathrm{Op}(b)(v_1, \dots, v_n) \\ &\quad = \mathrm{Op}(p(\xi_1 + \dots + \xi_n)b)(v_1, \dots, v_n) + \mathrm{Op}(c'_1)(v_1, \dots, v_n) \end{aligned}$$

with c'_1 *in the class* $\tilde{S}'_{1,0}(\prod_{j=1}^n \langle \xi_j \rangle^{-1} M_0(\xi), n)$ *of Definition* 3.1.1.

We shall use also:

Corollary B.2.8. *Let* $n \geq 2$. *Let* $M(\xi_1, \dots, \xi_n)$ *be an order function on* $\mathbb{R}^n$ *(independent of* x*) and let* $a(y, \xi_1, \dots, \xi_n)$ *be a symbol in* $S_{\kappa,0}(M, n)$, *independent of* x, *for some* κ *in* $\mathbb{N}$. *Let* Z *be a function in* $\mathcal{S}(\mathbb{R})$. *Denote*

$$\tilde{M}(\xi_1, \dots, \xi_{n-1}) = M(\xi_1, \dots, \xi_{n-1}, 0).$$

There is a symbol a' *in* $S'_{\kappa,0}(\tilde{M}, n-1)$, *independent of* x, *such that for any test functions* $v_1, \dots, v_{n-1}$,

$$\mathrm{Op}(a)[v_1, \dots, v_{n-1}, Z] = \mathrm{Op}(a')[v_1, \dots, v_{n-1}]. \quad \text{(B.46)}$$

Moreover, if Z *is odd and* $a(-y, -\xi_1, \dots, -\xi_n) = (-1)^{n-1} a(y, \xi_1, \dots, \xi_n)$, *then*

$$a'(-y, -\xi_1, \dots, -\xi_{n-1}) = (-1)^{n-2} a(y, \xi_1, \dots, \xi_{n-1}).$$

Proof. By (B.17), we have that (B.46) holds if we define

$$a'(y, \xi_1, \dots, \xi_{n-1}) = \frac{1}{2\pi} \int e^{iy\xi_n} a(y, \xi_1, \dots, \xi_{n-1}, \xi_n) \hat{Z}(\xi_n)\, d\xi_n. \quad \text{(B.47)}$$

If $\alpha' = (\alpha_1, \dots, \alpha_{n-1}) \in \mathbb{N}^{n-1}$ and $\xi' = (\xi_1, \dots, \xi_{n-1})$, we deduce from (B.12) with $\beta = 0$ that

$$|\partial_{\xi'}^{\alpha'} a'(y, \xi_1, \dots, \xi_{n-1})| \leq C \int M(\xi', \xi_n) M_0(\xi', \xi_n)^{\kappa|\alpha'|} |\hat{Z}(\xi_n)|\, d\xi_n.$$

Using (B.9) both for M and M_0, we obtain a bound in $\tilde{M}(\xi')M_0(\xi')^{\kappa|\alpha'|}$. To check that actually our symbol a' is in $S'_{\kappa,0}(\tilde{M}, n-1)$, i.e. that it is rapidly decaying in $(1 + M_0(\xi')^{-\kappa}|y|)^{-N}$, we just make in (B.47) ∂_{ξ_n}-integrations by parts, and perform the same estimate. One bounds ∂_y-derivatives in the same way. Finally, the last statement of the corollary follows from (B.47) and the oddness of $\hat{Z}$. ■