Appendix C

Bounds for forced linear Klein–Gordon equations

The goal of this appendix is to obtain some Sobolev or L^∞ estimates of solutions of half-Klein–Gordon equations with zero initial data and force term that is time oscillating. The kind of equations we want to study is of the form

$$\begin{aligned} \big(D_t - \sqrt{1 + D_x^2}\big)U &= e^{i\lambda t} t_\varepsilon^{-1} M(x), \\ U|_{t=1} &= 0, \end{aligned} \tag{C.1}$$

where M is in $\mathcal{S}(\mathbb{R})$, $t_\varepsilon^{-1} = \frac{\varepsilon^2}{1+t\varepsilon^2}$ and λ is a real number different from one. This restriction means that the right-hand side of the equation oscillates at a frequency which is non-characteristic when one restricts the symbol $\sqrt{1+\xi^2}$ of the operator on the left-hand side to frequency zero. Our goal is to prove estimates for U or $L_+ U = (x + t\frac{D_x}{\langle D_x\rangle})U$ for large times. Actually, we shall split the solution as $U = U' + U''$, where U' is obtained writing the Duhamel formula to express U and restricting the time integral to times that are $O(\sqrt{t})$. It turns out that, when time t stays smaller than ε^{-4+0}, $L_+ U'(t, \cdot)$ has L^2 estimates that are $o(t^{\frac{1}{4}})$, which is acceptable for our applications. On the other hand $L_+ U''$ would not enjoy such bounds, but it has good estimates in L^∞-like spaces.

Equation (C.1) is actually just a simplified model of the problem we study in this Appendix. For the applications to our main problem, i.e. the description of some approximate solutions (see Section 2.5 of Chapter 2), we need more general right-hand sides than in (C.1). Though, the method of proof of our estimates is quite the same as for the model above. It relies on the explicit writing of the solution using Duhamel formula and the stationary phase formula.

We shall close this appendix with explicit computations that are used in the main part of this text to check Fermi's golden rule.

C.1 Linear solutions to half-Klein–Gordon equations

We consider a function $(t, x) \mapsto M(t, x)$ that is C^1 in time, with values in $\mathcal{S}(\mathbb{R})$. If λ is in $\mathbb{R}$, $\lambda \neq 1$, we denote by $U(t, x)$ the solution to

$$\begin{aligned} (D_t - p(D_x))U &= e^{i\lambda t} M(t, x), \\ U|_{t=1} &= 0, \end{aligned} \tag{C.2}$$

where $p(D_x) = \sqrt{1 + D_x^2}$, and where we study the solution for t in an interval $[1, T]$. We write the solution by Duhamel formula as

$$U(t, x) = i \int_1^t e^{i(t-\tau)p(D_x) + i\lambda\tau} M(\tau, \cdot)\, d\tau. \tag{C.3}$$

We fix some function χ in $C^\infty(\mathbb{R})$, equal to one close to $]-\infty, \frac{1}{4}]$, supported in $]-\infty, \frac{1}{2}]$. Then for t larger than some constant (say $t \geq 16$), we may write (C.3) as $U = U' + U''$ where

$$\begin{aligned} U'(t,x) &= i\int_1^{+\infty} e^{i(t-\tau)p(D_x)+i\lambda\tau}\chi\Big(\frac{\tau}{\sqrt{t}}\Big)M(\tau,\cdot)\,d\tau \\ U''(t,x) &= i\int_{-\infty}^{t} e^{i(t-\tau)p(D_x)+i\lambda\tau}(1-\chi)\Big(\frac{\tau}{\sqrt{t}}\Big)M(\tau,\cdot)\,d\tau. \end{aligned} \tag{C.4}$$

Our goal is to obtain Sobolev and L^∞ estimates for U', U'' and for the result of the action on U', U'' of the operator

$$L_\pm = x \pm tp'(D_x) = x \pm t\frac{D_x}{\langle D_x\rangle}, \tag{C.5}$$

under two sets of assumptions on M, that we describe now. We shall take ε in $]0,1]$ and for $t \geq 1$, we recall that we defined in (4.1)

$$t_\varepsilon = \varepsilon^{-2}\langle t\varepsilon^2\rangle = (\varepsilon^{-4}+t^2)^{\frac{1}{2}}. \tag{C.6}$$

For ω in $[1,+\infty[$, $\theta' \in]0, \frac{1}{2}[$, close to $\frac{1}{2}$, we introduce the following:

Assumption (H1)$_\omega$. For any α, N in $\mathbb{N}$, any t in $[1,T]$, x in $\mathbb{R}$, ε in $]0,1]$, one has bounds

$$\begin{aligned} |\partial_x^\alpha M(t,x)| &\leq C_{\alpha,N} t_\varepsilon^{-\omega}\langle x\rangle^{-N}, \\ |\partial_x^\alpha \partial_t M(t,x)| &\leq C_{\alpha,N} t_\varepsilon^{-\omega+\frac{1}{2}}(t_\varepsilon^{-\frac{3}{2}} + t^{-\frac{3}{2}}(\varepsilon^2\sqrt{t})^{\frac{3}{2}\theta'})\langle x\rangle^{-N}. \end{aligned} \tag{C.7}$$

The second type of assumption we shall make on M is more technical. If $\lambda > 1$, we denote by $\pm\xi_\lambda$ the two roots of $\sqrt{1+\xi^2} = \lambda$ (with $\xi_\lambda > 0$) and set $\mathcal{W}_\lambda$ for a small open neighborhood of the set $\{\xi_\lambda, -\xi_\lambda\}$. We introduce:

Assumption (H2). For any α, N, the x-Fourier transform of $M(t,x)$ satisfies bounds

$$\begin{aligned} |\partial_\xi^\alpha \hat{M}(t,\xi)| &\leq C_{\alpha,N} t^{-\frac{1}{2}} t_\varepsilon^{-1}\langle\xi\rangle^{-N}, \\ |\partial_t\partial_\xi^\alpha \hat{M}(t,\xi)| &\leq C_{\alpha,N} t^{-\frac{3}{4}} t_\varepsilon^{-1}\langle\xi\rangle^{-N}. \end{aligned} \tag{C.8}$$

Moreover, for ξ in $\mathcal{W}_\lambda$, one may decompose

$$D_t\hat{M}(t,\xi) = (D_t + \lambda - \sqrt{1+\xi^2})\Phi(t,\xi) + \Psi(t,\xi), \tag{C.9}$$

where Φ, Ψ satisfy the following bounds:

$$\begin{aligned} |\Phi(t,\xi)| &\leq Ct^{-\frac{1}{2}}t_\varepsilon^{-1}, \\ |\Psi(t,\xi)| &\leq Ct^{-1}t_\varepsilon^{-1} \end{aligned} \tag{C.10}$$

and a similar decomposition holds for xM instead of M. Of course, conditions (C.9) and (C.10) are void if $\lambda < 1$.

For future reference, let us state some elementary inequalities that hold if $\theta' < \frac{1}{2}$ is close enough to $\frac{1}{2}$, $\varepsilon^2\sqrt{t} \leq 1$ and $\omega \geq 1$:

$$\begin{aligned} \int_1^{\sqrt{t}} \tau_\varepsilon^{-\omega+\frac{1}{2}} \big(\tau_\varepsilon^{-\frac{3}{2}} + \tau^{-\frac{3}{2}}(\varepsilon^2\sqrt{\tau})^{\frac{3}{2}\theta'}\big)\,d\tau &\leq C\varepsilon^{2\omega}(\varepsilon^2\sqrt{t} + \varepsilon^{3\theta'-1}) \\ &\leq C\varepsilon^{2\omega}, \end{aligned} \tag{C.11}$$

$$\begin{aligned} &\int_{\sqrt{t}}^{t} \tau_\varepsilon^{-\omega+\frac{1}{2}} \big(\tau_\varepsilon^{-\frac{3}{2}} + \tau^{-\frac{3}{2}}(\varepsilon^2\sqrt{\tau})^{\frac{3}{2}\theta'}\big)\,d\tau \\ &\quad\leq C\varepsilon^{2\omega}\Big(\frac{\varepsilon^2 t}{\langle\varepsilon^2 t\rangle} + \varepsilon^{\frac{3}{2}\theta'}(\varepsilon^2\sqrt{t})^{-\frac{1}{2}+\frac{3}{4}\theta'}\Big) \\ &\quad\leq C\min\Big(\varepsilon^{2\omega-1}\Big(\frac{\varepsilon^2 t}{\langle\varepsilon^2 t\rangle}\Big)^{\frac{1}{2}}, \varepsilon^{2\omega}\Big), \end{aligned} \tag{C.12}$$

$$\int_1^{\sqrt{t}} \tau^a \tau_\varepsilon^{-\omega}\,d\tau \leq C\varepsilon^{2\omega} t^{\frac{1}{2}+\frac{a}{2}}, \quad a > -1, \tag{C.13}$$

$$\int_{\sqrt{t}}^{t} \tau^{-a}\tau_\varepsilon^{-1}\,d\tau \leq C\varepsilon^{2a}\Big(\frac{\varepsilon^2 t}{\langle\varepsilon^2 t\rangle}\Big)^{1-a} \leq C\varepsilon\Big(\frac{\varepsilon^2 t}{\langle\varepsilon^2 t\rangle}\Big)^{\frac{1}{2}}, \quad \frac{1}{2} \leq a < 1, \tag{C.14}$$

$$\begin{aligned} &\int_{\sqrt{t}}^{t} \tau_\varepsilon^{-\omega+\frac{1}{2}} \big(\tau_\varepsilon^{-\frac{3}{2}} + \tau^{-\frac{3}{2}}(\varepsilon^2\sqrt{\tau})^{\frac{3}{2}\theta'}\big)\sqrt{\tau}\,d\tau \\ &\quad\leq C\varepsilon^{2\omega-1}\Big(\Big(\frac{\varepsilon^2 t}{\langle\varepsilon^2 t\rangle}\Big)^{\frac{3}{2}} + \varepsilon^{\frac{3}{2}\theta'}\Big(\frac{\varepsilon^2 t}{\langle\varepsilon^2 t\rangle}\Big)^{\frac{3}{4}\theta'}\Big) \\ &\quad\leq C\varepsilon^{2\omega-1}\Big(\frac{\varepsilon^2 t}{\langle\varepsilon^2 t\rangle}\Big)^{\frac{1}{2}}, \end{aligned} \tag{C.15}$$

$$\int_{\sqrt{t}}^{t} \tau^{\frac{1}{2}}\tau_\varepsilon^{-1}\,d\tau \leq C\sqrt{t}\,\frac{\varepsilon^2 t}{\langle\varepsilon^2 t\rangle}. \tag{C.16}$$

Let us state two propositions giving the bounds we shall get for U', U'' under either Assumption (H1)$_\omega$ or Assumption (H2). We denote below

$$\|v\|_{W^{\rho,\infty}} = \|\langle D_x\rangle^\rho v\|_{L^\infty} \tag{C.17}$$

for any $\rho \geq 0$.

Proposition C.1.1. *The following statements hold.*

(i) *Assume that* (H1)$_\omega$ *holds for some $\omega \geq 1$. Then for any $r \geq 0$, there is $C_r > 0$ such that U' given by* (C.4) *satisfies for any $\varepsilon \in]0,1]$, $t \in [1, \varepsilon^{-4}]$,*

$$\|U'(t,\cdot)\|_{H^r} \leq C_r\varepsilon(\varepsilon^{2(\omega-1)}(\varepsilon^2\sqrt{t})^{\frac{1}{2}}), \tag{C.18}$$

$$\|U'(t,\cdot)\|_{W^{r,\infty}} \leq C_r\varepsilon^{2\omega}, \tag{C.19}$$

$$\|L_+U'(t,\cdot)\|_{H^r} \leq C_r t^{\frac{1}{4}}(\varepsilon^{2(\omega-1)}(\varepsilon^2\sqrt{t})). \tag{C.20}$$

(ii) *Under Assumption* (H2), *there is, for any* $r \geq 1$, *a constant* $C_r > 0$ *such that* U' *satisfies for any* $\varepsilon \in]0,1]$, $t \in [1, \varepsilon^{-4}]$,

$$\|U'(t,\cdot)\|_{H^r} \leq C_r \varepsilon (\varepsilon^2 \sqrt{t})^{\frac{1}{2}}, \tag{C.21}$$

$$\|U'(t,\cdot)\|_{W^{r,\infty}} \leq C_r \varepsilon^2 t^{-\frac{1}{4}}, \tag{C.22}$$

$$\|L_+ U'(t,\cdot)\|_{H^r} \leq C_r t^{\frac{1}{4}} (\varepsilon^{\frac{1}{8}} (\varepsilon^2 \sqrt{t})^{\frac{7}{8}}). \tag{C.23}$$

Let us state now the bounds we shall prove for U''.

Proposition C.1.2. *The following statements hold.*

(i) *Under Assumption* $(H1)_\omega$ *with* $\omega \geq 1$, *one has for any* $r \geq 0$, *the following bounds:*

$$\|U''(t,\cdot)\|_{H^r} \leq C_r \varepsilon^{2\omega-1} \Big(\frac{\varepsilon^2 t}{\langle \varepsilon^2 t\rangle}\Big)^{\frac{1}{2}}, \tag{C.24}$$

$$\|U''(t,\cdot)\|_{W^{r,\infty}} \leq C_r \varepsilon^{2\omega} \log(1+t), \tag{C.25}$$

$$\|L_+ U''(t,\cdot)\|_{W^{r,\infty}} \leq C_r \log(1+t) \log(1+\varepsilon^2 t) \qquad \text{if } \omega = 1, \tag{C.26}$$

$$\|L_+ U''(t,\cdot)\|_{W^{r,\infty}} \leq C_r \varepsilon^{2(\omega-1)} \log(1+t) \Big(\frac{\varepsilon^2 t}{\langle \varepsilon^2 t\rangle}\Big) \qquad \text{if } \omega > 1. \tag{C.27}$$

(ii) *Under Assumption* (H2), *one has for any* $r \geq 0$, *the following bounds:*

$$\|U''(t,\cdot)\|_{H^r} \leq C_r \varepsilon \Big(\frac{\varepsilon^2 t}{\langle \varepsilon^2 t\rangle}\Big)^{\frac{1}{2}}, \tag{C.28}$$

$$\|U''(t,\cdot)\|_{W^{r,\infty}} \leq C_r \varepsilon^2 (\log(1+t))^2, \tag{C.29}$$

$$\|L_+ U''(t,\cdot)\|_{W^{r,\infty}} \leq C_r \log(1+t) \log(1+\varepsilon^2 t). \tag{C.30}$$

Remark. Notice that we obtain Sobolev estimates for $L_+ U'(t,\cdot)$ in (C.20), (C.23), while we bound $L_+ U''(t,\cdot)$ in $W^{r,\infty}$ spaces in (C.26), (C.27), (C.30). Actually, we could not obtain for the $L_+ U''$ contribution to $L_+ U$ as good Sobolev estimates as those that hold for $L_+ U'$, and this is the reason for our splitting $U = U' + U''$.

Study of the U' contribution. We shall prove Proposition C.1.1. By (C.4) and (C.5)

$$U'(t,x) = \frac{i}{2\pi} \int_1^{+\infty} \int e^{i((t-\tau)\sqrt{1+\xi^2} + \lambda\tau + x\xi]} \chi\Big(\frac{\tau}{\sqrt{t}}\Big) \hat{M}(\tau,\xi)\, d\xi\, d\tau \tag{C.31}$$

and

$$\begin{aligned} L_+ U'(t,x) = \frac{i}{2\pi} \int_1^{+\infty} \int & e^{i((t-\tau)\sqrt{1+\xi^2} + \lambda\tau + x\xi)} \chi\Big(\frac{\tau}{\sqrt{t}}\Big) \\ & \times \Big(\tau \frac{\xi}{\langle \xi\rangle} \hat{M}(\tau,\xi) + \widehat{xM}(\tau,\xi)\Big)\, d\xi\, d\tau. \end{aligned} \tag{C.32}$$

We shall estimate first the above integrals when either $\lambda < 1$, so that the coefficient of τ in the phase $\lambda - \sqrt{1+\xi^2}$ never vanishes, or when $\lambda > 1$ but $\hat{M}(\tau,\xi)$ is supported outside a neighborhood of the two roots $\pm\xi_\lambda$ of that expression.

Lemma C.1.3. *Assume that either $\lambda < 1$ or $\lambda > 1$ and there is a neighborhood $\mathcal{W}_\lambda$ of $\{-\xi_\lambda, \xi_\lambda\}$ such that $\hat{M}(\cdot, \xi)$ vanishes for ξ in $\mathcal{W}_\lambda$. Assume also $t \leq \varepsilon^{-4}$.*

(i) *Under Assumption (H1)$_\omega$, estimates (C.18)–(C.20) hold true.*

(ii) *Under Assumption (H2), estimates (C.21)–(C.23) hold true.*

Proof. Let us prove first the Sobolev bounds (C.18), (C.20), (C.21) and (C.23). By (C.31), $\hat{U}'(t, \xi)$ may be written as

$$e^{it\sqrt{1+\xi^2}} \int_1^{+\infty} e^{i(\lambda - \sqrt{1+\xi^2})\tau} \chi\Big(\frac{\tau}{\sqrt{t}}\Big) N(\tau, \xi)\, d\tau, \tag{C.33}$$

where $N(\tau, \xi)$ satisfies for any N, any α, according to (C.7) and (C.8),

$$|\partial_\xi^\alpha \partial_\tau^j N(\tau, \xi)| \leq C_{\alpha,N} \tau_\varepsilon^{-\omega + \frac{j}{2}} \big(\tau_\varepsilon^{-\frac{3}{2}} + \tau^{-\frac{3}{2}} (\varepsilon^2 \sqrt{\tau})^{\frac{3}{2}\theta'}\big)^j \langle \xi \rangle^{-N}, \quad j = 0, 1, \tag{C.34}$$

under Assumption (H1)$_\omega$ and

$$|\partial_\xi^\alpha \partial_\tau^j N(\tau, \xi)| \leq C_{\alpha,N} \tau^{-\frac{1}{2}} \tau_\varepsilon^{-1} \tau^{-\frac{j}{4}} \langle \xi \rangle^{-N}, \quad j = 0, 1, \tag{C.35}$$

under Assumption (H2). In the same way, by (C.32), $\widehat{L_+ U'}(t, \xi)$ may be written under the form (C.33), where N satisfies, according to (C.7) and (C.8),

$$|\partial_\xi^\alpha \partial_\tau^j N(\tau, \xi)| \leq C_{\alpha,N} \tau^{1-j} \tau_\varepsilon^{-\omega} \langle \xi \rangle^{-N}, \quad j = 0, 1, \tag{C.36}$$

under Assumption (H1)$_\omega$ and

$$|\partial_\xi^\alpha \partial_\tau^j N(\tau, \xi)| \leq C_{\alpha,N} \tau^{\frac{1}{2} - \frac{j}{4}} \tau_\varepsilon^{-1} \langle \xi \rangle^{-N}, \quad j = 0, 1, \tag{C.37}$$

under Assumption (H2).

Since $N(\tau, \xi)$ is supported outside a neighborhood of the zeros of $\sqrt{1+\xi^2} - \lambda$, we may perform in integral (C.33) one ∂_τ-integration by parts. Taking moreover an $L^2(\langle \xi \rangle^r d\xi)$ norm, we obtain quantities bounded in the following way:

- If N satisfies (C.34), we obtain a control of (C.33) in terms of $C\varepsilon^{2\omega}$ and of (C.11). This gives an $\varepsilon^{2\omega}$ estimate, better than the right-hand side (C.18).
- If N satisfies (C.35), we obtain an upper bound by the right-hand side of (C.13), which is better than (C.21).
- If N satisfies (C.36), the $L^2(\langle \xi \rangle^r d\xi)$ norm of (C.33) is bounded by (C.13) with $a = 0$, so by (C.20).
- If N satisfies (C.37), that same norm is bounded by (C.13), thus by the right-hand side of (C.23).

We have thus proved Lemma C.1.3 for Sobolev estimates. It remains to establish (C.19) and (C.22). Since $\hat{M}$ is rapidly decaying in ξ, it is sufficient to estimate the L^∞ norm of U'. Notice that the $d\xi$-integral in (C.31) may be written as

$$\int e^{it\left(\left(1-\frac{\tau}{t}\right)\sqrt{1+\xi^2} + \frac{x}{t}\xi\right)} \hat{M}(\tau, \xi)\, d\xi \tag{C.38}$$

and that on the support of $\chi(\tau/\sqrt{t})$, $|\tau/t| \ll 1$, so that the stationary phase formula implies that (C.38) is smaller in modulus than $Ct^{-\frac{1}{2}}\tau_\varepsilon^{-\omega}\mathbb{1}_{\tau<\sqrt{t}}$ under conditions (C.7) and $Ct^{-\frac{1}{2}}\tau^{-\frac{1}{2}}\tau_\varepsilon^{-1}\mathbb{1}_{\tau<\sqrt{t}}$ under condition (C.8). Integrating in τ, we get bounds in $O(\varepsilon^{2\omega})$ and $O(\varepsilon^2 t^{-\frac{1}{4}})$, respectively, as in (C.19) and (C.22). This concludes the proof. ∎

Lemma C.1.3 provides Proposition C.1.1 when either $\lambda < 1$ or $\lambda > 1$ and $\hat{M}$ in (C.31) and (C.32) is cut-off outside a neighborhood of $\sqrt{1+\xi^2} = \lambda$. We have thus to study now the case when $\lambda > 1$ and $\hat{M}$ is supported in a small neighborhood of one of the roots $\pm\xi_\lambda$ of that equation. More precisely, we have to study, in order to estimate the contribution to U', the expressions

$$\tilde{U}'_\pm(t,x) = \int_1^{+\infty}\int e^{it\left(\left(1-\frac{\tau}{t}\right)\sqrt{1+\xi^2}+\lambda\frac{\tau}{t}+\frac{x}{t}\xi\right)}\chi\Big(\frac{\tau}{\sqrt{t}}\Big)N_\pm(\tau,\xi)\,d\tau\,d\xi, \tag{C.39}$$

where $N_\pm$ is supported close to $\pm\xi_\lambda$ and satisfies (C.34) or (C.35), and, in order to estimate the contribution to L_+U', an expression of the form (C.39) with $N_\pm$ satisfying (C.36) or (C.37). We shall show actually the more precise result:

Proposition C.1.4. *For any α in $\mathbb{N}$, we have the following bounds:*

$$|\partial_x^\alpha \tilde{U}'_\pm(t,x)| \le C_\alpha \varepsilon^{2\omega}\langle t^{-\frac{1}{2}}(\lambda x \pm t\xi_\lambda)\rangle^{-1} \tag{C.40}$$

if $N_\pm$ satisfies (C.34),

$$|\partial_x^\alpha \tilde{U}'_\pm(t,x)| \le C_\alpha \varepsilon^{2} t^{-\frac{1}{4}}\langle t^{-\frac{7}{8}}(\lambda x \pm t\xi_\lambda)\rangle^{-1} \tag{C.41}$$

if $N_\pm$ satisfies (C.35),

$$|\partial_x^\alpha \tilde{U}'_\pm(t,x)| \le C_\alpha \varepsilon^{2\omega} t^{\frac{1}{2}}\langle t^{-\frac{1}{2}}(\lambda x \pm t\xi_\lambda)\rangle^{-1} \tag{C.42}$$

if $N_\pm$ satisfies (C.36), *and*

$$|\partial_x^\alpha \tilde{U}'_\pm(t,x)| \le C_\alpha \varepsilon^{2} t^{\frac{1}{4}}\langle t^{-\frac{7}{8}}(\lambda x \pm t\xi_\lambda)\rangle^{-1} \tag{C.43}$$

if $N_\pm$ satisfies (C.37).

It follows immediately from (C.40) (resp. (C.41)) that (C.18) and (C.19) (resp. (C.21) and (C.22)) hold true. In the same way, computing the L^2 norms of (C.42) (resp. (C.43)) we obtain upper bounds by (C.20) (resp. (C.23)). Consequently, Proposition C.1.1 will be proved if we establish Proposition C.1.4.

Lemma C.1.5. *One may write the derivatives of $\tilde{U}'_\pm$ given by* (C.39) *under the form*

$$\partial_x^\alpha \tilde{U}'_\pm(t,x) = \int_1^{+\infty} e^{i\psi_\pm(\tau,t,z_\pm)}\tilde{\chi}_\pm(t,\tau,z_\pm)J_\alpha(\tau,t,z_\pm)\,d\tau + R_\alpha^\pm, \tag{C.44}$$

where $\tilde{\chi}_\pm$ is supported for $\tau \le \sqrt{t}$ and for $|z_\pm| \le c$, and where

$$z_\pm = \frac{x}{t} \pm \frac{\xi_\lambda}{\lambda}, \quad \tilde{\chi}_\pm = O(1), \quad \partial_\tau\tilde{\chi}_\pm = O(t^{-\frac{1}{2}}), \tag{C.45}$$

where $\psi_\pm(\tau, t, z_\pm)$ *satisfies*

$$|\partial_\tau \psi_\pm(\tau, t, z_\pm)| \sim |z_\pm|, \quad \partial_\tau^2 \psi_\pm = 0 \tag{C.46}$$

on the support of the integrand, if t is large enough, and where J_α *satisfies the bounds*

$$\begin{aligned} |J_\alpha(\tau, t, z_\pm)| &\leq C_\alpha t^{-\frac{1}{2}} \tau_\varepsilon^{-\omega}, \\ |\partial_\tau J_\alpha(\tau, t, z_\pm)| &\leq C_\alpha t^{-\frac{1}{2}} \tau_\varepsilon^{-\omega+\frac{1}{2}} \big(\tau_\varepsilon^{-\frac{3}{2}} + t^{-1}\tau_\varepsilon^{-\frac{1}{2}} + \tau^{-\frac{3}{2}}(\varepsilon^2\sqrt{\tau})^{\frac{3}{2}\theta'}\big) \end{aligned} \tag{C.47}$$

if $N_\pm$ *satisfies* (C.34)*, and*

$$\begin{aligned} |J_\alpha(\tau, t, z_\pm)| &\leq C_\alpha t^{-\frac{1}{2}} \tau_\varepsilon^{-1} \tau^{-\frac{1}{2}}, \\ |\partial_\tau J_\alpha(\tau, t, z_\pm)| &\leq C_\alpha t^{-\frac{1}{2}} \tau_\varepsilon^{-1} \tau^{-\frac{3}{4}} \end{aligned} \tag{C.48}$$

if $N_\pm$ *satisfies* (C.35).

In the same way, $\partial_x^\alpha \tilde{U}_\pm'$ *is given by an integral of the form* (C.44) *with* J_α *satisfying*

$$\begin{aligned} |J_\alpha(\tau, t, z_\pm)| &\leq C_\alpha t^{-\frac{1}{2}} \tau_\varepsilon^{-\omega} \tau, \\ |\partial_\tau J_\alpha(\tau, t, z_\pm)| &\leq C_\alpha t^{-\frac{1}{2}} \tau_\varepsilon^{-\omega} \end{aligned} \tag{C.49}$$

if $N_\pm$ *satisfies* (C.36)*, and*

$$\begin{aligned} |J_\alpha(\tau, t, z_\pm)| &\leq C_\alpha t^{-\frac{1}{2}} \tau_\varepsilon^{-1} \tau^{\frac{1}{2}}, \\ |\partial_\tau J_\alpha(\tau, t, z_\pm)| &\leq C_\alpha t^{-\frac{1}{2}} \tau_\varepsilon^{-1} \tau^{\frac{1}{4}} \end{aligned} \tag{C.50}$$

if $N_\pm$ *satisfies* (C.37)*. Finally, the remainder* $R_\alpha^\pm$ *in* (C.44) *satisfies*

$$\begin{aligned} |R_\alpha^\pm| &\leq C_{\alpha,N} \varepsilon^{2\omega} t^{-N} \langle \lambda x \pm t\xi_\lambda \rangle^{-N} \quad \text{under (H1)}_\omega, \\ |R_\alpha^\pm| &\leq C_{\alpha,N} \varepsilon^{2} t^{-N} \langle \lambda x \pm t\xi_\lambda \rangle^{-N} \quad \text{under (H2)}, \end{aligned} \tag{C.51}$$

for any N *in* $\mathbb{N}$.

Proof. For t bounded, estimates of the form (C.51) follow from (C.34), (C.36) and ∂_ξ-integration by parts. Assume $t \gg 1$. We treat the case of sign $+$ and set z for z_+ in (C.45). We consider the $d\xi$ integral in (C.39), expressed in terms of z instead of x. The oscillatory phase may be written as $t\phi(t, \tau, z, \xi)$ with

$$\frac{\partial \phi}{\partial \xi}(t, \tau, z, \xi) = \Big(\frac{\xi}{\sqrt{1+\xi^2}} - \frac{\xi_\lambda}{\lambda}\Big) - \frac{\tau}{t}\frac{\xi}{\sqrt{1+\xi^2}} + z. \tag{C.52}$$

Since we assume $t \gg 1$, $\frac{\tau}{t} \leq \frac{1}{\sqrt{t}} \ll 1$ in (C.52). If $|z| \geq c > 0$, under this condition on t, and for $|\xi - \xi_\lambda| \ll 1$, we see from (C.52) that $\big|\frac{\partial \phi}{\partial \xi}(t, \tau, z, \xi)\big| \sim |z|$, so that, performing ∂_ξ-integration by parts, we get again estimates of the form (C.51).

We may thus assume from now on that $t \gg 1$, $|z| \ll 1$. For $z = 0$, $\frac{\tau}{t} = 0$, (C.52) vanishes at $\xi = \xi_\lambda$, and since the ∂_ξ-derivative at this point is $\lambda^{-3} \neq 0$, we have for $t \gg 1$, $|z| \ll 1$, a unique critical point $\xi(t, \tau, z)$ close to ξ_λ. Moreover, it follows from (C.52) that

$$\frac{\partial \xi}{\partial \tau}(t,\tau,z) = O\Big(\frac{1}{t}\Big), \quad \frac{\partial^2 \xi}{\partial \tau^2}(t,\tau,z) = O\Big(\frac{1}{t^2}\Big). \tag{C.53}$$

We rewrite the phase ϕ as

$$\phi(t,\tau,z,\xi) = \phi^c(t,\tau,z) + \frac{1}{2}A(t,\tau,z,\xi)^2(\xi - \xi(t,\tau,z))^2, \tag{C.54}$$

where the critical value $\phi^c(t,\tau,z)$ satisfies

$$|\partial_\tau \phi^c(t,\tau,z)| = O(t^{-1}), \quad |\partial_\tau^2 \phi^c(t,\tau,z)| = O(t^{-2}) \tag{C.55}$$

and where A is strictly positive for $\frac{\tau}{t} \ll 1$, $|z| \ll 1$, $|\xi - \xi_\lambda| \ll 1$ and satisfies for any γ,

$$|\partial_\tau \partial_\xi^\gamma A(t,\tau,z,\xi)| = O(t^{-1}). \tag{C.56}$$

We introduce the change of variables $\zeta = A(t,\tau,z,\xi)(\xi - \xi(t,\tau,z))$ for ξ close to ξ_λ and its inverse $\xi = \Xi(t,\tau,z,\zeta)$. By (C.53) and (C.56), we have

$$\frac{\partial \zeta}{\partial \tau} = O(t^{-1}), \quad \frac{\partial^{\gamma+1}\Xi}{\partial \zeta^\gamma \partial \tau} = O(t^{-1}) \tag{C.57}$$

for any γ. Then the expression of $\partial_x^\alpha \tilde{U}'_+$ may be written from (C.39)

$$\partial_x^\alpha \tilde{U}'_+(t,x) = \int_1^{+\infty} e^{it\phi^c(t,\tau,z)} \chi\Big(\frac{\tau}{\sqrt{t}}\Big) J_\alpha(t,\tau,z)\, d\tau, \tag{C.58}$$

where

$$J_\alpha(t,\tau,z) = \int e^{it\frac{\zeta^2}{2}} \tilde{N}_\alpha(t,\tau,z,\zeta)\, d\zeta, \tag{C.59}$$

where $\tilde{N}_\alpha$ is supported close to $\zeta = 0$ and satisfies when $\tau \le \sqrt{t}$, by (C.57), the following estimates for any γ in $\mathbb{N}$:

$$\begin{aligned} &|\partial_\zeta^\gamma \tilde{N}_\alpha(t,\tau,z,\zeta)| \le C\tau_\varepsilon^{-\omega}, \\ &|\partial_\tau \partial_\zeta^\gamma \tilde{N}_\alpha(t,\tau,z,\zeta)| \le C\tau_\varepsilon^{-\omega+\frac{1}{2}}\big(\tau_\varepsilon^{-\frac{3}{2}} + \tau^{-\frac{3}{2}}(\varepsilon^2\sqrt{\tau})^{\frac{3}{2}\theta'} + \tau_\varepsilon^{-\frac{1}{2}} t^{-1}\big) \end{aligned} \tag{C.60}$$

if $N_\pm$ in (C.39) satisfies (C.34),

$$|\partial_\zeta^\gamma \tilde{N}_\alpha(t,\tau,z,\zeta)| \le C\tau^{-\frac{1}{2}}\tau_\varepsilon^{-1}, \quad |\partial_\tau \partial_\zeta^\gamma \tilde{N}_\alpha(t,\tau,z,\zeta)| \le C\tau^{-\frac{3}{4}}\tau_\varepsilon^{-1} \tag{C.61}$$

if $N_\pm$ satisfies (C.35),

$$|\partial_\zeta^\gamma \tilde{N}_\alpha(t,\tau,z,\zeta)| \le C\tau\tau_\varepsilon^{-\omega}, \quad |\partial_\tau \partial_\zeta^\gamma \tilde{N}_\alpha(t,\tau,z,\zeta)| \le C\tau_\varepsilon^{-\omega} \tag{C.62}$$

if $N_\pm$ satisfies (C.36), and

$$|\partial_\zeta^\gamma \tilde{N}_\alpha(t,\tau,z,\zeta)| \le C\tau^{\frac{1}{2}}\tau_\varepsilon^{-1}, \quad |\partial_\tau \partial_\zeta^\gamma \tilde{N}_\alpha(t,\tau,z,\zeta)| \le C\tau^{\frac{1}{4}}\tau_\varepsilon^{-1} \tag{C.63}$$

if $N_\pm$ satisfies (C.37). If we apply the stationary phase formula to equation (C.59), we gain a factor $t^{-\frac{1}{2}}$, which, according to (C.60)–(C.63) provides bounds of the form (C.47)–(C.50). To get expressions of the form (C.44), we still have to replace the phase $t\phi^c$ of (C.58) by ψ_+. By the Taylor–Lagrange formula relatively to τ and (C.55),

$$\phi^c(t,\tau,z) = \phi^c(t,0,z) + \tau(\partial_\tau\phi^c)(t,0,z) + O\Big(\frac{\tau^2}{t^2}\Big).$$

Moreover, by the definition of the phase ϕ of (C.39),

$$\big(\partial_\tau\phi^c\big)(t,0,z) = \frac{1}{t}\Big(\lambda - \sqrt{1+\xi(t,0,z)^2}\Big)$$

and by (C.52), the critical point $\xi(t,0,z)$ satisfies

$$\frac{\xi(t,0,z)}{\langle\xi(t,0,z)\rangle} = \frac{\xi_\lambda}{\lambda} - z = \frac{\xi_\lambda}{\langle\xi_\lambda\rangle} - z$$

so that

$$\sqrt{1+\xi(t,0,z)^2} = \lambda - \lambda^2\xi_\lambda z + O(z^2), \quad z\to 0.$$

We thus get

$$\begin{aligned}\phi^c(t,\tau,z) &= \phi^c(t,0,z) + \frac{\tau}{t}\big(\lambda^2\xi_\lambda z + O(z^2)\big) + r(t,\tau,z),\\ r(t,\tau,z) &= O\Big(\frac{\tau^2}{t^2}\Big), \quad \partial_\tau r(t,\tau,z) = O\Big(\frac{\tau}{t^2}\Big).\end{aligned} \tag{C.64}$$

We define

$$\begin{aligned}\psi_+(t,\tau,z) &= t\big(\phi^c(t,\tau,z) - r(t,\tau,z)\big),\\ \tilde\chi_+(t,\tau,z) &= \chi\Big(\frac{\tau}{\sqrt{t}}\Big)e^{itr(t,\tau,z)}.\end{aligned} \tag{C.65}$$

Plugging (C.64) in (C.58), we deduce from (C.65) that for $|z|\ll 1$, the properties of $\tilde\chi_+,\psi_+$ in (C.45), (C.46) do hold. This concludes the proof of the lemma. ∎

Proof of Proposition C.1.4. Since $R_\alpha^\pm$ in (C.44) satisfy better estimates than those we want, by (C.51), we just consider the integral in the expansion of $\partial_x^\alpha \tilde U'_\pm$.

Under condition (C.34), J_α satisfies (C.47). It follows from (C.13) that the modulus of the integral in (C.44) is $O(\varepsilon^{2\omega})$. On the other hand, if we multiply (C.44) by $z_\pm$, use (C.46), integrate by parts in τ in (C.44) and use (C.45), we deduce from (C.11) and (C.13) a bound in $t^{-\frac{1}{2}}\varepsilon^{2\omega}$ for the resulting expression. Together with the definition (C.45) of $z_\pm$, this brings (C.40).

To prove (C.41), we proceed in the same way. Under estimates (C.35), (C.48) holds for J_α. By (C.13), this provides for (C.44) an estimate in $\varepsilon^2 t^{-\frac{1}{4}}$. On the other hand, if we multiply equation (C.44) by $z_\pm$ and integrate by parts, we get using (C.48) and (C.13) an estimate in $\varepsilon^2 t^{-\frac{3}{8}}$. Together with the first one, this implies (C.41).

One obtains (C.42) (resp. (C.43)) in the same way from (C.49) (resp. (C.50)) and (C.13). ∎

Study of the U'' contribution. According to (C.4) and (C.5) we have

$$U''(t,x) = \frac{i}{2\pi}\int_{-\infty}^{t}\int e^{i((t-\tau)\sqrt{1+\xi^2}+\lambda\tau+x\xi)}(1-\chi)\Big(\frac{\tau}{\sqrt{t}}\Big)\hat{M}(\tau,\xi)\,d\xi\,d\tau \quad \text{(C.66)}$$

and

$$L_+U''(t,x) = \frac{i}{2\pi}\int_{-\infty}^{t}\int e^{i((t-\tau)\sqrt{1+\xi^2}+\lambda\tau+x\xi)}(1-\chi)\Big(\frac{\tau}{\sqrt{t}}\Big) \times\Big(\tau\frac{\xi}{\langle\xi\rangle}\hat{M}(\tau,\xi)+\widehat{xM}(\tau,\xi)\Big)\,d\xi\,d\tau. \quad \text{(C.67)}$$

We treat first the case when $\lambda < 1$ or $\lambda > 1$ and $\hat{M}$ is supported for ξ outside a neighborhood of $\pm\xi_\lambda$.

Lemma C.1.6. *Assume $\lambda < 1$ or $\lambda > 1$ and $\hat{M}$ supported outside a neighborhood of $\{-\xi_\lambda, \xi_\lambda\}$.*

(i) *Under Assumption (H1)$_\omega$, estimates (C.24)–(C.27) hold true.*

(ii) *Under Assumption (H2), estimates (C.28)–(C.30) hold true.*

Proof. We write $\hat{U}''(t,\xi)$ as

$$\int_{-\infty}^{t} e^{i(\lambda-\sqrt{1+\xi^2})\tau}(1-\chi)\Big(\frac{\tau}{\sqrt{t}}\Big)N(\tau,\xi)\,d\tau\, e^{it\sqrt{1+\xi^2}} \quad \text{(C.68)}$$

with N satisfying condition (C.34) under Assumption (H1)$_\omega$ and condition (C.35) under Assumption (H2). In the same way, $\widehat{L_+U''}$ is given by (C.68) with N satisfying (C.36) when Assumption (H1)$_\omega$ holds and (C.37) under Assumption (H2).

We perform one ∂_τ-integration by parts in (C.68) and compute the $L^2(\langle\xi\rangle^r)$ norm. When N satisfies (C.34), we obtain from (C.12) (and from (C.13) if ∂_τ falls on $(1-\chi)(\tau/\sqrt{t})$) a bound of the form (C.24). If instead of computing the $L^2(\langle\xi\rangle^r d\xi)$ norm, we estimate the $L^1(\langle\xi\rangle^r d\xi)$ one, we get (C.25) from (C.12) and (C.13).

Under condition (C.35) we get an estimate of the $L^2(\langle\xi\rangle^r d\xi)$ norm of (C.68) by

$$C\int_{\sqrt{t}}^{t}\tau_\varepsilon^{-1}\tau^{-\frac{3}{4}}\,d\tau + C\varepsilon^2 t^{-\frac{1}{2}}$$

which is smaller than the right-hand side of (C.28) by (C.14).

We are left with proving (C.26), (C.27), (C.29) and (C.30). Integrating by parts in τ in (C.66) and (C.67), we have thus to bound the integrals

$$\int e^{i(\lambda t+x\xi)}N(t,\xi)\,d\xi, \quad \text{(C.69)}$$

$$\int_{-\infty}^{t}\int e^{i((t-\tau)\sqrt{1+\xi^2}+\lambda\tau+x\xi)}\partial_\tau\Big(N(\tau,\xi)(1-\chi)\Big(\frac{\tau}{\sqrt{t}}\Big)\Big)\,d\xi\,d\tau, \quad \text{(C.70)}$$

where N satisfies (C.35) (to get (C.29)) or (C.36) (to obtain (C.26)–(C.27)) or (C.37) (to get (C.30)). The $W^{r,\infty}$ norm of (C.69) is bounded from above by the L^1 norm of

$\langle\xi\rangle^r N(\tau,\xi)$, that has immediately the wanted estimates. Let us study (C.70). Since the integrand is in $\mathcal{S}(\mathbb{R})$ relatively to ξ, stationary phase shows that the $d\xi$-integral is $O(\langle t-\tau\rangle^{-\frac{1}{2}})$, with bounds given by the right-hand side of (C.35)–(C.37). Consequently, the contribution of (C.70) to (C.29) will be estimated by

$$C\int_{\sqrt{t}}^{t}\langle t-\tau\rangle^{-\frac{1}{2}}\frac{\varepsilon^2}{1+\tau\varepsilon^2}\tau^{-\frac{3}{4}}\,d\tau, \tag{C.71}$$

its contribution to (C.26)–(C.27) will be bounded by

$$C\int_{\sqrt{t}}^{t}\langle t-\tau\rangle^{-\frac{1}{2}}\frac{\varepsilon^{2\omega}}{(1+\tau\varepsilon^2)^{\omega}}\,d\tau, \tag{C.72}$$

and its contribution to (C.30) will be controlled by

$$C\int_{\sqrt{t}}^{t}\langle t-\tau\rangle^{-\frac{1}{2}}\frac{\varepsilon^2}{1+\tau\varepsilon^2}\tau^{\frac{1}{4}}\,d\tau. \tag{C.73}$$

One checks that (C.71) (resp. (C.72), resp. (C.73)) is bounded from above by the right-hand side of (C.29) (resp. (C.26)–(C.27), resp. (C.30)). This concludes the proof of the lemma. ■

We have obtained estimates (C.24)–(C.30) when $\hat{M}$ in (C.66)–(C.67) is supported away from the zeros of $\lambda-\sqrt{1+\xi^2}$. We shall next obtain these bounds for $\hat{M}$ supported in a small neighborhood of this set. We prove first these estimates under Assumption (H1)$_\omega$, i.e. those of (i) in the statement of Proposition C.1.2. We have to study again the integral

$$\int_{-\infty}^{t}\int e^{i((t-\tau)\sqrt{1+\xi^2}+\lambda\tau+x\xi)}(1-\chi)\Big(\frac{\tau}{\sqrt{t}}\Big)N(\tau,\xi)\,d\xi\,d\tau, \tag{C.74}$$

where N will satisfy (C.34) or (C.36) and is supported close to $\pm\xi_\lambda$.

Lemma C.1.7. *Assume $\lambda>1$ and N supported in a small enough neighborhood of $\{\xi_\lambda,-\xi_\lambda\}$. Then if N satisfies* (C.34) *(resp.* (C.36)*), estimates* (C.24) *and* (C.25) *(resp.* (C.26)–(C.27)*) hold true.*

Proof. Introduce $\Omega(\tau,\zeta)=\frac{e^{i\tau\zeta}-1}{i\zeta}$ and write (C.74), after making a ∂_τ-integration by parts, as the sum of the following quantities:

$$\int e^{i(t\sqrt{1+\xi^2}+x\xi)}\Omega\Big(t,\lambda-\sqrt{1+\xi^2}\Big)N(t,\xi)\,d\xi, \tag{C.75}$$

$$\begin{aligned}-\int_{-\infty}^{t}\int e^{i(t\sqrt{1+\xi^2}+x\xi)}\Omega\Big(\tau,\lambda-\sqrt{1+\xi^2}\Big)\\ \times\,\partial_\tau\Big((1-\chi)\Big(\frac{\tau}{\sqrt{t}}\Big)N(\tau,\xi)\Big)\,d\xi\,d\tau.\end{aligned} \tag{C.76}$$

Assume for instance that ξ stays in a small neighborhood of ξ_λ on the support of N, and make the change of variables $\zeta = \lambda - \sqrt{1+\xi^2}$ in the integrals, with ζ staying close to zero.

Consider first the case when N satisfies (C.34) and let us prove (C.25). We estimate the modulus of (C.75) by

$$\int_{|\zeta|\ll 1} |\Omega(t,\zeta)| \frac{\varepsilon^{2\omega}}{(1+t\varepsilon^2)^\omega}\, d\zeta \le \frac{C\varepsilon^{2\omega}}{(1+t\varepsilon^2)^\omega} \log t$$

which is controlled by the right-hand side of (C.25). In the same way, we bound the modulus of (C.76) by

$$C\int_{\sqrt{t}}^t \Big(\tau_\varepsilon^{-\omega+\frac{1}{2}}\Big(\tau_\varepsilon^{-\frac{3}{2}} + \tau^{-\frac{3}{2}}(\varepsilon^2\sqrt{\tau})^{\frac{3}{2}\theta'}\Big) + \frac{1}{\sqrt{t}}\tau_\varepsilon^{-\omega}\mathbb{1}_{\tau\sim\sqrt{t}}\Big)\int_{|\zeta|\ll 1}|\Omega(\tau,\zeta)|\, d\zeta\, d\tau.$$

As

$$\int_{|\zeta|\ll 1} |\Omega(\tau,\zeta)|\, d\zeta = O(\log \tau) = O(\log t),$$

we obtain using (C.12) and (C.13) a bound in $\varepsilon^{2\omega}\log(1+t)$ as wanted. Assume next that N satisfies (C.36), and let us show (C.26)–(C.27). We estimate then (C.75) by

$$\frac{C\varepsilon^{2\omega}t}{(1+t\varepsilon^2)^\omega}\int_{|\zeta|\ll 1}|\Omega(t,\zeta)|\, d\zeta$$

that is bounded by (C.26)–(C.27). On the other hand, (C.76) may be controlled by

$$\int_{\sqrt{t}}^t \log\tau \frac{\varepsilon^{2\omega}}{(1+\tau\varepsilon^2)^\omega}\, d\tau,$$

that is bounded by (C.26) if $\omega = 1$, (C.27) if $\omega > 1$.

To finish the proof of the lemma, we still need to get (C.24). The H^r norm of (C.75) and (C.76) is bounded from above respectively by

$$\big\|\Omega\big(t, \lambda - \sqrt{1+\xi^2}\big)N(t,\xi)\big\|_{L^2(\langle\xi\rangle^r d\xi)} \tag{C.77}$$

and by

$$\int_{\sqrt{t}}^t \Big\|\Omega\big(\tau,\lambda-\sqrt{1+\xi^2}\big)\partial_\tau\Big((1-\chi)\Big(\frac{\tau}{\sqrt{t}}\Big)N(\tau,\xi)\Big)\Big\|_{L^2(\langle\xi\rangle^r d\xi)} d\tau. \tag{C.78}$$

We consider again the case when N is supported in a small neighborhood of ξ_λ and use $\zeta = \lambda - \sqrt{1+\xi^2}$ as the variable of integration. Since

$$\|\Omega(\tau,\zeta)\mathbb{1}_{|\zeta|\ll 1}\|_{L^2(d\zeta)} = O(\sqrt{\tau}),$$

we estimate, in view of (C.34), (C.77) and (C.78) by (C.24) again using (C.15) and (C.13). This concludes the proof. ■

Lemma C.1.7 concludes the proof of (i) of Proposition C.1.2. In order to finish the proof of (ii), we need to show the following.

Lemma C.1.8. *Consider equation* (C.66) *(resp.* (C.67)*) when* $\hat{M}$ *is supported close to* $\{-\xi_\lambda, \xi_\lambda\}$ *and when Assumption* (H2) *holds i.e. under conditions* (C.8)–(C.10)*. Then estimates* (C.28) *and* (C.29) *(resp.* (C.30)*) hold true.*

Proof. Notice first that the term $\widehat{xM}$ under the integral (C.67) satisfies the same hypothesis as $\hat{M}$ under integral (C.66) (see the lines below (C.10)). Since the right-hand side of (C.30) is larger than the one in (C.29), it suffices to show (C.28) and (C.29) for expression (C.66), and (C.30) for (C.67) where one forgets the $\widehat{xM}$ term. We thus have to study an expression

$$\int_{-\infty}^{t}\int e^{i((t-\tau)\sqrt{1+\xi^2}+\lambda\tau+x\xi)}(1-\chi)\Big(\frac{\tau}{\sqrt{t}}\Big)\tau^j N(\tau,\xi)\,d\xi\,d\tau, \tag{C.79}$$

where, according to conditions (C.8)–(C.10), N is supported in a small neighborhood of $\{-\xi_\lambda, \xi_\lambda\}$ and there are functions ϕ, ψ such that the following estimates hold:

$$\begin{aligned} |N(t,\xi)| + |\phi(t,\xi)| &\le C t^{-\frac{1}{2}} t_\varepsilon^{-1}, \\ |\partial_t N(t,\xi)| &\le C t^{-\frac{3}{4}} t_\varepsilon^{-1}, \\ |\psi(t,\xi)| &\le C t^{-1} t_\varepsilon^{-1}, \\ D_t N(t,\xi) &= (D_t + \lambda - \sqrt{1+\xi^2})\phi(t,\xi) + \psi(t,\xi), \end{aligned} \tag{C.80}$$

and where $j = 0$ in the case of bounds (C.28)–(C.29) and $j = 1$ for (C.30).

Let χ_0 be in $C_0^\infty(\mathbb{R})$, equal to one close to zero, and write the integral in (C.79) as $I_L^j + I_R^j$, where

$$\begin{aligned} I_L^j = \int_{-\infty}^{t}\int & e^{i((t-\tau)\sqrt{1+\xi^2}+\lambda\tau+x\xi)}\chi_0\Big(\big(\lambda-\sqrt{1+\xi^2}\big)\sqrt{t}\Big) \\ & \times (1-\chi)\Big(\frac{\tau}{\sqrt{t}}\Big)\tau^j N(\tau,\xi)\,d\tau\,d\xi. \end{aligned} \tag{C.81}$$

Since $\lambda > 1$, the $d\xi$ integral is $O(t^{-\frac{1}{2}})$, and using the estimate of N in (C.80), we get by (C.14) and (C.16)

$$|I_L^0| \le C\frac{\varepsilon}{\sqrt{t}}\Big(\frac{t\varepsilon^2}{\langle t\varepsilon^2\rangle}\Big)^{\frac{1}{2}}, \quad |I_L^1| \le C\frac{t\varepsilon^2}{\langle t\varepsilon^2\rangle}$$

which are better than the right-hand side of (C.29), (C.30), respectively. To study I_R^j, we make a ∂_τ-integration by parts and write this term as a sum of

$$-i\sqrt{t}\int e^{i(\lambda t+x\xi)}\chi_1\Big(\sqrt{t}\big(\lambda-\sqrt{1+\xi^2}\big)\Big)t^j N(t,\xi)\,d\xi, \tag{C.82}$$

where $\chi_1(z) = \frac{1-\chi_0(z)}{z}$, of

$$\begin{aligned} i\sqrt{t}\int_{-\infty}^{t}\int & e^{i((t-\tau)\sqrt{1+\xi^2}+\lambda\tau+x\xi)} \\ & \times \chi_1\Big(\big(\lambda-\sqrt{1+\xi^2}\big)\sqrt{t}\Big)\partial_\tau\Big((1-\chi)\Big(\frac{\tau}{\sqrt{t}}\Big)\tau^j\Big)N(\tau,\xi)\,d\xi\,d\tau \end{aligned} \tag{C.83}$$

and of

$$-\sqrt{t}\int_{-\infty}^{t}\int e^{i((t-\tau)\sqrt{1+\xi^2}+\lambda\tau+x\xi)} \times \chi_1\Big(\big(\lambda-\sqrt{1+\xi^2}\big)\sqrt{t}\Big)(1-\chi)\Big(\frac{\tau}{\sqrt{t}}\Big)\tau^j D_\tau N(\tau,\xi)\,d\xi\,d\tau. \tag{C.84}$$

We plug the last equality (C.80) in (C.84). We get on the one hand

$$-\sqrt{t}\int_{-\infty}^{t}\int e^{i((t-\tau)\sqrt{1+\xi^2}+\lambda\tau+x\xi)} \times \chi_1\Big(\big(\lambda-\sqrt{1+\xi^2}\big)\sqrt{t}\Big)(1-\chi)\Big(\frac{\tau}{\sqrt{t}}\Big)\tau^j \psi(\tau,\xi)\,d\xi\,d\tau \tag{C.85}$$

and, after another integration by parts, the terms

$$i\sqrt{t}\int e^{i(\lambda t+x\xi)}\chi_1\Big(\sqrt{t}\big(\lambda-\sqrt{1+\xi^2}\big)\Big)t^j\phi(t,\xi)\,d\xi \tag{C.86}$$

and

$$-i\sqrt{t}\int_{\sqrt{t}}^{t}\int e^{i((t-\tau)\sqrt{1+\xi^2}+\lambda\tau+x\xi)} \times \chi_1\Big(\big(\lambda-\sqrt{1+\xi^2}\big)\sqrt{t}\Big)\partial_\tau\Big((1-\chi)\Big(\frac{\tau}{\sqrt{t}}\Big)\tau^j\Big)\phi(\tau,\xi)\,d\xi\,d\tau. \tag{C.87}$$

Notice that since N and ϕ satisfy the same bound (C.80), a bound for (C.82) will also provide a bound for (C.86). In the same way, an estimate for (C.83) will bring one for (C.87). We are just reduced, in order to get (C.29) and (C.30), to estimate the L^∞ norms of (C.82), (C.83) and (C.85).

We estimate the modulus of (C.82) by

$$C\frac{\varepsilon^2 t^j}{\langle t\varepsilon^2\rangle}\int_{|\zeta|<c}\frac{d\zeta}{\langle\sqrt{t}\zeta\rangle}\le C\frac{\varepsilon^2 t^{j-\frac{1}{2}}}{\langle t\varepsilon^2\rangle}\log(1+t)$$

which is better than the right-hand side of (C.29) (resp. (C.30)) if $j=0$ (resp. $j=1$). We bound (C.83) by

$$C\sqrt{t}\int_{|\zeta|<c}\frac{d\zeta}{\langle\sqrt{t}\zeta\rangle}\int_{\sqrt{t}}^{t}\tau^{-\frac{1}{2}}\frac{\varepsilon^2}{1+\tau\varepsilon^2}\Big|\partial_\tau\Big(\tau^j(1-\chi)\Big(\frac{\tau}{\sqrt{t}}\Big)\Big)\Big|\,d\tau.$$

If $j=0$, we get a bound in $\log(1+t)\varepsilon^2 t^{-\frac{1}{4}}$, better than (C.29), and if $j=1$, we obtain using (C.13), a bound in

$$\varepsilon^2 t^{\frac{1}{4}}\log(1+t)$$

which is better than (C.30) since $t\le\varepsilon^{-4}$.

Finally, we estimate (C.85) by, using (C.80),

$$\log(1+t)\int_{\sqrt{t}}^{t}\tau^{j-1}\frac{\varepsilon^2}{1+\tau\varepsilon^2}\,d\tau$$

which is bounded by (C.29) if $j=0$ and by (C.30) if $j=1$. We have thus established these two estimates. To get the remaining bound (C.28), we just plug inside (C.66) bound (C.8) of $\hat{M}$ and use (C.14). This concludes the proof. ■

C.2 Action of linear and bilinear operators

The goal of this section is to study the action of some operators on a function of the form (C.3), and on its decomposition $U = U' + U''$ given by (C.4). These operators will be of the form $\mathrm{Op}(m')$, given by the non-semiclassical quantization (B.17), for symbols $m'(y,\xi)$ that do not depend on x and belong to the class $\tilde{S}'_{\kappa,0}(1,j)$, $j=1,2$, defined in Definition 3.1.1.

We study first linear operators.

Proposition C.2.1. *Let $(t,x)\mapsto M(t,x)$ be a function satisfying Assumption $(\mathrm{H1})_\omega$, i.e. inequalities (C.7). Assume moreover that M is an odd function of x. Let m' be a symbol in the class $\tilde{S}'_{0,0}(1,1)$ of Definition 3.1.1, i.e. a function $m'(y,\xi)$ on $\mathbb{R}\times\mathbb{R}$ such that*

$$|\partial_y^{\alpha'_0}\partial_\xi^{\alpha}m'(y,\xi)| \le C(1+|y|)^{-N} \tag{C.88}$$

for any N,α'_0,α, and that m' satisfies $m'(-y,-\xi)=m'(y,\xi)$, so that $\mathrm{Op}(m')$ will preserve odd functions. Then, for U'' defined from M by (C.4), we have

$$\mathrm{Op}(m')U'' = e^{i\lambda t}M_1(t,x) + r(t,x), \tag{C.89}$$

where $M_1(t,x)$ is an odd function of x, satisfying for any $\alpha, N\in\mathbb{N}$,

$$\begin{aligned}|\partial_x^\alpha M_1(t,x)| &\le C_{\alpha,N}t_\varepsilon^{-\omega}\langle x\rangle^{-N},\\ |\partial_x^\alpha\partial_t M_1(t,x)| &\le C_{\alpha,N}t_\varepsilon^{-\omega+\frac12}\Big(t_\varepsilon^{-\frac32}+t^{-\frac32}(\varepsilon^2\sqrt{t})^{\frac32\theta'}\Big)\langle x\rangle^{-N}\end{aligned} \tag{C.90}$$

and where $r(t,x)$ is such that for any α,N,

$$|\partial_x^\alpha r(t,x)| \le C_{\alpha,N}\big(\varepsilon^{2\omega}t^{-1}\log(1+t)\big)\langle x\rangle^{-N}. \tag{C.91}$$

Moreover, if L_+ is the operator (C.5), for any $\alpha\in\mathbb{N}$, $k=0,1$,

$$\begin{aligned}&\int_{-1}^{1}\|\partial_x^\alpha\mathrm{Op}(m')((L_+^kU')(t,\mu\cdot))\|_{L^\infty}\,d\mu \le C_\alpha\varepsilon^{2\omega},\\ &\int_{-1}^{1}\|\partial_x^\alpha\mathrm{Op}(m')((L_+^kU')(t,\mu\cdot))\|_{L^2}\,d\mu \le C_\alpha\varepsilon^{2\omega}.\end{aligned} \tag{C.92}$$

Proof. The definition (B.17) of $\operatorname{Op}(m')$ and the expression (C.4) of U'' imply that

$$\operatorname{Op}(m')U'' = \frac{i}{2\pi}\int_{-\infty}^{t}\int e^{i(x\xi+(t-\tau)\sqrt{1+\xi^2}+\lambda\tau)}m'(x,\xi) \times (1-\chi)\Big(\frac{\tau}{\sqrt{t}}\Big)\hat{M}(\tau,\xi)\,d\xi\,d\tau. \tag{C.93}$$

We decompose $\hat{M}(\tau,\xi) = \hat{M}'(\tau,\xi) + \hat{M}''(\tau,\xi)$, where $\hat{M}'$ is supported for ξ in a small neighborhood of the two roots $\pm\xi_\lambda$ of $\sqrt{1+\xi^2} = \lambda$ and $\hat{M}''$ vanishes close to that set when $\lambda > 1$, and $\hat{M}' = 0$ if $\lambda < 1$. Moreover, $\hat{M}'(\tau,\xi)$, $\hat{M}''(\tau,\xi)$ are odd in ξ, because M is odd in x. We define then

$$\begin{aligned} B'(x,\tau,\xi) &= e^{ix\xi}m'(x,\xi)\hat{M}'(\tau,\xi),\\ B''(x,\tau,\xi) &= e^{ix\xi}m'(x,\xi)\hat{M}''(\tau,\xi). \end{aligned} \tag{C.94}$$

By the evenness of m', we have

$$B'(-x,\tau,-\xi) = -B'(x,\tau,\xi), \quad B''(-x,\tau,-\xi) = -B''(x,\tau,\xi). \tag{C.95}$$

Let us study first the contribution of $\hat{M}''$ to (C.93), given by

$$\frac{i}{2\pi}\int_{-\infty}^{t}\int e^{i((t-\tau)\sqrt{1+\xi^2}+\lambda\tau)}B''(x,\tau,\xi)(1-\chi)\Big(\frac{\tau}{\sqrt{t}}\Big)\,d\xi\,d\tau. \tag{C.96}$$

We perform one ∂_τ-integration by parts, that provides on the one hand $e^{i\lambda t}M_1(t,x)$, where

$$M_1(t,x) = \frac{1}{2\pi}\int\Big(\lambda-\sqrt{1+\xi^2}\Big)^{-1}B''(x,t,\xi)\,d\xi$$

satisfies (C.90) by (C.94), (C.88) and (C.7), and is odd in x by (C.95), and on the other hand a contribution

$$\frac{1}{2\pi}\int_{-\infty}^{t}\int e^{i((t-\tau)\sqrt{1+\xi^2}+\lambda\tau)}N(x,\tau,\xi)\,d\xi\,d\tau, \tag{C.97}$$

where

$$N(x,\tau,\xi) = -\partial_\tau\Big(B''(x,\tau,\xi)(1-\chi)\Big(\frac{\tau}{\sqrt{t}}\Big)\Big)\Big(\lambda-\sqrt{1+\xi^2}\Big)^{-1}$$

satisfies by (C.88) and (C.7)

$$|\partial_x^\alpha\partial_\xi^\beta N(x,\tau,\xi)| \le C\langle x\rangle^{-N}\langle\xi\rangle^{-N}\tau_\varepsilon^{-\omega} \times\Big(\tau_\varepsilon^{-1}+\tau^{-1}\mathbb{1}_{\tau\sim\sqrt{t}}+\tau_\varepsilon^{\frac{1}{2}}\tau^{-\frac{3}{2}}(\varepsilon^2\sqrt{\tau})^{\frac{3}{2}\theta'}\Big). \tag{C.98}$$

By the oddness of $\hat{M}$ in ξ, $N(x,\tau,0)\equiv 0$. Consequently, if we apply the stationary phase formula to the ∂_ξ-integral in (C.97) at the unique (non-degenerate) critical point

$\xi = 0$, we gain a decaying factor in $\langle t-\tau\rangle^{-1}$ instead of $\langle t-\tau\rangle^{-\frac{1}{2}}$. Taking (C.98) into account, and using (C.12), we obtain for (C.97) and its ∂_x-derivatives a bound in

$$C_N\langle x\rangle^{-N}\int_{\sqrt{t}}^{t}\langle t-\tau\rangle^{-1}\tau_\varepsilon^{-\omega}\big(\tau_\varepsilon^{-1}+\tau^{-1}\mathbb{1}_{\tau\sim\sqrt{t}}+\tau_\varepsilon^{\frac{1}{2}}\tau^{-\frac{3}{2}}(\varepsilon^2\sqrt{\tau})^{\frac{3}{2}\theta'}\big)\,d\tau$$
$$\leq C_N\langle x\rangle^{-N}\varepsilon^{2\omega}t^{-1}\log(1+t)$$

which is bounded by (C.91).

Let us study next the contribution of $\hat{M}'$ to (C.93). We get

$$\int_1^t\int e^{i((t-\tau)\sqrt{1+\xi^2}+\lambda\tau)}B'(x,\tau,\xi)(1-\chi)\Big(\frac{\tau}{\sqrt{t}}\Big)\,d\xi\,d\tau. \tag{C.99}$$

Write for $1\leq\tau\leq t$

$$B'(x,\tau,\xi)=B'(x,t,\xi)+(\tau-t)\tilde{B}'(x,\tau,t,\xi), \tag{C.100}$$

where $\tilde{B}'$ satisfies by (C.7) and (C.88)

$$|\partial_x^\alpha\partial_\xi^\beta\tilde{B}'(x,\tau,t,\xi)|\leq C\tau_\varepsilon^{-\omega}\big(\tau_\varepsilon^{-1}+\tau_\varepsilon^{\frac{1}{2}}\tau^{-\frac{3}{2}}(\varepsilon^2\sqrt{t})^{\frac{3}{2}\theta'}\big)\langle x\rangle^{-N}$$

and is supported for ξ close to $\{-\xi_\lambda,\xi_\lambda\}$. If we substitute in the integral (C.99) expression $(\tau-t)\tilde{B}'$ to B', and use that, since $\xi_\lambda\neq 0$, $\tilde{B}'$ is supported far away the critical point $\xi=0$ of the phase, we may gain a factor $\langle t-\tau\rangle^{-N}$ for any N by ∂_ξ-integration by parts. We thus get a contribution to (C.99) and to its ∂_x-derivatives bounded by

$$C_N\langle x\rangle^{-N}\int_{\sqrt{t}}^{t}\langle t-\tau\rangle^{-N}\tau_\varepsilon^{-\omega}\big(\tau_\varepsilon^{-1}+\tau_\varepsilon^{\frac{1}{2}}\tau^{-\frac{3}{2}}(\varepsilon^2\sqrt{t})^{\frac{3}{2}\theta'}\big)\,d\tau.$$

This again provides a contribution to (C.91). We are left with studying (C.99) with $B'(x,\tau,\xi)$ replaced by $B'(x,t,\xi)$ according to (C.100), i.e.

$$\begin{aligned}&\int_1^t\int e^{i((t-\tau)\sqrt{1+\xi^2}+\lambda\tau)}(1-\chi)\Big(\frac{\tau}{\sqrt{t}}\Big)B'(x,t,\xi)\,d\xi\,d\tau\\&\quad=e^{i\lambda t}\int T\big(t,\sqrt{1+\xi^2}-\lambda\big)B'(x,t,\xi)\,d\xi\end{aligned} \tag{C.101}$$

with

$$T(t,\zeta)=T_1(t,\zeta)+T_2(t,\zeta)$$

and

$$T_1(t,\zeta)=\int_0^{t-1}e^{i\tau\zeta}\,d\tau,$$
$$T_2(t,\zeta)=-\int_0^{t-1}e^{i\tau\zeta}\chi\Big(\frac{t-\tau}{\sqrt{t}}\Big)\,d\tau.$$

Note that if $\varphi \in \mathcal{S}(\mathbb{R})$,

$$\begin{aligned}\int T_1(t,\zeta)\varphi(\zeta)\,d\zeta &= \int_0^{t-1} \hat{\varphi}(-\tau)\,d\tau = \int_0^{+\infty} \hat{\varphi}(-\tau)\,d\tau + O(t^{-\infty}),\\ \int T_2(t,\zeta)\varphi(\zeta)\,d\zeta &= O(t^{-\infty}).\end{aligned} \tag{C.102}$$

Using that B' is supported close to $\xi = \pm\xi_\lambda$, and that $\xi_\lambda \neq 0$, we may use in the last integral in (C.101) $\zeta = \sqrt{1+\xi^2} - \lambda$ as a variable of integration close to this point. We express thus (C.101) from integrals of the form (C.102), with φ expressed from B'. The definition (C.94) of B' and (C.88), (C.7) imply that the principal term on the first line (C.102) brings to (C.101) a contribution in $e^{i\lambda t}M_1(t,x)$ with M_1 satisfying estimates (C.90). The other contributions, as well as their ∂_x-derivatives, are $O(t_\varepsilon^{-\omega}t^{-N}\langle x\rangle^{-N})$ for any N, so satisfy (C.91).

It remains to prove (C.92). We express L_+U' from (C.32), which allows us to write $\mathrm{Op}(m')((L_+U')(\mu\cdot))$ as the sum of two expressions

$$\frac{i}{2\pi}\int_1^{+\infty}\int e^{i((t-\tau)\sqrt{1+\xi^2}+\lambda\tau)}\chi\Big(\frac{\tau}{\sqrt{t}}\Big)B_j^\mu(x,\tau,\xi)\,d\tau\,d\xi, \quad j = 1,2, \tag{C.103}$$

with

$$\begin{aligned}B_1^\mu(x,\tau,\xi) &= e^{ix\xi\mu}m'(x,\mu\xi)\widehat{xM}(\tau,\xi),\\ B_2^\mu(x,\tau,\xi) &= e^{ix\xi\mu}m'(x,\mu\xi)\tau\frac{\xi}{\langle\xi\rangle}\hat{M}(\tau,\xi).\end{aligned} \tag{C.104}$$

When $j = 1$, we use the stationary phase formula in ξ to make appear a $\langle t-\tau\rangle^{-\frac{1}{2}}$ factor. Using also (C.7) and (C.88), we get for any ∂_x-derivative of (C.103) with $j = 1$ a bound in

$$C\int_1^{\sqrt{t}}\langle t-\tau\rangle^{-\frac{1}{2}}\tau_\varepsilon^{-\omega}\,d\tau\langle x\rangle^{-N} \le C\varepsilon^{2\omega}\langle x\rangle^{-N}. \tag{C.105}$$

When $j = 2$, we notice that because $\hat{M}$ is odd in ξ, $B_2^\mu(x,\tau,\xi)$ vanishes at second order at $\xi = 0$. Consequently, stationary phase formula in (C.103) makes gain a factor in $\langle t-\tau\rangle^{-\frac{3}{2}}$, so that (C.103) is controlled, using again (C.13), by

$$C\int_1^{\sqrt{t}}\langle t-\tau\rangle^{-\frac{3}{2}}\tau\tau_\varepsilon^{-\omega}\,d\tau\langle x\rangle^{-N} \le C\varepsilon^{2\omega}\langle x\rangle^{-N}.$$

Bounds (C.92) follow from this inequality and (C.105). This concludes the proof of (C.92) when $k = 1$. If $k = 0$, the estimate is similar to the one with B_1^μ above. ■

Let us prove a similar result to Proposition C.2.1 for some bilinear operators.

Proposition C.2.2. *Let M and U'' be as in the statement of Proposition C.2.1. Let m' be a symbol in $\tilde{S}'_{\kappa,0}(\prod_{j=1}^2\langle\xi_j\rangle^{-1},2)$ for some $\kappa \ge 0$, satisfying*

$$m'(-y,-\xi_1,-\xi_1) = -m'(y,\xi_1,\xi_2).$$

Then for any function v,

$$\operatorname{Op}(m')(U'', v) = e^{i\lambda t}\operatorname{Op}(b_1)v + \operatorname{Op}(b_2)v, \tag{C.106}$$

where b_1, b_2 *satisfy for any* α_0', α, N *the following estimates:*

$$\begin{aligned} |\partial_y^{\alpha_0'}\partial_\xi^\alpha b_1(t,y,\xi)| &\le C t_\varepsilon^{-\omega}\langle y\rangle^{-N}\langle\xi\rangle^{-1}, \\ |\partial_y^{\alpha_0'}\partial_\xi^\alpha \partial_t b_1(t,y,\xi)| &\le C t_\varepsilon^{-\omega+\frac{1}{2}}\big(t_\varepsilon^{-\frac{3}{2}} + t^{-\frac{3}{2}}(\varepsilon^2\sqrt{t})^{\frac{3}{2}\theta'}\big)\langle y\rangle^{-N}\langle\xi\rangle^{-1}, \\ |\partial_y^{\alpha_0'}\partial_\xi^\alpha b_2(t,y,\xi)| &\le C\varepsilon^{2\omega}t^{-1}\log(1+t)\langle y\rangle^{-N}\langle\xi\rangle^{-1}. \end{aligned} \tag{C.107}$$

Moreover, $b_j(t,-y,-\xi) = b_j(t,y,\xi)$.

Proof. By expression (C.4) of U'', we have

$$\begin{aligned} \operatorname{Op}(m')(U'', v) &= \frac{i}{(2\pi)^2}\int_{-\infty}^{t}\iint e^{i(x(\xi_1+\xi)+(t-\tau)\sqrt{1+\xi_1^2}+\lambda\tau)} \\ &\qquad \times m'(x,\xi_1,\xi)(1-\chi)\Big(\frac{\tau}{\sqrt{t}}\Big)\hat{M}(\tau,\xi_1)\hat{v}(\xi)\,d\xi\,d\xi_1\,d\tau \\ &= \operatorname{Op}(b)v \end{aligned}$$

if

$$\begin{aligned} b(t,x,\xi) = \frac{i}{2\pi}\int_{-\infty}^{t}\iint & e^{i(x\xi_1+(t-\tau)\sqrt{1+\xi_1^2}+\lambda\tau)} \\ & \times m'(x,\xi_1,\xi)(1-\chi)\Big(\frac{\tau}{\sqrt{t}}\Big)\hat{M}(\tau,\xi_1)\,d\xi_1\,d\tau. \end{aligned} \tag{C.108}$$

We notice that if we consider ξ as a parameter, the function

$$(y,\xi_1) \mapsto m'(y,\xi_1,\xi)\hat{M}(\tau,\xi_1)$$

satisfies estimates of the form (C.88) for every τ, as the losses in

$$M_0(\xi_1,\xi)^\kappa = O(\langle\xi_1\rangle^\kappa)$$

appearing when one takes derivatives in the definition of symbol classes in (B.13) are compensated by the rapid decay of $\hat{M}(\tau,\xi_1)$. We obtain thus an integral of the form (C.93) (with ξ replaced by ξ_1), depending on an extra parameter ξ. By (the proof of) Proposition C.2.1, we obtain thus that (C.108) has an expression of the form (C.89), i.e. $e^{i\lambda t}b_1 + b_2$, with b_1, (resp. b_2) satisfying bounds of the form (C.90) (resp. (C.91)), which gives (C.107), using also that $m'(x,\xi_1,\xi)$ in equation (C.108) is $O(\langle\xi\rangle^{-1})$. The evenness of b_j in (y,ξ) comes from the oddness of m' and $\hat{M}$. This concludes the proof. ∎

Corollary C.2.3. *Under the assumptions of Proposition* C.2.2, *one has the following estimates for any* α, N:

$$|\partial_x^\alpha \operatorname{Op}(m')(U'', U'')| \le C\langle x\rangle^{-N}\big(t_\varepsilon^{-2\omega} + \varepsilon^{4\omega}t^{-2}(\log(1+t))^2\big). \tag{C.109}$$

Proof. By (C.106), we may write

$$\mathrm{Op}(m')(U'', U'') = e^{i\lambda t}\mathrm{Op}(b_1)U'' + \mathrm{Op}(b_2)U''$$

with b_1, b_2 satisfying (C.107). We may apply (C.89) to each term above, using that b_1, b_2 satisfy estimates of the form (C.88), with an extra pre-factor given by the first and last estimates (C.107). Using the first bound (C.90) and (C.91), we reach the conclusion. ∎

We have obtained in the preceding results estimates under assumptions of the form (C.7) for the function M in (C.4), i.e. under Assumption $(\mathrm{H1})_\omega$. We shall need also variants of the preceding results when Assumption (H2), i.e. (C.8) holds instead. In this case, we shall split the function U defined in (C.3) in a different way than in (C.4), cutting at time of order $\tau \sim ct$ instead of $\tau \sim \sqrt{t}$. More precisely, we set

$$\begin{aligned} U &= U_1' + U_1'', \\ U_1'(t,x) &= i\int_1^{+\infty} e^{i(t-\tau)p(D_x)+i\lambda\tau}\chi\Big(\frac{\tau}{t}\Big)M(\tau,\cdot)\,d\tau, \\ U_1''(t,x) &= i\int_{-\infty}^{t} e^{i(t-\tau)p(D_x)+i\lambda\tau}(1-\chi)\Big(\frac{\tau}{t}\Big)M(\tau,\cdot)\,d\tau \end{aligned} \tag{C.110}$$

Proposition C.2.4. *Let us assume that M is odd in x, satisfies the first inequality of (C.8) and that m' satisfies (C.88). We have then the following estimates for any $\alpha, N \in \mathbb{N}$:*

$$|\partial_x^\alpha \mathrm{Op}(m')U_1''| \le C_{\alpha N}\langle x\rangle^{-N} t_\varepsilon^{-\frac{1}{2}} t^{-1}\log(1+t) \tag{C.111}$$

and for $\ell = 0, 1$,

$$\begin{aligned} \int_{-1}^{1}\Big(&\|\partial_x^\alpha \mathrm{Op}(m')\big((L_+^\ell\, U_1')(t,\mu\cdot)\big)\|_{L^2} \\ &+ \|\partial_x^\alpha \mathrm{Op}(m')\big((L_+^\ell U_1')(t,\mu\cdot)\big)\|_{L^\infty}\Big)\,d\mu \le C_\alpha \varepsilon^2. \end{aligned} \tag{C.112}$$

Estimate (C.112) holds as soon as (C.88) is true for some large enough N.

Proof. We denote

$$B(x,\tau,\xi_1) = e^{ix\xi_1}m'(x,\xi_1)\hat{M}(\tau,\xi_1),$$

that satisfies by the first inequality of (C.8) and (C.88)

$$|\partial_x^{\alpha_0}\partial_{\xi_1}^{\alpha} B(x,\tau,\xi_1)| \le C_{\alpha_0,\alpha}\langle x\rangle^{-N}\langle \xi_1\rangle^{-N}\tau^{-\frac{1}{2}}\tau_\varepsilon^{-1}$$

and that vanishes at $\xi_1 = 0$ as M is odd. Then as in (C.93), (C.96)

$$\begin{aligned} \mathrm{Op}(m')U_1'' = \frac{i}{2\pi}\int_{-\infty}^{t}\int & e^{i((t-\tau)\sqrt{1+\xi_1^2}+\lambda\tau)} \\ &\times (1-\chi)\Big(\frac{\tau}{t}\Big)B(x,\tau,\xi_1)\,d\xi_1\,d\tau. \end{aligned} \tag{C.113}$$

Using stationary phase in ξ_1 and the fact that B vanishes at $\xi_1 = 0$, we get for some $a \in]0,1[$,

$$|\partial_x^\alpha \mathrm{Op}(m')U_1''(t,x)| \le C \int_{at}^{t} \langle t-\tau\rangle^{-1} \tau_\varepsilon^{-1} \tau^{-\frac{1}{2}}\, d\tau \langle x\rangle^{-N}$$

which is bounded by the right-hand side of (C.111).

To prove estimate (C.112) with $\ell = 1$, we express $\mathrm{Op}(m')((L_+U')(\mu\cdot))$ under form (C.103), except that the cut-off $\chi(\tau/\sqrt{t})$ has to be replaced by $\chi(\tau/t)$, i.e. we have to study

$$\frac{i}{2\pi} \int_1^{+\infty} \int e^{i((t-\tau)\sqrt{1+\xi_1^2}+\lambda\tau)} \chi\Big(\frac{\tau}{t}\Big) B_j^\mu(x,\tau,\xi_1)\, d\xi_1\, d\tau, \tag{C.114}$$

where B_j^μ, $j = 1, 2$, is given by (C.104). If $j = 1$, we get from the first inequality of (C.8), (C.88) and stationary phase in ξ_1 a bound of ∂_x-derivatives of (C.114) by

$$C\langle x\rangle^{-N} \int_1^{at} \langle t-\tau\rangle^{-\frac{1}{2}} \tau^{-\frac{1}{2}} \tau_\varepsilon^{-1}\, d\tau \tag{C.115}$$

for some $a \in]0,1[$, whence the $O(\varepsilon^2)$ wanted bound for the L^2 and L^∞ norms. If $j = 2$, using stationary phase and the fact that B_2^μ vanishes at order 2 at $\xi = 0$, we get an estimate in

$$C\langle x\rangle^{-N} \int_1^{at} \langle t-\tau\rangle^{-\frac{3}{2}} \tau^{\frac{1}{2}} \tau_\varepsilon^{-1}\, d\tau \tag{C.116}$$

which is also $O(\varepsilon^2)$. This concludes the proof of (C.112) when $\ell = 1$. If $\ell = 0$, we may use directly (C.115) to get the estimate. Notice that to get (C.112), we do not need that (C.115) and (C.116) hold for any N, but just for a large enough N (actually $N = 1$ suffices), so that (C.88) has to be assumed only for some large enough N. ■

Let us write a version of Proposition C.2.2 under Assumption (H2) as well.

Proposition C.2.5. *Let M be as in Proposition C.2.4 and m' in $\tilde{S}'_{\kappa,0}(\prod_{j=1}^2 \langle\xi_j\rangle^{-1}, 2)$. Then $\mathrm{Op}(m')(U_1', v)$ and $\mathrm{Op}(m')(U_1'', v)$ may be written as $\mathrm{Op}(b)v$ for all symbols $b(t,y,\xi)$ satisfying the estimates*

$$|\partial_y^{\alpha_0'} \partial_\xi^\alpha b(t,y,\xi)| \le C t_\varepsilon^{-\frac{1}{2}} t^{-1} \log(1+t) \langle y\rangle^{-N} \langle\xi\rangle^{-1}. \tag{C.117}$$

Proof. Consider first $\mathrm{Op}(m')(U_1'', v)$ that may be written using expression (C.110) of U_1'' as

$$\mathrm{Op}(m')(U_1'', v) = \frac{1}{2\pi} \int e^{ix\xi} b(t,x,\xi)\hat{v}(\xi)\, d\xi \tag{C.118}$$

with

$$b(t,x,\xi) = \frac{i}{2\pi} \int_{-\infty}^{t} \int e^{ix\xi_1 + i((t-\tau)\sqrt{1+\xi_1^2}+\lambda\tau)} \times m'(x,\xi_1,\xi)\hat{M}(\tau,\xi_1)(1-\chi)\Big(\frac{\tau}{t}\Big)\, d\xi_1\, d\tau.$$

Using again stationary phase with respect to ξ_1 and the fact that $\hat{M}(\tau, 0) = 0$ to gain a decaying factor in $\langle t-\tau\rangle^{-1}$, we obtain for the $\partial_x^{\alpha'_0}\partial_\xi^{\alpha}$-derivatives of b an upper bound in

$$C\int_{at}^{t}\langle t-\tau\rangle^{-1}\tau^{-\frac{1}{2}}\tau_\varepsilon^{-1}\,d\tau\langle x\rangle^{-N}\langle\xi\rangle^{-1}\quad (a\in]0,1[) \tag{C.119}$$

since, as seen at the beginning of the proof of Proposition C.2.2,

$$(y,\xi_1)\mapsto m'(y,\xi_1,\xi)\hat{M}(\tau,\xi_1)$$

and its derivatives have bounds in

$$C\langle y\rangle^{-N}\tau^{-\frac{1}{2}}\tau_\varepsilon^{-1}\langle\xi_1\rangle^{-N}\langle\xi\rangle^{-1}$$

according to (C.8). As (C.119) is bounded by the right-hand side of (C.117), we get the wanted conclusion for $\mathrm{Op}(m')(U_1'', v)$.

Consider now the case of $\mathrm{Op}(m')(U_1', v)$, i.e.

$$\frac{1}{(2\pi)^2}\int e^{ix(\xi_1+\xi)}m'(x,\xi_1,\xi)\hat{U}_1'(\xi_1)\hat{v}(\xi)\,d\xi_1\,d\xi.$$

We may rewrite it as

$$\frac{1}{2\pi}\int e^{ix\xi}b(t,x,\xi)\hat{v}(\xi)\,d\xi$$

with, for any N,

$$b(t,x,\xi)=\int K_N(t,x-y,x,\xi)\langle D_y\rangle^{2N-1}U_1'(y)\,dy, \tag{C.120}$$

where

$$K_N(t,z,x,\xi)=\frac{1}{2\pi}\int e^{iz\xi_1}\langle\xi_1\rangle^{-2N+1}m'(x,\xi_1,\xi)\,d\xi_1.$$

By the assumption on m', estimates of the form (B.13) hold (with y on the right-hand side of this inequality replaced by x) whence

$$|\partial_x^{\alpha'_0}\partial_\xi^{\alpha}\partial_{\xi_1}^{\alpha_1}m'(x,\xi_1,\xi)|\le C(1+|x|\langle\xi_1\rangle^{-\kappa})^{-N'}\langle\xi\rangle^{-1}\langle\xi_1\rangle^{-1+\kappa(|\alpha|+|\alpha_1|)}$$

for any N'. We conclude that for any α,β,N',N'', one has estimates

$$|\partial_x^{\alpha}\partial_\xi^{\beta}K_N(t,z,x,\xi)|\le C\langle x\rangle^{-N'}\langle z\rangle^{-N''}\langle\xi\rangle^{-1}$$

if N is taken large enough relatively to N',N'',α,β. Plugging this in (C.120), we conclude that for any N',N'',α,β, there is N such that

$$|\partial_x^{\alpha}\partial_\xi^{\beta}b(t,x,\xi)|\le C\langle x\rangle^{-N'}\sup_y|\langle y\rangle^{-N''}\langle D_y\rangle^{2N-1}U_1'(y)|\langle\xi\rangle^{-1}. \tag{C.121}$$

Since U_1' is odd, we may write

$$\begin{aligned}\langle D_y\rangle^{2N-1}U_1'(y)&=i\frac{y}{2}\int_{-1}^{1}(D_x\langle D_x\rangle^{2N-1}U_1')(\mu y)\,d\mu\\&=i\frac{y}{2t}\int_{-1}^{1}\big((L_+\langle D_x\rangle^{2N}U_1')(\mu y)-\mu y(\langle D_x\rangle^{2N}U_1')(\mu y)\big)\,d\mu\end{aligned}$$

using the definition (C.5) of L_+. We get finally

$$\begin{aligned}|\langle y\rangle^{-N''}\langle D_y\rangle^{2N-1}U_1'(y)| \\ \leq \frac{C}{t}\big(\|\langle y\rangle^{-N''+1}L_+\langle D_x\rangle^{2N}U_1'\|_{L^\infty} + \|\langle y\rangle^{-N''+2}\langle D_x\rangle^{2N}U_1'\|_{L^\infty}\big).\end{aligned} \tag{C.122}$$

We may apply estimate (C.112) with U_1' replaced by $\langle D_x\rangle^{2N}U_1'$ (as $\langle D_x\rangle^{2N}M(\tau,\cdot)$ in (C.110) satisfies the same assumption as $M(\tau,\cdot)$), and the pre-factor $\langle y\rangle^{-N''+1}$, $\langle y\rangle^{-N''+2}$ on the right-hand side of (C.122) satisfies estimates of the form (C.88) with some large fixed N (instead of for any N). By the last statement in Proposition C.2.4, this is enough to apply (C.112). Plugging this in (C.121), we get for that expression a bound in $\varepsilon^2 t^{-1}\langle x\rangle^{-N'}\langle\xi\rangle^{-1}$, which is controlled by the right-hand side of (C.117) since $t \leq \varepsilon^{-4}$. This concludes the proof. ∎

C.3 An explicit computation

In this last section of this chapter, we make an explicit computation that will be used in relation with Fermi's golden rule.

Let χ be in $C_0^\infty(\mathbb{R})$, even, equal to one close to zero. If $\lambda > 1$ and if $\pm\xi_\lambda$ are still the two roots of $\sqrt{1+\xi^2}-\lambda = 0$, set

$$\chi_\lambda(\xi) = \chi(\xi-\xi_\lambda) + \chi(\xi+\xi_\lambda). \tag{C.123}$$

If $\lambda < 1$, set $\chi_\lambda \equiv 0$.

Proposition C.3.1. *Let M be a function satisfying (C.7) with $\omega = 1$, that is odd in x. Let U be defined from M by (C.3) ant let Z be an odd function in $\mathcal{S}(\mathbb{R})$. Then*

$$\begin{aligned}\int \hat{U}(t,\xi)\hat{Z}(\xi)\,d\xi \\ = \lim_{\sigma\to 0+} ie^{i\lambda t}\int_0^{+\infty}\int e^{i\tau(\sqrt{1+\xi^2}-\lambda+i\sigma)}\chi_\lambda(\xi)\hat{M}(t,\xi)\hat{Z}(\xi)\,d\xi\,d\tau \\ + e^{i\lambda t}\int \frac{(1-\chi_\lambda)(\xi)}{\lambda-\sqrt{1+\xi^2}}\hat{M}(t,\xi)\hat{Z}(\xi)\,d\xi + r(t),\end{aligned} \tag{C.124}$$

where r satisfies

$$|r(t)| \leq C\big(\varepsilon^2 t^{-\frac{3}{2}} + t_\varepsilon^{-2} + \varepsilon t^{-\frac{3}{2}}(\varepsilon^2\sqrt{t})^{\frac{3}{2}\theta'}\big). \tag{C.125}$$

Remark. It is clear that the limit on the right-hand side of (C.124) exists and may be computed from $(\sqrt{1+\xi^2}-\lambda+i0)^{-1}$. We keep it nevertheless under the form (C.124) as this will be more convenient for us when using the proposition.

To prove the proposition, we shall write the left-hand side of (C.124), according to (C.3), under the form

$$i\int_1^t\int e^{i(t-\tau)\sqrt{1+\xi^2}+i\lambda\tau}\hat{M}(\tau,\xi)\hat{Z}(\xi)\,d\xi\,d\tau. \tag{C.126}$$

We decompose

$$\begin{aligned}\hat{M}(\tau,\xi) &= \hat{M}'(\tau,\xi)+\hat{M}''(\tau,\xi),\\ \hat{M}'(\tau,\xi) &= \hat{M}(\tau,\xi)\chi_\lambda(\xi),\\ \hat{M}''(\tau,\xi) &= \hat{M}(\tau,\xi)(1-\chi_\lambda)(\xi).\end{aligned} \tag{C.127}$$

We notice that $\hat{M}''$ vanishes at order one at $\xi = 0$ by the oddness assumption on M.

Lemma C.3.2. *Expression* (C.126) *with* $\hat{M}$ *replaced by* $\hat{M}''$ *may be written as*

$$e^{i\lambda t}\int \frac{(1-\chi_\lambda)(\xi)}{\lambda-\sqrt{1+\xi^2}}\hat{M}(t,\xi)\hat{Z}(\xi)\,d\xi \tag{C.128}$$

modulo a remainder satisfying (C.125).

Proof. The expression under study is the sum of (C.128) and of

$$-\int e^{i(t-1)\sqrt{1+\xi^2}+i\lambda}\hat{M}(1,\xi)\frac{(1-\chi_\lambda)(\xi)}{\lambda-\sqrt{1+\xi^2}}\hat{Z}(\xi)\,d\xi \tag{C.129}$$

and

$$-\int_1^t\int e^{i(t-\tau)\sqrt{1+\xi^2}+i\lambda\tau}\partial_\tau\hat{M}(\tau,\xi)\frac{(1-\chi_\lambda)(\xi)}{\lambda-\sqrt{1+\xi^2}}\hat{Z}(\xi)\,d\xi\,d\tau. \tag{C.130}$$

In (C.129) and (C.130), the integrand vanishes at order 2 at $\xi = 0$ by the oddness of M and Z. The stationary phase formula in ξ allows thus to gain a factor $t^{-\frac{3}{2}}$ or $\langle t-\tau\rangle^{-\frac{3}{2}}$. Taking into account (C.7) with $\omega = 1$, we thus bound (C.129) by $C\varepsilon^2 t^{-\frac{3}{2}}$ and (C.130) from

$$\begin{aligned}&\int_1^t \langle t-\tau\rangle^{-\frac{3}{2}}\Big(\frac{\varepsilon^4}{(1+\tau\varepsilon^2)^2}+\frac{\varepsilon^{1+3\theta'}}{(1+\tau\varepsilon^2)^{\frac{1}{2}}}\tau^{-\frac{3}{2}(1-\frac{\theta'}{2})}\Big)\,d\tau\\ &\quad\le C\big(t_\varepsilon^{-2}+\varepsilon t^{-\frac{3}{2}}(\varepsilon^2\sqrt{t})^{\frac{3}{2}\theta'}\big)\end{aligned}$$

(using $t \le \varepsilon^{-4}$). We thus get quantities controlled as in (C.125). ∎

The lemma implies the proposition when $\lambda < 1$. We shall assume from now on that $\lambda > 1$ and study (C.126) with $\hat{M}$ replaced by $\hat{M}'$.

End of the proof of Proposition C.3.1. By the Taylor formula, we write for $1 \le \tau \le t$,

$$\hat{M}'(\tau,\xi) = \hat{M}'(t,\xi)+(\tau-t)H(t,\tau,\xi),$$

where according to (C.7) with $\omega = 1$, H satisfies for any α,

$$|\partial_\xi^\alpha H(t,\tau,\xi)| \le C_\alpha \tau_\varepsilon^{-\frac{1}{2}}\big(\tau_\varepsilon^{-\frac{3}{2}}+\tau^{-\frac{3}{2}}(\varepsilon^2\sqrt{t})^{\frac{3}{2}\theta'}\big).$$

Integral (C.126) with $\hat{M}$ replaced by $\hat{M}'$ may be written as the sum $J_1 + J_2$, where

$$\begin{aligned} J_1 &= i\int_1^t \int e^{i(t-\tau)\sqrt{1+\xi^2}+i\lambda\tau}\hat{M}'(t,\xi)\hat{Z}(\xi)\,d\xi\,d\tau, \\ J_2 &= i\int_1^t \int e^{i(t-\tau)\sqrt{1+\xi^2}+i\lambda\tau}(\tau - t)H(t,\tau,\xi)\hat{Z}(\xi)\,d\xi\,d\tau. \end{aligned} \tag{C.131}$$

Since H is supported close to $\pm\xi_\lambda$, so far away from zero, we can make in J_2 any number of integrations by parts in ξ in order to gain a decaying factor in $\langle t-\tau\rangle^{-N}$ for any N, so that

$$|J_2| \le C\int_1^t \langle t-\tau\rangle^{-N}\big(\tau_\varepsilon^{-2} + \tau_\varepsilon^{-\frac{1}{2}}\tau^{-\frac{3}{2}}(\varepsilon^2\sqrt{t})^{\frac{3}{2}\theta'}\big)\,d\tau$$

which is better than the right-hand side of (C.125). On the other hand, we may write

$$\begin{aligned} J_1 &= ie^{i\lambda t}\int_0^{t-1}\int e^{i\tau(\sqrt{1+\xi^2}-\lambda)}\hat{M}'(t,\xi)\hat{Z}(\xi)\,d\xi\,d\tau \\ &= \lim_{\sigma\to 0+} ie^{i\lambda t}\int_0^{+\infty}\int e^{i\tau(\sqrt{1+\xi^2}-\lambda+i\sigma)}\hat{M}'(t,\xi)\hat{Z}(\xi)\,d\xi\,d\tau + J_1', \end{aligned} \tag{C.132}$$

where

$$J_1' = -ie^{i\lambda t}\lim_{\sigma\to 0+}\int_{t-1}^{+\infty}\int e^{i\tau(\sqrt{1+\xi^2}-\lambda+i\sigma)}\hat{M}'(t,\xi)\hat{Z}(\xi)\,d\xi\,d\tau.$$

The first term on the right-hand side of (C.132) provides the first term on the right-hand side of (C.124). Moreover, in the expression of J_1', we can make as many integrations by parts in ξ as we want to get a decaying factor in $\langle\tau\rangle^{-N}$ for any N. This shows that J_1' is $O(\varepsilon^2 t^{-N})$, so may be incorporated to r in (C.124). This concludes the proof. ∎