Appendix D

Action of multilinear operators on Sobolev and Hölder spaces

In Appendix B, we have introduced multilinear operators that generalize the linear operators (B.3). In this appendix, we want to discussed Sobolev boundedness properties of such operators. For linear ones like (B.3), given in terms of symbols satisfying (B.1) with $M(x,\xi)\equiv 1$, such bounds are well known: see for instance Dimassi and Sjöstrand [24]. We generalize these bounds to multilinear operators, under the form

$$\|\mathrm{Op}_h(a)(\underline{v}_1,\dots,\underline{v}_n)\|_{H^s_h} \leq C\sum_{j=1}^{n}\prod_{\ell\neq j}\|\underline{v}_\ell\|_{W_h^{\rho_0,\infty}}\|\underline{v}_j\|_{H^s_h}, \tag{D.1}$$

where $\|\underline{v}\|_{W_h^{\rho_0,\infty}} = \|\langle hD_x\rangle^{\rho_0}\underline{v}\|_{L^\infty}$ and $\|\underline{v}\|_{H^s_h} = \|\langle hD_x\rangle^{s}\underline{v}\|_{L^2}$ with $s\geq 0$ and ρ_0 a large enough number independent of s. Notice that such an estimate is the natural generalization of the standard bound $\|uv\|_{H^s}\leq \|u\|_{L^\infty}\|v\|_{H^s}+\|u\|_{H^s}\|v\|_{L^\infty}$, that holds for any $s\geq 0$, to a framework of multilinear operators more general than the product.

We give also, in the case when the symbol $a(\frac{x}{h},x,\xi_1,\dots,\xi_n)$ in (D.1) is rapidly decaying in $\frac{x}{h}$, other estimates of the form

$$\|\mathrm{Op}_h(a)(\underline{v}_1,\dots,\underline{v}_n)\|_{L^2} \leq Ch\prod_{j=1}^{n-1}\|\underline{v}_j\|_{W_h^{\rho_0,\infty}}(\|\mathcal{L}_\pm\underline{v}_n\|_{L^2}+\|\underline{v}_n\|_{L^2}) \tag{D.2}$$

for any *odd* functions $\underline{v}_1,\dots,\underline{v}_n$, where

$$\mathcal{L}_\pm = x\pm\frac{D_x}{\langle D_x\rangle}.$$

The important point here is that the rapid decay in $\frac{x}{h}$ of the symbol a allows one to gain on the right-hand side a small factor h. We have already explained in Chapter 2 where this gain comes from: The quantity inside the norm on the left-hand side of (D.2) is $h=t^{-1}$ times a generalization of expression (2.64). We have seen that thanks to (2.65), one may express any of the functions $\underline{v}_j$, say $\underline{v}_n$, from $\mathcal{L}_\pm\underline{v}_n$, up to a loss of $\frac{x}{h}$ that is compensated by the rapid decay of a relatively to that variable. Such properties explain why terms like r_1' in (B.8) may be considered somewhat as remainders: they do not involve a factor h in their estimate, but the fact that they decay rapidly in $\frac{x}{h}$ allows one to use (D.2) and thus to recover in that way an $O(h)$ bound.

Let us indicate more precisely what are the Sobolev bounds we shall get with respect to the symbols defined in Appendix B. Recall that we introduced classes of symbols $\tilde{S}_{\kappa,0}(M,p)$, $\tilde{S}'_{\kappa,0}(M,p)$ in Definition 3.1.1 and their (generalized) semiclassical counterparts $S_{\kappa,\beta}(M,p)$, $S'_{\kappa,\beta}(M,p)$ in Definition B.1.2. We shall study first the action of operators associated to the $\tilde{S}_{\kappa,0}(M,p)$, $S_{\kappa,\beta}(M,p)$ classes and then, in the second section of this appendix, the case of operators associated to classes of decaying symbols $\tilde{S}'_{\kappa,0}(M,p)$, $S'_{\kappa,\beta}(M,p)$.

D.1 Action of quantization of non-space-decaying symbols

We introduce the following notation. If $\underline{v}$ is a function depending on the semiclassical parameter $h \in]0,1]$, we set

$$\|\underline{v}\|_{H^s_h} = \|\langle hD_x\rangle^s \underline{v}\|_{L^2} \tag{D.3}$$

for any $s \in \mathbb{R}$. For ρ in $\mathbb{N}$, we define

$$\|\underline{v}\|_{W^{\rho,\infty}_h} = \|\langle hD_x\rangle^\rho \underline{v}\|_{L^\infty}. \tag{D.4}$$

Proposition D.1.1. *Let n be in $\mathbb{N}^*$, κ in $\mathbb{N}$, $\nu \geq 0$. There is ρ_0 in $\mathbb{N}$ such that, for any $\beta \geq 0$, any symbol a in the class $S_{\kappa,\beta}(M_0^\nu, n)$ of Definition B.1.2 (with M_0 given by (B.10)), the following holds true, under the restriction that, for* (i) *and* (ii)*, either $(\kappa,\beta) = (0,0)$ or $0 < \kappa\beta \leq 1$ or $a(y,x,\xi_1,\dots,\xi_n)$ is independent of x:*

(i) *Assume moreover that $a(y,x,\xi_1,\dots,\xi_n)$ is supported in the domain*

$$|\xi_1| + \cdots + |\xi_{n-1}| \leq K(1+|\xi_n|)$$

for some constant K. Then, for any $s \geq 0$, there is $C > 0$ such that, for any test functions $\underline{v}_1,\dots,\underline{v}_n$,

$$\|\mathrm{Op}_h(a)(\underline{v}_1,\dots,\underline{v}_n)\|_{H^s_h} \leq C \prod_{j=1}^{n-1} \|\underline{v}_j\|_{W^{\rho_0,\infty}_h} \|\underline{v}_n\|_{H^s_h} \tag{D.5}$$

uniformly in $h \in]0,1]$.

(ii) *Without any support condition on the symbol, we have instead*

$$\|\mathrm{Op}_h(a)(\underline{v}_1,\dots,\underline{v}_n)\|_{H^s_h} \leq C \sum_{j=1}^{n} \prod_{\ell\neq j} \|\underline{v}_\ell\|_{W^{\rho_0,\infty}_h} \|\underline{v}_j\|_{H^s_h}. \tag{D.6}$$

(iii) *For any $j = 1,\dots,n$, we have also the estimate (without any restriction on (κ,β) or a)*

$$\|\mathrm{Op}_h(a)(\underline{v}_1,\dots,\underline{v}_n)\|_{L^2} \leq C \prod_{\ell\neq j} \|\underline{v}_\ell\|_{W^{\rho_0,\infty}_h} \|\underline{v}_j\|_{L^2}. \tag{D.7}$$

Moreover, the above estimates hold true under a weaker assumption than in Definition B.1.2 of the symbols: namely it is enough to assume that bounds (B.13) hold with $N = 2$ (instead of for all N) for the last exponent in this formula.

Before giving the proof, we establish a lemma.

Lemma D.1.2. *Let a be in the class $S'_{\kappa,0}(M_0^\nu, n)$ of Definition B.1.2 (or more generally a symbol satisfying (B.13) for any $\alpha'_0, \alpha_0, k \in \mathbb{N}, \alpha \in \mathbb{N}^p$, with the last factor replaced by $(1 + M_0^{-\kappa}|y|)^{-2}$). There are ρ_0 in $\mathbb{N}$ depending only on ν, and a family of functions $a_{k_1,\dots,k_{n-1}}(\underline{v}_1, \dots, \underline{v}_{n-1}, y, x, \xi)$ indexed by $(k_1, \dots, k_{n-1}) \in \mathbb{N}^{n-1}$ satisfying bounds*

$$
\begin{aligned}
&|\partial_x^\alpha \partial_\xi^{\alpha'} a_{k_1,\dots,k_{n-1}}(\underline{v}_1, \dots, \underline{v}_{n-1}, y, x, \xi)| \\
&\qquad \le C 2^{-\max(k_1,\dots,k_{n-1})} \langle y \rangle^{-2} \prod_{j=1}^{n-1} \|\underline{v}_j\|_{W_h^{\rho_0,\infty}}
\end{aligned} \tag{D.8}
$$

for $0 \le \alpha, \alpha' \le 2$, such that if we set for any y

$$
\begin{aligned}
&a(y, x, hD_1, \dots, hD_n)(\underline{v}_1, \dots, \underline{v}_{n-1}, \underline{v}_n) \\
&= \frac{1}{(2\pi)^n} \int e^{ix(\xi_1+\dots+\xi_n)} a(y, x, h\xi_1, \dots, h\xi_n) \prod_{j=1}^{n} \hat{\underline{v}}_j(\xi_j)\, d\xi_1 \cdots d\xi_n
\end{aligned} \tag{D.9}
$$

and use a similar notation for $a_{k_1,\dots,k_{n-1}}(\underline{v}_1, \dots, \underline{v}_{n-1}, y, x, hD_x)\underline{v}_n$, then

$$
\begin{aligned}
&a(y, x, hD_1, \dots, hD_n)(\underline{v}_1, \dots, \underline{v}_{n-1}, \underline{v}_n) \\
&= \sum_{k_1=0}^{+\infty} \cdots \sum_{k_{n-1}=0}^{+\infty} a_{k_1,\dots,k_{n-1}}(\underline{v}_1, \dots, \underline{v}_{n-1}, y, x, hD_x)\underline{v}_n.
\end{aligned} \tag{D.10}
$$

Proof. We take a Littlewood–Paley decomposition of the identity, $\mathrm{Id} = \sum_{k=0}^{+\infty} \Delta_k^h$, where $\Delta_0^h = \mathrm{Op}_h(\psi(\xi))$, $\Delta_k^h = \mathrm{Op}_h(\varphi(2^{-k}\xi))$ for $k > 0$, with convenient functions $\psi \in C_0^\infty(\mathbb{R})$, $\varphi \in C_0^\infty(\mathbb{R} - \{0\})$. We also take $\tilde{\psi}$ in $C_0^\infty(\mathbb{R})$, $\tilde{\varphi}$ in $C_0^\infty(\mathbb{R} - \{0\})$ with $\tilde{\psi}\psi = \psi$, $\tilde{\varphi}\varphi = \varphi$. We set $\tilde{\varphi}_k(\xi) = \tilde{\varphi}(2^{-k}\xi)$ for $k > 0$, $\tilde{\varphi}_0(\xi) = \tilde{\psi}(\xi)$. Plugging this decomposition on each factor $\underline{v}_j$, $j = 1, \dots, n-1$ in (D.9), we obtain an expression of the form (D.10) if we define

$$
\begin{aligned}
&a_{k_1,\dots,k_{n-1}}(\underline{v}_1, \dots, \underline{v}_{n-1}, y, x, \xi) \\
&\quad = \frac{1}{(2\pi)^{n-1}} \int e^{ix(\xi_1+\dots+\xi_{n-1})} a(y, x, h\xi_1, \dots, h\xi_{n-1}, \xi) \\
&\qquad \times \prod_{j=1}^{n-1} \tilde{\varphi}_{k_j}(h\xi_j) \widehat{\Delta_{k_j}^h \underline{v}_j}(\xi_j)\, d\xi_1 \cdots d\xi_{n-1}.
\end{aligned} \tag{D.11}
$$

We may rewrite this as

$$
\begin{aligned}
&a_{k_1,\dots,k_{n-1}}(\underline{v}_1, \dots, \underline{v}_{n-1}, y, x, \xi) \\
&\quad = h^{-(n-1)} \int K_{k_1,\dots,k_{n-1}}\Big(y, x, \frac{x - x'_1}{h}, \dots, \frac{x - x'_{n-1}}{h}, \xi\Big) \\
&\qquad \times \prod_{j=1}^{n-1} \Delta_{k_j}^h \underline{v}_j(x'_j)\, dx'_1 \cdots dx'_{n-1}
\end{aligned} \tag{D.12}
$$

with

$$
\begin{aligned}
&K_{k_1,\dots,k_{n-1}}(y,x,z_1,\dots,z_{n-1},\xi)\\
&\quad=\frac{1}{(2\pi)^{n-1}}\int e^{i(z_1\xi_1+\cdots+z_{n-1}\xi_{n-1})}a(y,x,\xi_1,\dots,\xi_{n-1},\xi)\\
&\qquad\times\prod_{j=1}^{n-1}\tilde{\varphi}_{k_j}(\xi_j)\,d\xi_1\cdots d\xi_{n-1}.
\end{aligned}
\tag{D.13}
$$

By the definition of $M_0(\xi_1,\dots,\xi_{n-1},\xi_n)$, on the support of $\prod_{j=1}^{n-1}\tilde{\varphi}_{k_j}(\xi_j)$, one has

$$
M_0(\xi_1,\dots,\xi_{n-1},\xi_n)=O(2^{\hat{k}})\quad\text{if } \hat{k}=\max(k_1,\dots,k_{n-1}).
$$

As a is in the class $S'_{\kappa,0}(M_0^{\nu},n)$, this implies that a in (D.13) is $O(2^{\nu\hat{k}})$. Moreover, if we perform two ∂_{ξ_j}-integrations by parts in (D.13), we gain a factor in $\langle 2^{-\hat{k}\kappa}z_j\rangle^{-2}$ under the integral, for $j=1,\dots,n-1$, according to (B.13). In addition, we have also a decaying factor in $\langle 2^{-\hat{k}\kappa}|y|\rangle^{-2}$. It follows that for $\alpha,\alpha'\le 1$,

$$
\begin{aligned}
&|\partial_x^{\alpha}\partial_{\xi}^{\alpha'}K_{k_1,\dots,k_{n-1}}(y,x,z_1,\dots,z_{n-1},\xi)|\\
&\qquad\le C2^{(\kappa(\alpha+\alpha'+2)+\nu+n-1)\hat{k}}\prod_{j=1}^{n-1}\langle 2^{-\kappa\hat{k}}z_j\rangle^{-2}\langle y\rangle^{-2}.
\end{aligned}
\tag{D.14}
$$

Plugging this estimate in (D.12) and using

$$
|\Delta_{k_j}^{h}\underline{v}_j(x'_j)|\le C2^{-k_j\rho_0}\|\langle hD_x\rangle^{\rho_0}\underline{v}_j\|_{L^\infty}
$$

we see that if ρ_0 has been taken large enough relatively to ν,κ, we get bounds of the form (D.8). This concludes the proof. ∎

Proof of Proposition D.1.1. (i) We reduce first to the case $s=0$. Actually, by Corollary B.2.4, that applies under the restrictions in the statement on (κ,β) or a, the operator

$$
(\underline{v}_1,\dots,\underline{v}_n)\mapsto\langle hD_x\rangle^{s}\mathrm{Op}_h(a)(\underline{v}_1,\dots,\underline{v}_{n-1},\langle hD_x\rangle^{-s}\underline{v}_n)
$$

may be written as $\mathrm{Op}_h(\tilde{a})(\underline{v}_1,\dots,\underline{v}_n)$ for some symbol $\tilde{a}$ in $S_{\kappa,\beta}(M_0^{\nu'},n)$ for some ν' that does not depend on s. It is thus sufficient to show that

$$
\|\mathrm{Op}_h(\tilde{a})(\underline{v}_1,\dots,\underline{v}_n)\|_{L^2}\le C\prod_{j=1}^{n-1}\|\underline{v}_j\|_{W_h^{\rho_0,\infty}}\|\underline{v}_n\|_{L^2}.
\tag{D.15}
$$

By expression (B.14), we have

$$
\begin{aligned}
\mathrm{Op}_h(\tilde{a})(\underline{v}_1,\dots,\underline{v}_n)&=\tilde{a}\Big(\frac{x}{h},x,hD_1,\dots,hD_n\Big)(\underline{v}_1,\dots,\underline{v}_n)\\
&=\tilde{a}(-\infty,x,hD_1,\dots,hD_n)(\underline{v}_1,\dots,\underline{v}_n)\\
&\qquad+\int_{-\infty}^{\frac{x}{h}}(\partial_y\tilde{a})(y,x,hD_1,\dots,hD_n)(\underline{v}_1,\dots,\underline{v}_n)\,dy.
\end{aligned}
\tag{D.16}
$$

As $\partial_y \tilde{a}$ is in $S'_{\kappa,0}(M_0^\nu, n)$ (for some ν), we may apply at any fixed y expansion (D.10) to $\partial_y \tilde{a}$. The symbols $a_{k_1,\dots,k_{n-1}}$ on the right-hand side satisfy (D.8), so that we may apply to them the Calderón–Vaillancourt theorem [9] in the version of Cordes [12], considering $y, \underline{v}_1, \dots, \underline{v}_{n-1}$ as parameters. One gets in that way for any $y, \underline{v}_1, \dots, \underline{v}_n$,

$$\begin{aligned}&\|\partial_y \tilde{a}(y, x, hD_1, \dots, hD_n)(\underline{v}_1, \dots, \underline{v}_n)\|_{L^2}\\ &\le C \sum_{k_1} \cdots \sum_{k_{n-1}} 2^{-\max(k_1,\dots,k_{n-1})} \langle y\rangle^{-2} \prod_{j=1}^{n-1} \|\underline{v}_j\|_{W_h^{\rho_0,\infty}} \|\underline{v}_n\|_{L^2}.\end{aligned} \tag{D.17}$$

The fact that the L^2 norm of the last term in (D.16) is bounded from above by the right-hand side of (D.5) (with $s = 0$) follows from that inequality. If we apply the version of Lemma D.1.2 without parameter y to $\tilde{a}(-\infty, x, \xi_1, \dots, \xi_n)$, we obtain also an inequality of the form (D.17) (without factor $\langle y\rangle^{-2}$ on the right-hand side), which implies for the first term on the right-hand side of (D.16) the wanted estimate. This concludes the proof.

(ii) We just split a as a sum of symbols for which

$$\sum_{\ell \ne j} |\xi_\ell| \le K(1 + |\xi_j|), \quad j = 1, \dots, n,$$

and apply (i) to each of them.

(iii) It is enough to prove (D.7) with $j = n$ for instance. Remember that in the proof of (i), we use that the support condition on a and the restrictions on (κ, β) or a only to reduce the case of H_h^s to L^2 estimates. Once this has been done, inequality (D.15) has been proved without any support condition on $\tilde{a}$, nor on (κ, β), so that it implies (D.7). This concludes the proof, the last statement of the Proposition coming from the fact that Lemma D.1.2 has been proved for symbols satisfying the indicated property and that Corollary B.2.4 used at the beginning of the proof holds also under such a condition. ∎

It will be useful to be able to decompose a symbol belonging to $S_{\kappa,0}(M_0^\nu, n)$ as a sum of a symbol in $S_{\kappa,\beta}(M_0^\nu, n)$ for some small $\beta > 0$ and a symbol whose quantization satisfies better estimates than (D.6) and (D.7). Define

$$\mathcal{L}_\pm = \frac{1}{h} \mathrm{Op}_h(x \pm p'(\xi)). \tag{D.18}$$

Corollary D.1.3. *Let $a(y, x, \xi_1, \dots, \xi_n)$ be in $S_{\kappa,0}(M_0^\nu, n)$ for some $\kappa \ge 0$, some $\nu \ge 0$, some $n \ge 2$. Let $\beta > 0$ (small), $r \in \mathbb{R}_+$. One may decompose $a = a_1 + a_2$, where a_1 is in $S_{\kappa,\beta}(M_0^\nu, n)$ and a_2 is such that if s satisfies $(s - \rho_0 - 1)\beta \ge r + \frac{n+1}{2}$,*

$$\|\mathrm{Op}_h(a_2)(\underline{v}_1, \dots, \underline{v}_n)\|_{H_h^s} \le C h^r \prod_{j=1}^n \|\underline{v}_j\|_{H_h^s}, \tag{D.19}$$

$$\|\mathcal{L}_\pm \mathrm{Op}_h(a_2)(\underline{v}_1, \dots, \underline{v}_n)\|_{L^2} \le C h^r \prod_{j=1}^{n-1} \|\underline{v}_j\|_{H_h^s} (\|\underline{v}_n\|_{L^2} + \|\mathcal{L}_\pm \underline{v}_n\|_{L^2}) \tag{D.20}$$

and

$$\|\mathcal{L}_\pm \mathrm{Op}_h(a_2)(\underline{v}_1,\dots,\underline{v}_n)\|_{L^2} \leq Ch^r \prod_{j=1}^{n-1} \|\underline{v}_j\|_{H_h^s}(\|\underline{v}_n\|_{L^2} + \|\mathcal{L}_\pm \underline{v}_n\|_{W_h^{\rho_0,\infty}}). \quad \text{(D.21)}$$

(In the last two estimates, we could make play the special role devoted to n to any other index).

A similar statement holds replacing classes $S_{\kappa,0}$ (resp. $S_{\kappa,\beta}$) by $S'_{\kappa,0}$ (resp. $S'_{\kappa,\beta}$).

Proof. Take χ in $C_0^\infty(\mathbb{R})$ equal to one close to zero and define $a_1 = a\chi(h^\beta M_0(\xi))$, $a_2 = a(1-\chi)(h^\beta M_0(\xi))$. Then a_1 is in $S_{\kappa,\beta}(M_0^\nu, n)$ as it satisfies (B.12)–(B.13). Let us show that a_2 obeys (D.19)–(D.20). Decomposing a_2 in a sum of several symbols, we may assume for instance that it is supported for $|\xi_1| + \dots + |\xi_{n-1}| \leq K\langle \xi_n \rangle$. Then, by the definition of a_2, there is at least one index j, $1 \leq j \leq n-1$, such that $|\xi_j| \geq ch^{-\beta}$ on the support of a_2, for instance $j = n-1$. Applying (D.5), we get

$$\begin{aligned} &\|\mathrm{Op}_h(a_2)(\underline{v}_1,\dots,\underline{v}_n)\|_{H_h^s} \\ &\quad \leq C \prod_{j=1}^{n-1} \|\underline{v}_j\|_{W_h^{\rho_0,\infty}} \|\mathrm{Op}_h((1-\tilde{\chi})(h^{-\beta}\xi))\underline{v}_{n-1}\|_{W_h^{\rho_0,\infty}} \|\underline{v}_n\|_{H_h^s} \end{aligned} \quad \text{(D.22)}$$

for some new function $\tilde{\chi}$ equal to one close to zero. By semiclassical Sobolev injection,

$$\|\underline{v}_j\|_{W_h^{\rho_0,\infty}} \leq Ch^{-\frac{1}{2}} \|\underline{v}_j\|_{H_h^s}$$

if $s > \rho_0 + \frac{1}{2}$, and

$$\begin{aligned} &\|\mathrm{Op}_h((1-\tilde{\chi})(h^\beta\xi))\underline{v}_{n-1}\|_{W_h^{\rho_0,\infty}} \\ &\quad \leq Ch^{-\frac{1}{2}} \|\mathrm{Op}_h((1-\tilde{\chi})(h^{-\beta}\xi))\underline{v}_{n-1}\|_{H_h^{\rho_0+1}} \\ &\quad \leq Ch^{-\frac{1}{2}+(s-\rho_0-1)\beta} \|\underline{v}_{n-1}\|_{H_h^s}. \end{aligned} \quad \text{(D.23)}$$

If s is as in the statement, we get (D.19).

To obtain (D.20), we notice that

$$\begin{aligned} \mathcal{L}_\pm \mathrm{Op}_h(a_2)(\underline{v}_1,\dots,\underline{v}_n) = &\pm \frac{1}{h}\mathrm{Op}_h(p'(\xi))\mathrm{Op}_h(a_2)(\underline{v}_1,\dots,\underline{v}_n) \\ &+ i\,\mathrm{Op}_h\Big(\frac{\partial a_2}{\partial \xi_n}\Big)(\underline{v}_1,\dots,\underline{v}_n) \\ &+ \mathrm{Op}_h(a_2)\Big(\underline{v}_1,\dots,\underline{v}_{n-1},\frac{x}{h}\underline{v}_n\Big). \end{aligned} \quad \text{(D.24)}$$

The L^2 norm of the first two terms on the right-hand side is bounded from above by $Ch^r \prod_{j=1}^{n-1} \|\underline{v}_j\|_{H_h^s} \|\underline{v}_n\|_{L^2}$ if we use (D.7) and (D.23), for s as in the statement. On the other hand, in the third term, the last argument of $\mathrm{Op}_h(a_2)$ in (D.24) may be written $\mathcal{L}_\pm \underline{v}_n \mp \frac{1}{h}\mathrm{Op}_h(p'(\xi))$, so that we get an upper bound by the right-hand side of (D.20) using again (D.7) and (D.23).

We may also estimate the last term in (D.24) using (D.7), but putting the L^2 norm on $\underline{v}_{n-1}$, i.e. writing

$$\begin{aligned}&\|\mathrm{Op}_h(a_2)(\underline{v}_1,\dots,\underline{v}_{n-1},\mathcal{L}_\pm\underline{v}_n)\|_{L^2}\\&\quad\le C\prod_{j=1}^{n-2}\|\underline{v}_j\|_{W_h^{\rho_0,\infty}}\|\mathrm{Op}_h((1-\tilde{\chi})(h^\beta\xi))\underline{v}_{n-1}\|_{L^2}\|\mathcal{L}_\pm\underline{v}_n\|_{W_h^{\rho_0,\infty}}.\end{aligned}$$

Bounding the last but one factor by $h^{\beta s}\|\underline{v}_{n-1}\|_{H_h^s}$, we get as well (D.21). The last statement of the corollary concerning classes $S'_{\kappa,0}$, $S'_{\kappa,\beta}$ holds in the same way. ∎

Let us state next a corollary of Proposition D.1.1.

Corollary D.1.4. *Let $\nu\ge 0, n\in\mathbb{N}^*$. There is $\rho_0\in\mathbb{N}$ such that for any $\kappa\ge 0$, any $\beta\ge 0$, for any $j=1,\dots,n$, any a in $S_{\kappa,\beta}(M_0^\nu,n)$, there is $C>0$ such that for any $\underline{v}_1,\dots,\underline{v}_n$,*

$$\Big\|\frac{x}{h}\mathrm{Op}_h(a)(\underline{v}_1,\dots,\underline{v}_n)\Big\|_{L^2}\le C\prod_{\ell\ne j}\|\underline{v}_\ell\|_{W_h^{\rho_0,\infty}}(h^{-1}\|\underline{v}_j\|_{L^2}+\|\mathcal{L}_\pm\underline{v}_j\|_{L^2})\quad\text{(D.25)}$$

and for any $j\ne j'$, $1\le j,j'\le n$,

$$\begin{aligned}\Big\|\frac{x}{h}\mathrm{Op}_h(a)(\underline{v}_1,\dots,\underline{v}_n)\Big\|_{L^2}&\le C\Big(\prod_{\ell\ne j,j'}\|\underline{v}_\ell\|_{W_h^{\rho_0,\infty}}\Big)\|\underline{v}_{j'}\|_{L^2}\\&\quad\times\big(h^{-1}\|\underline{v}_j\|_{W_h^{\rho_0,\infty}}+\|\mathcal{L}_\pm\underline{v}_j\|_{W_h^{\rho_0,\infty}}\big).\end{aligned}\quad\text{(D.26)}$$

Proof. Let us prove (D.25) with $j=n$ for instance. By the definition of the quantization

$$\frac{x}{h}\mathrm{Op}_h(a)(\underline{v}_1,\dots,\underline{v}_n)=\mathrm{Op}_h(a)\Big(\underline{v}_1,\dots,\underline{v}_{n-1},\frac{x}{h}\underline{v}_n\Big)+i\,\mathrm{Op}_h\Big(\frac{\partial a}{\partial\xi_n}\Big)(\underline{v}_1,\dots,\underline{v}_n).$$

If we write $\frac{x}{h}=\mathcal{L}_\pm\mp h^{-1}p'(D_x)$, and apply (D.7) with $j=n$, we obtain (D.25). One obtains (D.26) in the same way, applying estimate (D.7) with j replaced by j', and using that $p'(hD_x)$ is bounded from $W_h^{\rho'_0,\infty}$ to $W_h^{\rho_0,\infty}$ if $\rho'_0>\rho_0$. This concludes the proof. ∎

We shall also use some L^∞ estimates.

Proposition D.1.5. *Let $\nu\in[0,+\infty[$, $\kappa\ge 0$, $n\in\mathbb{N}^*$, $\beta\ge 0$. Let $q>1$ and let a be a symbol in $S_{\kappa,\beta}(M_0^\nu\prod_{j=1}^n\langle\xi_j\rangle^{-q},n)$. (It is actually enough to assume that in estimates (B.13), the last exponent N is equal to 2). Assume that $(\kappa,\beta)=(0,0)$ or $0<\kappa\beta\le 1$, or that $a(y,x,\xi)$ is independent of x. Then there are ρ_0 in $\mathbb{N}$ and, for any integer $\rho\ge\rho_0$, a constant $C>0$ such that for any $\underline{v}_1,\dots,\underline{v}_n$,*

$$\|\mathrm{Op}_h(a)(\underline{v}_1,\dots,\underline{v}_n)\|_{W_h^{\rho,\infty}}\le C\prod_{j=1}^n\|\underline{v}_j\|_{W_h^{\rho,\infty}}.\quad\text{(D.27)}$$

If we have just $a \in S_{\kappa\beta}(M_0^\nu \prod_{j=1}^n \langle\xi_j\rangle^{-1}, n)$, *we get for any* r *in* $\mathbb{N}$, *any* $\sigma > 0$, *any* s, ρ *with* $(s-\rho-1)\sigma \geq r + \frac{1}{2}$ *and* $\rho \geq \rho_0$, *the bound*

$$\begin{aligned} &\|\mathrm{Op}_h(a)(\underline{v}_1, \dots, \underline{v}_n)\|_{W_h^{\rho,\infty}} \\ &\quad \leq C h^{-\sigma} \prod_{j=1}^n \|\underline{v}_j\|_{W_h^{\rho,\infty}} + C h^r \sum_{j=1}^n \prod_{\ell \neq j} \|\underline{v}_\ell\|_{W_h^{\rho,\infty}} \|\underline{v}_j\|_{H_h^s}. \end{aligned} \tag{D.28}$$

Proof. One may assume that a is supported for $|\xi_1| + \cdots + |\xi_{n-1}| \leq K(1+|\xi_n|)$. One may use Corollary B.2.4, whose assumptions are satisfied, in order to reduce (D.27) to estimate

$$\|\mathrm{Op}_h(a)(\underline{v}_1, \dots, \underline{v}_n)\|_{L^\infty} \leq C \prod_{j=1}^{n-1} \|\underline{v}_j\|_{W_h^{\rho_0,\infty}} \|\underline{v}_n\|_{L^\infty}. \tag{D.29}$$

We apply (D.16) to reduce (D.29) to bounds of the form

$$\begin{aligned} &\|a(-\infty, x, hD_1, \dots, hD_n)(\underline{v}_1, \dots, \underline{v}_n)\|_{L^\infty} \\ &\quad \leq C \prod_{j=1}^{n-1} \|\underline{v}_j\|_{W_h^{\rho_0,\infty}} \|\underline{v}_n\|_{L^\infty}, \\ &\int_{-\infty}^{+\infty} \|\partial_y a(y, x, hD_1, \dots, hD_n)(\underline{v}_1, \dots, \underline{v}_n)\|_{L^\infty} \\ &\quad \leq C \prod_{j=1}^{n-1} \|\underline{v}_j\|_{W_h^{\rho_0,\infty}} \|\underline{v}_n\|_{L^\infty}. \end{aligned} \tag{D.30}$$

We may decompose $\partial_y a(y, x, hD_1, \dots, hD_n)$ using equality (D.10). Each contribution in the sum is given by a symbol satisfying estimate (D.8), with an extra factor $\langle\xi_n\rangle^{-q}$ on the right-hand side, coming from the fact that our symbol a was in $S_{\kappa,\beta}(M_0^\nu \prod_{j=1}^n \langle\xi_j\rangle^{-q}, n)$. The kernel of the corresponding operator will then be bounded in modulus by

$$C h^{-1} G\Big(\frac{x-x'}{h}\Big) 2^{-\max(k_1, \dots, k_{n-1})} \langle y\rangle^{-2} \prod_{j=1}^{n-1} \|\underline{v}_j\|_{W_h^{\rho_0,\infty}}$$

with some L^1 function G. The second estimate (D.30) follows from that. The first one is proved in the same way.

Finally, to get (D.28), we assume again a supported as above and decompose it as $a = a_1 + a_2$, with $a_1 = a\chi(h^\sigma \xi_n)$ for some $\sigma > 0$ and χ in $C_0^\infty(\mathbb{R})$ equal to one close to zero. Then a_1 is in $h^{-\sigma} S_{\kappa\beta}(M_0^\nu \prod_{j=1}^n \langle\xi_j\rangle^{-2}, n)$ (for a new value of ν), so that (D.27) applies, with a loss $h^{-\sigma}$, which provides the first term on the right-hand side of (D.28). On the other hand, we estimate $\|\mathrm{Op}_h(a_2)(\underline{v}_1, \dots, \underline{v}_n)\|_{W_h^{\rho,\infty}}$ from $C h^{-\frac{1}{2}} \|\mathrm{Op}_h(a_2)(\underline{v}_1, \dots, \underline{v}_n)\|_{H_h^{\rho+1}}$ by semiclassical Sobolev injection, and then this quantity by the last term on the right-hand side of (D.28) with $r = \sigma(s-\rho-1) - \frac{1}{2}$. This concludes the proof. ∎

Let us translate the preceding results in the non-semiclassical case using the transformation Θ_t defined in (B.15) and (B.16)–(B.17). We translate first Proposition D.1.1.

Proposition D.1.6. *Let a be a symbol satisfying the assumptions of Proposition D.1.1 and (κ, β) satisfying also the assumptions of that proposition in the case of statements* (i) *and* (ii) *below (in particular, if a is independent of x, these statements hold for any (κ, β) with $\kappa \geq 0, \beta \geq 0$).*

(i) *If moreover a is supported for $|\xi_1| + \cdots + |\xi_{n-1}| \leq K(1 + |\xi_n|)$, one has for any $s \geq 0$ the bound*

$$\|\mathrm{Op}^t(a)(v_1, \dots, v_n)\|_{H^s} \leq C \prod_{j=1}^{n-1} \|v_j\|_{W^{\rho_0,\infty}} \|v_n\|_{H^s} \tag{D.31}$$

with some ρ_0 independent of s, Op^t being defined in (B.16)*.*

(ii) *Without any support assumption on the symbol of a, one has*

$$\|\mathrm{Op}^t(a)(v_1, \dots, v_n)\|_{H^s} \leq C \sum_{j=1}^{n} \prod_{\ell \neq j} \|v_\ell\|_{W^{\rho_0,\infty}} \|v_j\|_{H^s}. \tag{D.32}$$

(iii) *For any $j = 1, \dots, n$, one has also*

$$\|\mathrm{Op}^t(a)(v_1, \dots, v_n)\|_{L^2} \leq C \prod_{\ell \neq j} \|v_\ell\|_{W^{\rho_0,\infty}} \|v_j\|_{L^2}. \tag{D.33}$$

Proof. One combines Proposition D.1.1, (B.16) and the fact that by (B.15),

$$\|\Theta_t \underline{v}\|_{H^s} = \|\underline{v}\|_{H^s_h}$$

and

$$\|\Theta_t \underline{v}\|_{W^{\rho,\infty}} = h^{\frac{1}{2}} \|\underline{v}\|_{W^{\rho,\infty}_h}$$

if $h = t^{-1}$. ∎

To get non-semiclassical versions of Corollaries D.1.3 and D.1.4, let us notice that by (B.15)

$$L_\pm \Theta_t \underline{v} = \frac{1}{\sqrt{t}} (\mathcal{L}_\pm \underline{v})\Big(\frac{x}{t}\Big)$$

is $\mathcal{L}_\pm$ is defined by (D.18) and

$$L_\pm = x \pm t p'(D_x). \tag{D.34}$$

We have then:

Corollary D.1.7. *Let $a(y, x, \xi_1, \dots, \xi_n)$ be a symbol in $S_{\kappa,0}(M_0^\nu, n)$ for some $\kappa \geq 0$, some $\nu \geq 0$, some $n \geq 2$. Let $\beta > 0$ be small and r in $\mathbb{R}_+$. One may decompose $a = a_1 + a_2$, where a_1 is in $S_{\kappa,\beta}(M_0^\nu, n)$ and a_2 satisfies, if $(s - \rho_0)\beta$ is large* *enough relatively to* r, n,

$$\|\mathrm{Op}^t(a_2)(v_1,\dots,v_n)\|_{H^s} \le Ct^{-r}\prod_{j=1}^{n}\|v_j\|_{H^s},$$

$$\|L_\pm \mathrm{Op}^t(a_2)(v_1,\dots,v_n)\|_{L^2} \le Ct^{-r}\prod_{j=1}^{n-1}\|v_j\|_{H^s}\big(\|v_n\|_{L^2}+\|L_\pm v_n\|_{L^2}\big), \tag{D.35}$$

$$\|L_\pm \mathrm{Op}^t(a_2)(v_1,\dots,v_n)\|_{L^2} \le Ct^{-r}\Big(\prod_{j=1}^{n-1}\|v_j\|_{H^s}\Big)\big(\|v_n\|_{L^2}+\|L_\pm v_n\|_{W^{\rho,\infty}}\big).$$

Moreover, in the last two estimates, one may make play the special role devoted to n *to any other index.*

Proof. Again, we combine (B.15)–(B.16) and the estimates in (D.19)–(D.21) (up to a change of notation for r). ∎

In the same way, we get from Corollary D.1.4:

Corollary D.1.8. *With the notation of Corollary D.1.4, we have*

$$\|x\mathrm{Op}^t(a)(v_1,\dots,v_n)\|_{L^2} \le C\prod_{\ell\neq j}\|v_\ell\|_{W^{\rho_0,\infty}}(t\|v_j\|_{L^2}+\|L_\pm v_j\|_{L^2}) \tag{D.36}$$

for any $1\le j\le n$*. Moreover, for any* $j\neq j'$, $1\le j,j'\le n$,

$$\begin{aligned}&\|x\mathrm{Op}^t(a)(v_1,\dots,v_n)\|_{L^2}\\ &\quad\le C\prod_{\ell\neq j,j'}\|v_\ell\|_{W^{\rho_0,\infty}}\|v_{j'}\|_{L^2}(t\|v_j\|_{W^{\rho_0,\infty}}+\|L_\pm v_j\|_{W^{\rho_0,\infty}}).\end{aligned} \tag{D.37}$$

Finally, it follows from Proposition D.1.5:

Proposition D.1.9. *Under the assumptions and with notation of Proposition D.1.5, one has for* $\rho\ge\rho_0$,

$$\|\mathrm{Op}^t(a)(v_1,\dots,v_n)\|_{W^{\rho,\infty}} \le C\prod_{j=1}^{n}\|v_j\|_{W^{\rho,\infty}} \tag{D.38}$$

if a *is in* $S_{\kappa,\beta}(M_0^\nu\prod_{j=1}^n\langle\xi_j\rangle^{-q},n)$ *for some* $q>1$ *and*

$$\begin{aligned}&\|\mathrm{Op}^t(a)(v_1,\dots,v_n)\|_{W^{\rho,\infty}}\\ &\quad\le Ct^{\sigma}\prod_{j=1}^{n}\|v_j\|_{W^{\rho,\infty}}+Ct^{-r}\sum_{j=1}^{n}\prod_{\ell\neq j}\|v_\ell\|_{W^{\rho,\infty}}\|v_j\|_{H^s}\end{aligned} \tag{D.39}$$

if $q=1$, $\sigma>0$ *and* $(s-\rho)\sigma$ *is large enough relatively to* r.

D.2 Action of quantization of space decaying symbols

In this section we study the action of operators associated to symbols belonging to the classes $S'_{\kappa,\beta}(M_0^\nu, n)$ on Sobolev or Hölder spaces of *odd* functions. The oddness of the functions, together with the fact that elements in the S' class are symbols $a(y, x, \xi)$ rapidly decaying in y, will allow us to re-express the functions $\underline{v}$ on which acts the operator from $h\mathcal{L}_\pm \underline{v}$ (using notation (D.18)), thus gaining a power of h. Actually, it is not necessary that a be rapidly decaying in y, and we shall give statements with less stringent decay assumptions.

Proposition D.2.1. *Let n be in $\mathbb{N}^*$, κ in $\mathbb{N}$, $\nu \geq 0$. There is ρ_0 in $\mathbb{N}$ such that, for any $\beta \geq 0$, any symbol $a(y, x, \xi_1, \dots, \xi_n)$, supported in the domain*

$$|\xi_1| + \cdots + |\xi_{n-1}| \leq K(1 + |\xi_n|)$$

for some constant K, and such that for some ℓ, $1 \leq \ell \leq n-1$, a belongs to the class $S'^{2\ell+2}_{\kappa,\beta}(M_0^\nu, n)$ introduced at the end of Definition B.1.2, with $\kappa \geq 0$ and either $(\kappa, \beta) = (0, 0)$ or $0 < \kappa\beta \leq 1$ or a is independent of x, the following holds true:

(i) *For any $s \geq 0$, any* odd *test functions $\underline{v}_1, \dots, \underline{v}_n$, and any choice of signs $\varepsilon_j \in \{-, +\}$, $j = 1, \dots, \ell$,*

$$\begin{aligned} &\|\mathrm{Op}_h(a)(\underline{v}_1, \dots, \underline{v}_n)\|_{H^s_h} \\ &\quad\leq C h^\ell \prod_{j=1}^{\ell} \big(\|\mathcal{L}_{\varepsilon_j} \underline{v}_j\|_{W_h^{\rho_0,\infty}} + \|\underline{v}_j\|_{W_h^{\rho_0,\infty}}\big) \\ &\qquad\times \prod_{j=\ell+1}^{n-1} \|\underline{v}_j\|_{W_h^{\rho_0,\infty}} \|\underline{v}_n\|_{H^s_h}. \end{aligned} \tag{D.40}$$

(ii) *Assume in addition to the preceding assumptions that $\beta > 0$. Then, for any $0 \leq \ell' \leq \ell$, one has*

$$\begin{aligned} &\|\mathrm{Op}_h(a)(\underline{v}_1, \dots, \underline{v}_n)\|_{H^s_h} \\ &\quad\leq C h^{\ell - \frac{1}{2}\ell' - \sigma(\beta)} \prod_{j=1}^{\ell'} \big(\|\mathcal{L}_{\varepsilon_j} \underline{v}_j\|_{L^2} + \|\underline{v}_j\|_{L^2}\big) \\ &\qquad\times \prod_{j=\ell'+1}^{\ell} \big(\|\mathcal{L}_{\varepsilon_j} \underline{v}_j\|_{W_h^{\rho_0,\infty}} + \|\underline{v}_j\|_{W_h^{\rho_0,\infty}}\big) \\ &\qquad\times \prod_{j=\ell+1}^{n-1} \|\underline{v}_j\|_{W_h^{\rho_0,\infty}} \|\underline{v}_n\|_{H^s_h}, \end{aligned} \tag{D.41}$$

where $\sigma(\beta) > 0$ goes to zero when β goes to zero ($\sigma(\beta) = \ell'(\rho_0 + \frac{1}{2})\beta$ holds).

Proof. We shall prove (i) and (ii) simultaneously. We notice first that, by our support condition on $(\xi_1, \dots, \xi_n)$, $M_0(\xi) \sim 1 + |\xi_1| + \cdots + |\xi_{n-1}|$, so that, up to changing ν, we may study the H_h^s norm of

$$\mathrm{Op}_h(\tilde{a})\big(\mathrm{Op}_h(\langle\xi\rangle^{-1})\underline{v}_1, \dots, \mathrm{Op}_h(\langle\xi\rangle^{-1})\underline{v}_\ell, \underline{v}_{\ell+1}, \dots, \underline{v}_n\big) \tag{D.42}$$

for a new symbol $\tilde{a}$ satisfying the same assumptions as a. Moreover, when $\beta > 0$, this symbol is rapidly decaying in $h^\beta M_0(\xi)$ according to (B.12)–(B.13), so that, modifying again $\tilde{a}$, we rewrite (D.42) as

$$\begin{aligned}\mathrm{Op}_h(\tilde{a})\big(\mathrm{Op}_h(\langle\xi\rangle^{-1}\langle\beta h^\beta\xi\rangle^{-\gamma})\underline{v}_1, \dots, \mathrm{Op}_h(\langle\xi\rangle^{-1}\langle\beta h^\beta\xi\rangle^{-\gamma})\underline{v}_\ell,&\\ \underline{v}_{\ell+1}, \dots, \underline{v}_n\big)&\end{aligned} \tag{D.43}$$

with $\gamma > 0$ to be chosen. We use now that if f is an odd function, we may write

$$f(x) = \frac{x}{2}\int_{-1}^{1} (\partial f)(\mu x)\, d\mu.$$

Consequently, for $j = 1, \dots, \ell$,

$$\mathrm{Op}_h\big(\langle\xi\rangle^{-1}\langle\beta h^\beta\xi\rangle^{-\gamma}\big)\underline{v}_j = \frac{ix}{2h}\int_{-1}^{1}\Big(\mathrm{Op}_h\Big(\langle\beta h^\beta\xi\rangle^{-\gamma}\frac{\xi}{\langle\xi\rangle}\Big)\underline{v}_j\Big)(\mu_j x)\, d\mu_j, \tag{D.44}$$

that we rewrite using (D.18)

$$\begin{aligned}&\mathrm{Op}_h\big(\langle\xi\rangle^{-1}\langle\beta h^\beta\xi\rangle^{-\gamma}\big)\underline{v}_j\\ &\quad = ih\frac{\varepsilon_j}{2}\frac{x}{h}\int_{-1}^{1}\big(\mathrm{Op}_h\big(\langle\beta h^\beta\xi\rangle^{-\gamma}\big)\mathcal{L}_{\varepsilon_j}\underline{v}_j\big)(\mu_j x)\, d\mu_j\\ &\qquad\quad - ih\frac{\varepsilon_j}{2}\frac{x}{h}\int_{-1}^{1}\Big(\mathrm{Op}_h\big(\langle\beta h^\beta\xi\rangle^{-\gamma}\big)\frac{x}{h}\underline{v}_j\Big)(\mu_j x)\, d\mu_j.\end{aligned} \tag{D.45}$$

We may thus write (D.45) as a linear combination of expressions of the form

$$h\Big(\frac{x}{h}\Big)^q\int_{-1}^{1}\mu_j^{q'}V_j(\mu_j x)\, d\mu_j, \tag{D.46}$$

where $q = 0, 1, 2$, $q' \in \mathbb{N}$ and $V_j(x)$ is of the form

$$V_j(x) = \mathrm{Op}_h\big(b_j(\beta h^\beta\xi)\big)\mathcal{L}_{\varepsilon_j}\underline{v}_j \quad \text{or} \quad V_j(x) = \mathrm{Op}_h\big(b_j(\beta h^\beta\xi)\big)\underline{v}_j \tag{D.47}$$

with $|\partial^k b_j(\xi)| = O(\langle\xi\rangle^{-\gamma-k})$. We plug these expressions inside (D.43). We remark that when we commute each factor $\frac{x}{h}$ with $\tilde{a}$, we get again an operator given by a symbol similar to $\tilde{a}$, up to changing ν. Moreover, the $\langle M_0^{-\kappa}y\rangle^{-2\ell-2}$ decay of $\tilde{a}(y, x, \xi)$ that we assume shows that for $q \le 2\ell$, $(\frac{x}{h})^q\tilde{a}(\frac{x}{h}, x, \xi)$ may be written $\tilde{a}_1(\frac{x}{h}, x, \xi)$ with $\tilde{a}_1(y, x, \xi)$ in $S'^{2}_{\kappa,\beta}(M_0^\nu, n)$ (for a new ν). Consequently, we may write (D.43) as a combination of quantities of the form

$$h^\ell \int_{-1}^{1} \cdots \int_{-1}^{1} \mathrm{Op}_h(\tilde{a}_1)\big(V_1(\mu_1 \cdot), \dots, V_\ell(\mu_\ell \cdot), \underline{v}_{\ell+1}, \dots, \underline{v}_n\big) \times P(\mu_1, \dots, \mu_\ell)\, d\mu_1 \cdots d\mu_\ell, \tag{D.48}$$

where V_j are given by (D.47) and P is some polynomial.

If we apply (D.5) (together with the remark at the end of the statement of Proposition D.1.1) and use that $\mathrm{Op}_h(b_j(\beta h^\beta \xi))$ is bounded from $W_h^{\rho_0,\infty}$ to itself, uniformly in h, we obtain (D.40). To prove (D.41), we apply again (D.5) and use that, for factors indexed by $j = 1, \dots, \ell'$, we may write if $\gamma \geq \rho_0 + 1$ and $\beta > 0$

$$\begin{aligned}
\|\mathrm{Op}_h\big(b_j(\beta h^\beta \xi)\big) w\|_{W_h^{\rho_0,\infty}} &= \|\mathrm{Op}_h\big(\langle \xi\rangle^{\rho_0} b_j(\beta h^\beta \xi)\big) w\|_{L^\infty} \\
&\leq C h^{-\frac{1}{2}} \|\mathrm{Op}_h\big(\langle\xi\rangle^{\rho_0} \langle \beta h^\beta \xi\rangle^{-\gamma}\big) w\|_{L^2}^{\frac{1}{2}} \\
&\quad \times \|\mathrm{Op}_h\big(\langle\xi\rangle^{\rho_0} \xi \langle \beta h^\beta \xi\rangle^{-\gamma}\big) w\|_{L^2}^{\frac{1}{2}} \\
&\leq C h^{-\frac{1}{2} - \beta(\rho_0 + \frac{1}{2})} \|w\|_{L^2}
\end{aligned}$$

if $\gamma \geq \rho_0$. This brings (D.41) with $\sigma(\beta) = \ell'(\rho_0 + \frac{1}{2})\beta$. ■

When we want to estimate only the L^2 norms, instead of the H^s ones, we have the following statement:

Proposition D.2.2. *Let n be in $\mathbb{N}^*$, $\kappa \in \mathbb{N}$, $\beta \geq 0$, $\nu \geq 0$. There is $\rho_0 \in \mathbb{N}$ such that, for any symbol a in $S'_{\kappa,\beta}(M_0^\nu \prod_{j=1}^n \langle \xi_j\rangle^{-1}, n)$ and for any odd functions $\underline{v}_1, \dots, \underline{v}_n$, one has the following estimate:*

$$\|\mathrm{Op}_h(a)(\underline{v}_1, \dots, \underline{v}_n)\|_{L^2} \leq C h \prod_{j=1}^{n-1} \|\underline{v}_j\|_{W_h^{\rho_0,\infty}} [\|\mathcal{L}_\pm \underline{v}_n\|_{L^2} + \|\underline{v}_n\|_{L^2}]. \tag{D.49}$$

Moreover, when $n \geq 2$, we have also the bound

$$\begin{aligned}
&\|\mathrm{Op}_h(a)(\underline{v}_1, \dots, \underline{v}_n)\|_{L^2} \\
&\quad \leq C h \prod_{j=1}^{n-2} \|\underline{v}_j\|_{W_h^{\rho_0,\infty}} \big[\|\mathcal{L}_\pm \underline{v}_{n-1}\|_{W_h^{\rho_0,\infty}} + \|\underline{v}_n\|_{W_h^{\rho_0,\infty}}\big] \|\underline{v}_n\|_{L^2}.
\end{aligned} \tag{D.50}$$

Estimate (D.49) (resp. (D.50)) holds as well for n (resp. $(n-1, n)$) replaced by any $j \in \{1, \dots, n\}$ (resp. $j, j' \in \{1, \dots, n\}$, $j \neq j'$). Moreover, it suffices to assume that a is in $S'^4_{\kappa,\beta}(M_0^\nu \prod_{j=1}^n \langle\xi_j\rangle^{-1}, n)$ instead of $a \in S'_{\kappa,\beta}(M_0^\nu \prod_{j=1}^n \langle \xi_j\rangle^{-1}, n)$.

Proof. Because of the assumption on a, we may write

$$\mathrm{Op}_h(a)(\underline{v}_1, \dots, \underline{v}_n) = \mathrm{Op}_h(\tilde{a})(\underline{v}_1, \dots, \underline{v}_{n-1}, \mathrm{Op}_h(\langle\xi\rangle^{-1})\underline{v}_n)) \tag{D.51}$$

with $\tilde{a}$ in $S'_{\kappa,\beta}(M_0^\nu \prod_{j=1}^{n-1}\langle\xi_j\rangle^{-1}, n)$ (or $\tilde{a}$ in $S'^4_{\kappa,\beta}(M_0^\nu \prod_{j=1}^{n-1}\langle\xi_j\rangle^{-1}, n)$). We use next equation (D.45) (with $\gamma = 0$) in order to express $\mathrm{Op}_h(\langle\xi\rangle^{-1})\underline{v}_n$ as a combination of terms of the form (D.46) with $j = n$ and V_n given by (D.47). We obtain thus for (D.51) an expression in terms of integrals

$$h\int_{-1}^{1} \mathrm{Op}_h(\tilde{a}_1)[\underline{v}_1, \dots, \underline{v}_{n-1}, V_n(\mu_n\cdot)]P(\mu_n)\,d\mu_n \tag{D.52}$$

for some polynomial P, some $\tilde{a}_1 \in S'^2_{\kappa,\beta}(M_0^\nu \prod_{j=1}^{n-1}\langle\xi_j\rangle^{-1}, n)$. Applying (D.7), we get (D.49).

To obtain (D.50), we make appear the $\mathrm{Op}_h(\langle\xi\rangle^{-1})$ operator on argument $\underline{v}_{n-1}$ instead of $\underline{v}_n$ in (D.51), use (D.45) with $j = n-1$, obtain an expression of the form (D.52) with the roles of n and $n-1$ interchanged, and apply again (D.7). ■

Let us also establish some corollaries and variants of the above results.

Corollary D.2.3. *Let n, κ, β, ν be as in Proposition D.2.2. Let a be a symbol in the class $S_{\kappa,\beta}(M_0^\nu \prod_{j=1}^{n+1}\langle\xi_j\rangle^{-1}, n+1)$. Let Z be in $\mathcal{S}(\mathbb{R})$. Then for any odd functions $\underline{v}_1, \dots, \underline{v}_n$,*

$$\begin{aligned}&\Big\|\mathrm{Op}_h(a)\Big[Z\Big(\frac{x}{h}\Big), \underline{v}_1, \dots, \underline{v}_n\Big]\Big\|_{L^2}\\ &\quad\le Ch\prod_{j=1}^{n-1}\|\underline{v}_j\|_{W_h^{\rho_0,\infty}}\big(\|\mathcal{L}_\pm \underline{v}_n\|_{L^2} + \|\underline{v}_n\|_{L^2}\big).\end{aligned} \tag{D.53}$$

If $n \ge 2$, we have also

$$\begin{aligned}&\Big\|\mathrm{Op}_h(a)\Big[Z\Big(\frac{x}{h}\Big), \underline{v}_1, \dots, \underline{v}_n\Big]\Big\|_{L^2}\\ &\quad\le Ch\prod_{j=1}^{n-2}\|\underline{v}_j\|_{W_h^{\rho_0,\infty}}\big(\|\mathcal{L}_\pm \underline{v}_{n-1}\|_{W_h^{\rho_0,\infty}} + \|\underline{v}_{n-1}\|_{W_h^{\rho_0,\infty}}\big)\|v_n\|_{L^2}.\end{aligned} \tag{D.54}$$

Proof. We write

$$a(y, x, \xi) = \langle y\rangle^4 \tilde{a}(y, x, \xi).$$

Then, according to the last remark in the statement, Proposition D.2.2 applies to $\tilde{a}$. Moreover, we may write $\mathrm{Op}_h(a)[Z(\frac{x}{h}), \underline{v}_1, \dots, \underline{v}_n]$ as a sum of expressions

$$\Big(\frac{x}{h}\Big)^q \mathrm{Op}_h(\tilde{a})\Big[Z\Big(\frac{x}{h}\Big), \underline{v}_1, \dots, \underline{v}_n\Big], \quad 0 \le q \le 4. \tag{D.55}$$

The commutator

$$\frac{x}{h}\mathrm{Op}_h(\tilde{a})\Big[Z\Big(\frac{x}{h}\Big), \underline{v}_1, \dots, \underline{v}_n\Big] - \mathrm{Op}_h(\tilde{a})\Big[\frac{x}{h}Z\Big(\frac{x}{h}\Big), \underline{v}_1, \dots, \underline{v}_n\Big]$$

is again of the form $\mathrm{Op}_h(\tilde{a}_1)[Z(\frac{x}{h}), \underline{v}_1, \dots, \underline{v}_n]$, for a new symbol satisfying the same assumptions as a, eventually with a different ν. Finally, we express (D.55) as a sum of expressions $\mathrm{Op}_h(\tilde{a}_1)[Z_1(\frac{x}{h}), \underline{v}_1, \dots, \underline{v}_n]$, for new symbols $\tilde{a}_1$ and a new $\mathcal{S}(\mathbb{R})$ function Z_1. If we apply (D.49) (resp. (D.50)), we get (D.53) (resp. (D.54)). ■

We have also the following variant of Proposition D.2.2, that we state only for bilinear operators.

Proposition D.2.4. *Let $\nu, \kappa \geq 0$. There is $\rho_0 \in \mathbb{N}$ such that, for any a in the class $S'_{\kappa,0}(M_0^\nu \prod_{j=1}^2 \langle \xi_j \rangle^{-1}, 2)$, any odd functions $\underline{v}_1, \underline{v}_2$, one has the following estimates:*

$$\begin{aligned} &\|\mathrm{Op}_h(a)(\underline{v}_1, \underline{v}_2)\|_{L^2} \\ &\quad \leq C h^2 \big(\|\mathcal{L}_\pm \underline{v}_1\|_{W_h^{\rho_0,\infty}} + \|\underline{v}_1\|_{W_h^{\rho_0,\infty}} \big) \big(\|\mathcal{L}_\pm \underline{v}_2\|_{L^2} + \|\underline{v}_2\|_{L^2} \big) \end{aligned} \tag{D.56}$$

for any choice of the signs $\pm$ on the right-hand side. The symmetric inequality holds as well.

If moreover s, σ are positive with $s\sigma \geq 2(\rho_0 + 1)$, we get

$$\|\mathrm{Op}_h(a)(\underline{v}_1, \underline{v}_2)\|_{L^2} \leq C h^{\frac{3}{2} - \sigma} \prod_{j=1}^2 \big(\|\mathcal{L}_\pm \underline{v}_j\|_{L^2} + \|\underline{v}_j\|_{H_h^s} \big). \tag{D.57}$$

Proof. To get (D.56), we write

$$\mathrm{Op}_h(a)(\underline{v}_1, \underline{v}_2) = \mathrm{Op}_h(\tilde{a}) \big(\mathrm{Op}_h(\langle \xi \rangle^{-1}) \underline{v}_1, \mathrm{Op}_h(\langle \xi \rangle^{-1}) \underline{v}_2 \big)$$

with some $\tilde{a}$ in $S_{\kappa,0}(M_0^\nu, 2)$. We use next (D.45) (with $\gamma = 0$) for $j = 1, 2$ in order to reduce ourselves to expressions of the form (D.48) with $\ell = 2$. Applying (D.7), we get the conclusion.

To obtain (D.57), we may assume that a is supported for $|\xi_1| \leq 2(1 + |\xi_2|)$ for instance. Let $\beta > 0$, $\chi \in C_0^\infty(\mathbb{R})$, equal to one close to zero and decompose

$$a(y, x, \xi_1, \xi_2) = a(y, x, \xi_1, \xi_2)\chi(h^{-\beta}\xi_1) + a(y, x, \xi_1, \xi_2)(1 - \chi)(h^{-\beta}\xi_1).$$

If we apply (D.7) to the second symbol, we obtain an estimate to the corresponding contribution to (D.57) by

$$C \|\mathrm{Op}_h((1 - \chi)(h^\beta \xi)) \underline{v}_1\|_{W_h^{\rho_0,\infty}} \|\underline{v}_2\|_{L^2}.$$

By semiclassical Sobolev injection, this is bounded from above by

$$C h^{-\frac{1}{2} + \beta(s - \rho_0 - 1)} \|\underline{v}_1\|_{H_h^s} \|\underline{v}_2\|_{L^2},$$

so by the right-hand side of (D.57) if $\beta(s - (\rho_0 + 1)) \geq 2 - \sigma$.

Consider next $\mathrm{Op}_h(a_1)(\underline{v}_1, \underline{v}_2)$ with $a_1 = a\chi(h^{-\beta}\xi_1)$, so that a_1 is in the class $S'_{\kappa,\beta}(M_0^\nu \prod_{j=1}^2 \langle \xi_j \rangle^{-1}, 2)$. Since $\beta > 0$, we may rewrite as in (D.43), $\mathrm{Op}_h(a_1)(\underline{v}_1, \underline{v}_2)$ as

$$\mathrm{Op}_h(\tilde{a}_1) \big[\mathrm{Op}_h\big(\langle \xi \rangle^{-1} \langle h^\beta \xi \rangle^{-\gamma} \big) \underline{v}_1, \mathrm{Op}_h\big(\langle \xi \rangle^{-1} \big) \underline{v}_2 \big]$$

with $\tilde{a}_1$ in $S'^2_{\kappa,\beta}(M_0^\nu, 2)$, hence under form (D.48) with $\ell = 2$, V_1 (resp. V_2) being given by (D.47) with $b_j = O(\langle \xi \rangle^{-\gamma})$ (resp. $O(1)$). Applying (D.7), we get, in view