Appendix E

Wave operators for time dependent potentials

The goal of this chapter is to construct wave operators for some time dependent perturbations of a constant coefficients operator. We consider a reference operator P_0 independent of time, and a perturbation of P_0 of the form $P(t) = P_0 + \mathcal{V}(t)$, given in terms of a time depending potential $\mathcal{V}(t)$. Our goal is to construct a "wave operator" $B(t)$ such that

$$(D_t - P(t))B(t) = B(t)(D_t - P_0). \tag{E.1}$$

We did something similar in Appendix A in the autonomous case, when $\mathcal{V}(t)$ does not depend on time, and is given by a potential smooth and decaying in space. Here, we shall have to consider a potential $\mathcal{V}(t)$ that depends on time. As mentioned in the introduction of Chapter 6, a scalar model for the kind of operators $P(t)$ we want to consider is given by

$$D_t - p(D_x) - t_\varepsilon^{-\frac{1}{2}} \operatorname{Re}\big(c(x)\langle D_x\rangle^{-1} e^{it\frac{\sqrt{3}}{2}}\big), \tag{E.2}$$

where $p(\xi) = \sqrt{1+\xi^2}$ and c is in $\mathcal{S}(\mathbb{R})$. The potential perturbing the autonomous problem is given here in terms of

$$t_\varepsilon^{-\frac{1}{2}} c(x)\langle D_x\rangle^{-1} e^{\pm it\frac{\sqrt{3}}{2}}.$$

As a function of x, this is still a smooth rapidly decaying function, but we have now also t dependence. On the one hand, this time dependence might be considered as an advantage, since it makes the potential smaller and smaller as time growth. On the other side, it makes impossible to use stationary arguments in order to construct wave operators. Of course, there are well known results concerning scattering by time dependent potentials. We refer for instance to the book of Dereziński and Gérard [23], in particular Sections 3.3 and 3.4. Though, these results would not apply to our problem, as they demand better time decay of the potential and of its space derivatives as the one we have in (E.2). We thus have to construct $B(t)$ by hand, composing (E.1) at the left with Fourier transform, at the right with inverse Fourier transform and defining a wave operator through iterated integrals.

E.1 Statement of the result

In order to state the result, we have to introduce some notation.

Definition E.1.1. Let a, b be in $\mathbb{N}$, $m \geq 0$, $\iota \geq 0$. We denote by $\Sigma_{0,0}^{\iota,m}$ the space of functions $(t,\xi,\eta) \mapsto q(t,\xi,\eta)$ defined on $[1,+\infty[\times\mathbb{R}\times\mathbb{R}$, with values in $\mathbb{C}$, that are Lipschitz in time, smooth in (ξ,η), and satisfy for any N in $\mathbb{N}$, any $j = 0, 1$, any

$t \geq 1$, any $(\xi, \eta) \in \mathbb{R}^2$, any $(\alpha, \alpha') \in \mathbb{N}^2$,

$$|\partial_t^j \partial_\xi^\alpha \partial_\eta^{\alpha'} q(t, \eta, \xi)| \leq C_{\alpha\alpha' N} \varepsilon^\iota t^{-m-j} \langle |\xi| - |\eta| \rangle^{-N}. \tag{E.3}$$

We denote by $\Sigma_{a,b}^{\iota,m}$ the space of functions q of the form $q = (\frac{\xi}{\langle \xi \rangle})^a (\frac{\eta}{\langle \eta \rangle})^b q_1$ with q_1 in $\Sigma_{0,0}^{\iota,m}$.

Example. Let us give an example of functions in the preceding class. Let $q = q_{j,(k,\ell)}$, where $q_{j,(k,\ell)}$ is one of the functions defined in Lemma 6.1.1. Assume that these functions are defined and satisfy (6.18) or (6.19) for t in some interval $[1, T]$ with $4 \leq T \leq \varepsilon^{-4+c}$. Extend this function to $[1, +\infty[$ by

$$q(t, \xi, \eta) \mathbb{1}_{t<T} + q(2T - t, \xi, \eta) \mathbb{1}_{t>T} \chi_0\Big(\frac{t}{T}\Big), \tag{E.4}$$

where $\chi_0 \in C^\infty(\mathbb{R})$ is equal to one on $]-\infty, \frac{5}{4}]$ and to zero on $[\frac{7}{4}, +\infty[$. If we denote this extension still by q, we get a Lipschitz function of time on $[1, +\infty[$ that satisfies (6.18) or (6.19) for any $t \geq 1$. Notice that these inequalities imply estimates of the form (E.3) when we take T in (E.4) smaller than ε^{-4+c} for some $c > 0$, so that (E.4) is supported for $t \leq C\varepsilon^{-4+c}$. Actually, writing for any $m \in]0, \frac{1}{2}[$, $t_\varepsilon^{-1/2} \leq t^{-m} \varepsilon^{1-2m}$, it follows from (6.18) that q belongs to $\Sigma_{0,0}^{\iota,m}$ if $\iota = \min(1 - 2m, 3c\theta'/4) > 0$. In the same way, under condition (6.19), we obtain an element of $\Sigma_{0,0}^{\iota,m+1/2}$. The matrix Q_j of Lemma 6.1.1 has thus entries in $\Sigma_{1,1}^{\iota,m}$.

We consider in this section an operator $\mathcal{V}$ defined in the following way. Assume that we are given matrices Q_j with entries in $\Sigma_{0,0}^{\iota,m}$ for $m > 0, \iota > 0$ and $-2 \leq j \leq 2$. Let $\lambda_j = j\frac{\sqrt{3}}{2}$ and define

$$\mathcal{V}(t) = \sum_{j=-2}^{2} e^{i\lambda_j t} K_{Q_j}, \tag{E.5}$$

where, when q is in $\Sigma_{0,0}^{\iota,m}$, and f is a scalar-valued function, $K_q f$ is defined by

$$\widehat{K_q f}(\xi) = \int q(t, \xi, \eta) \hat{f}(\eta)\, d\eta, \tag{E.6}$$

and when Q_j is a 2×2 matrix, and f is $\mathbb{C}^2$-valued, $K_{Q_j} f$ is defined in the natural way. We shall assume also that operator $\mathcal{V}$ satisfies

$$\overline{\mathcal{V}(t)} N_0 = -N_0 \mathcal{V}(t) \tag{E.7}$$

with $N_0 = [\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}]$ (see (6.9)) and that $\mathcal{V}(t)$ preserves the space of odd functions. If

$$P_0 = \begin{bmatrix} p(D_x) & 0 \\ 0 & -p(D_x) \end{bmatrix},$$

we define

$$P(t) = P_0 + \mathcal{V}(t). \tag{E.8}$$

We want to construct a family of operators $B(t)$ so that, for any f in $L^2(\mathbb{R})$ such that $(D_t - P_0)f$ is in $L^2(\mathbb{R})$ for any t,

$$\big(D_t - P(t)\big)B(t)f = B(t)(D_t - P_0)f. \tag{E.9}$$

We shall prove:

Proposition E.1.2. *For any $t \geq 1$, let $\mathcal{V}(t)$ be a bounded operator on $L^2(\mathbb{R})$. Assume that $t \mapsto \mathcal{V}(t)$ is compactly supported and define for any $t \geq 1$, $n \in \mathbb{N}^*$,*

$$B_n(t) = (-i)^n \int \prod_{j=1}^{n} e^{-i\tau_j P_0}\mathcal{V}(t+\tau_j)e^{i\tau_j P_0}\mathbb{1}_{0<\tau_1<\cdots<\tau_n}\, d\tau_1 \cdots d\tau_n, \tag{E.10}$$

where, for non-commuting variables $A_1, \dots, A_n$, $\prod_{j=1}^n A_j$ denotes $A_1 A_2 \cdots A_n$. Set also $B_0(t) = \mathrm{Id}$. Assume that for any f in $L^2(\mathbb{R})$, one may find a sequence $(\alpha_n)_n$ in ℓ^1 such that one has

$$\sup_{t\geq 1} \|B_n(t)f\|_{L^2} \leq \alpha_n. \tag{E.11}$$

Define

$$B(t)f = \sum_{n=0}^{+\infty} B_n(t)f, \tag{E.12}$$

that exists because of our assumptions. Then $B(t)$ solves equation (E.9). Moreover, define $C_0(t) = \mathrm{Id}$ and for n in $\mathbb{N}^*$,

$$C_n(t) = i^n \int \prod_{j=1}^{n} e^{-i\tau_j P_0}\mathcal{V}(t+\tau_j)e^{i\tau_j P_0}\mathbb{1}_{0<\tau_n<\cdots<\tau_1}\, d\tau_1 \cdots d\tau_n. \tag{E.13}$$

If we assume that the analogous of (E.11) holds for C_n, and define then $C(t)$ as in (E.12), one has

$$B(t)C(t) = C(t)B(t) = \mathrm{Id}. \tag{E.14}$$

Proof. Let us denote $A(t,s) = -ie^{-isP_0}\mathcal{V}(t+s)e^{isP_0}$. Then

$$[D_t - D_s, A(t,s)] = [P_0, A(t,s)]$$

and by (E.10)

$$B_n(t) = \int \prod_{j=1}^{n} A(t,\tau_j)\mathbb{1}_{0<\tau_1<\cdots<\tau_n}\, d\tau_1 \cdots d\tau_n \tag{E.15}$$

so that

$$\begin{aligned}
[D_t - P_0, B_n] &= \int (D_{\tau_1} + \cdots + D_{\tau_n})\Big(\prod_{j=1}^{n} A(t,\tau_j)\Big)\mathbb{1}_{0<\tau_1<\cdots<\tau_n}\, d\tau_1 \cdots d\tau_n \\
&= -\int \prod_{j=1}^{n} A(t,\tau_j)(D_{\tau_1} + \cdots + D_{\tau_n})\mathbb{1}_{0<\tau_1<\cdots<\tau_n}\, d\tau_1 \cdots d\tau_n \\
&= iA(t,0)B_{n-1}(t).
\end{aligned}$$

Using (E.8), and making the convention $B_{-1}(t) = 0$, we rewrite this as

$$(D_t - P(t))B_n(t) = B_n(t)(D_t - P_0) - \mathcal{V}(t)(B_n(t) - B_{n-1}(t)).$$

If we denote by $S_n(t) = \sum_{n'=0}^{n} B_{n'}(t)$ the partial sum, we get

$$(D_t - P(t))S_n(t) = S_n(t)(D_t - P_0) - \mathcal{V}(t)B_n(t). \tag{E.16}$$

If we make act this on a function f in $L^2(\mathbb{R})$ such that $(D_t - P_0)f$ is in L^2, we get when n goes to infinity, in view of (E.11) and (E.12), the conclusion (E.9).

We still have to show that $C(t)$ is the inverse of $B(t)$. To this end, let us denote for $j = 0, \dots, n-1$, $\varphi_j(\tau_j, \tau_{j+1}) = \mathbb{1}_{\tau_{j+1} > \tau_j}$ and rewrite the definition of $B_n(t)$ given in (E.15) as

$$B_n(t) = \int \prod_{j=1}^{n} A(t, \tau_j)\chi(\tau_1, \dots, \tau_n) \prod_{j'=1}^{n-1} \varphi_{j'}(\tau_{j'}, \tau_{j'+1})\, d\tau_1 \cdots d\tau_n,$$

where $\chi(\tau_1, \dots, \tau_n) = \prod_{\ell=1}^{n} \mathbb{1}_{0<\tau_\ell}$. In the same way, (E.13) may be written as

$$C_n(t) = (-1)^n \int \prod_{j=1}^{n} A(t, \tau_j)\chi(\tau_1, \dots, \tau_n) \prod_{j'=1}^{n-1} (1 - \varphi_{j'})(\tau_{j'}, \tau_{j'+1})\, d\tau_1 \cdots d\tau_n.$$

We thus get for $1 \le \ell \le n$,

$$C_\ell(t) \circ B_{n-\ell}(t) = (-1)^\ell \int \prod_{j=1}^{n} A(t, \tau_j)\chi(\tau_1, \dots, \tau_n) \prod_{j'=1}^{\ell-1} (1 - \varphi_{j'})(\tau_{j'}, \tau_{j'+1}) \times \prod_{j'=\ell+1}^{n-1} \varphi_{j'}(\tau_{j'}, \tau_{j'+1})\, d\tau_1 \cdots d\tau_n$$

using the convention $\prod_{j=1}^{0} = \prod_{j=n}^{n-1} = 1$. This may be rewritten for $\ell = 1, \dots, n-1$,

$$\begin{aligned} C_\ell(t) \circ B_{n-\ell}(t) = (-1)^\ell \int \prod_{j=1}^{n} A(t, \tau_j)\chi(\tau_1, \dots, \tau_n) \prod_{j'=1}^{\ell} (1 - \varphi_{j'})(\tau_{j'}, \tau_{j'+1}) \\ \times \prod_{j'=\ell+1}^{n-1} \varphi_{j'}(\tau_{j'}, \tau_{j'+1})\, d\tau_1 \cdots d\tau_n \\ - (-1)^{\ell-1} \int \prod_{j=1}^{n} A(t, \tau_j)\chi(\tau_1, \dots, \tau_n) \prod_{j'=1}^{\ell-1} (1 - \varphi_{j'})(\tau_{j'}, \tau_{j'+1}) \\ \times \prod_{j'=\ell}^{n-1} \varphi_{j'}(\tau_{j'}, \tau_{j'+1})\, d\tau_1 \cdots d\tau_n. \end{aligned}$$

It follows that $\sum_{\ell=0}^{n} C_\ell(t)B_{n-\ell}(t) = 0$ when $n \geq 1$, which implies $C(t) \circ B(t) = \mathrm{Id}$. In the same way $B(t) \circ C(t) = \mathrm{Id}$. ■

In the rest of this chapter, we shall show that the preceding proposition may be applied to an operator of the form (E.5), if one makes convenient assumptions on the Q_j. Moreover, we shall obtain for the operators $B(t)$ and $C(t)$ estimates in other spaces than L^2. More precisely, we shall prove the proposition below, where we use the following notation. Set, according to (D.34),

$$L_\pm = x \pm tp'(D_x), \quad L = \begin{bmatrix} L_+ & 0 \\ 0 & L_- \end{bmatrix} \tag{E.17}$$

so that

$$[D_t - P_0, L] = 0. \tag{E.18}$$

In the following sections, we shall prove:

Proposition E.1.3. *Let $B_n(t)$ and $C_n(t)$ be defined respectively by (E.10) and (E.13), in terms of $\mathcal{V}$ given by (E.5) with Q_j a 2×2 matrix of elements of $\Sigma_{1,1}^{\iota,m}$, for some $\iota > 0$ small, some $m \in]0, \frac{1}{2}[$, close to $\frac{1}{2}$. Then for ε small enough, (E.11) and the corresponding inequality for $C_n(t)$ holds, so that*

$$\sum_{n=0}^{+\infty} B_n(t) = B(t) \quad \text{and} \quad \sum_{n=0}^{+\infty} C_n(t) = C(t)$$

are well defined as operators acting on $L^2(\mathbb{R})$. Moreover, the operators $B(t)$, $C(t)$ are bounded on $H^s(\mathbb{R})$ for any $s \geq 0$ and satisfy for small $\delta' > 0$,

$$\begin{aligned} \|B(t) - \mathrm{Id}\|_{\mathcal{L}(H^s)} &\leq C\varepsilon^\iota t^{-m+\delta'+\frac{1}{4}}, \\ \|C(t) - \mathrm{Id}\|_{\mathcal{L}(H^s)} &\leq C\varepsilon^\iota t^{-m+\delta'+\frac{1}{4}}. \end{aligned} \tag{E.19}$$

One may also write for any f in $L^2(\mathbb{R};\mathbb{C}^2)$ such that $Lf \in L^2(\mathbb{R};\mathbb{C}^2)$,

$$L \circ C(t)f = \tilde{C}(t)Lf + \tilde{C}_1(t)f, \tag{E.20}$$

where

$$\|\tilde{C}(t) - \mathrm{Id}\|_{\mathcal{L}(L^2)} \leq C\varepsilon^\iota t^{-m+\delta'+\frac{1}{4}}, \tag{E.21}$$

$$\|\tilde{C}_1(t)\|_{\mathcal{L}(L^2)} \leq C\varepsilon^\iota t^{\frac{1}{2}-m}. \tag{E.22}$$

Moreover, under condition (E.7), one has

$$\overline{B(t)}N_0 = N_0B(t), \quad \overline{C(t)}N_0 = N_0C(t) \tag{E.23}$$

and if $\mathcal{V}(t)$ preserves the space of odd functions, so do $B(t)$ and $C(t)$.

E.2 Technical lemmas

In this section, we prove some technical lemmas that will be used to obtain Proposition E.1.3.

Lemma E.2.1. *For ξ, η, λ real, denote*

$$\phi_{\pm}(\xi, \eta, \lambda) = \langle \xi \rangle \pm \langle \eta \rangle + \lambda. \tag{E.24}$$

There is $C > 0$ such that for any λ in $\mathbb{R}$, any $t \geq 1$,

$$\int_{|\phi_{\pm}(\xi,\eta,\lambda)|<1} \langle t\phi_{\pm}(\xi, \eta, \lambda)\rangle^{-1} \, d\eta \leq Ct^{-\frac{1}{2}}, \tag{E.25}$$

$$\int_{|\phi_{\pm}(\xi,\eta,\lambda)|<1} \langle t\phi_{\pm}(\xi, \eta, \lambda)\rangle^{-1} \frac{|\eta|}{\langle \eta \rangle} \, d\eta \leq Ct^{-1} \log(1+t). \tag{E.26}$$

Proof. We compute first the integrals over the domain $\eta \geq c$ or $\eta \leq -c$ for some constant $c > 0$. On these domains, $\eta \mapsto \zeta = \phi_{\pm}(\xi, \eta, \lambda)$ is a change of variables, whose Jacobian has uniform lower and upper bounds. The corresponding integrals are thus bounded by

$$C \int_{|\zeta|<1} \langle t\zeta \rangle^{-1} \, d\zeta \leq Ct^{-1} \log(t+1).$$

We compute next the integrals for $|\eta| < c$. If c is small enough, we may write on this domain

$$\phi_{\pm}(\xi, \eta, \lambda) = \phi_{\pm}(\xi, 0, \lambda) + g(\eta)^2,$$

where $g(0) = 0$, $g'(0) \neq 0$, so that we may bound the two integrals (E.25) and (E.26), respectively, by

$$C \int_{|\zeta|<c'} \langle \rho + t\zeta^2 \rangle^{-1} \, d\zeta, \quad C \int_{|\zeta|<c'} \langle \rho + t\zeta^2 \rangle^{-1} |\zeta| \, d\zeta,$$

where $c' > 0$ is some constant, and ρ is some real number depending on ξ, λ, t. These two integrals are smaller than the right-hand side of (E.25) and (E.26), respectively, uniformly in ρ. ∎

We study now composition of operators defined by (E.6) from symbols in the classes of Definition E.1.1, and we prove also Sobolev estimates for such operators.

Lemma E.2.2. *The following statements hold.*

(i) *If ℓ is in $\mathbb{N}$, set $\mu(\ell) = \frac{1}{2}$ if $\ell = 0$ and let $\mu(\ell)$ be strictly smaller than 1 if $\ell \geq 1$. Let $N \geq 2$. There is a constant $C > 0$ such that if two functions q_1, q_2 satisfy estimates*

$$\begin{aligned} |q_1(\xi, \eta)| &\leq K_1 \langle |\xi| - |\eta| \rangle^{-N} \left(\frac{|\eta|}{\langle \eta \rangle} \right)^b, \\ |q_2(\xi, \eta)| &\leq K_2 \langle |\xi| - |\eta| \rangle^{-N} \left(\frac{|\xi|}{\langle \xi \rangle} \right)^a, \end{aligned} \tag{E.27}$$

where a, b are in $\{0,1\}$, then the function given by

$$q_3(\xi,\eta) = \int q_1(\xi,\zeta)q_2(\zeta,\eta)\langle t\phi_\pm(\xi,\zeta,\lambda)\rangle^{-1}\,d\zeta \tag{E.28}$$

satisfies

$$|q_3(\xi,\eta)| \le CK_1K_2t^{-\mu(b+a)}\langle|\xi|-|\eta|\rangle^{-N}. \tag{E.29}$$

(ii) *Let s be in $\mathbb{R}_+$, $\delta' > 0$, $N \ge s+2$. There is $C > 0$ such that if a function $(\xi,\eta)\mapsto q(\xi,\eta)$ satisfies*

$$|q(\xi,\eta)| \le K\langle|\xi|-|\eta|\rangle^{-N}\Big(\frac{|\xi|}{\langle\xi\rangle}+\frac{|\eta|}{\langle\eta\rangle}\Big), \tag{E.30}$$

then the operator K_q defined by (E.6) *satisfies*

$$\|K_q\|_{\mathcal{L}(H^s)} \le CKt^{-\frac{3}{4}+\delta'}. \tag{E.31}$$

(iii) *If instead of* (E.30)*, q satisfies*

$$|q(\xi,\eta)| \le K\langle|\xi|-|\eta|\rangle^{-N}\frac{|\xi|}{\langle\xi\rangle}\frac{|\eta|}{\langle\eta\rangle}, \tag{E.32}$$

one gets instead of (E.31)

$$\|K_q\|_{\mathcal{L}(H^s)} \le CKt^{-1+\delta'}. \tag{E.33}$$

Proof. (i) If in (E.28) we integrate for $\phi_\pm(\xi,\zeta,\lambda) \ge 1$, then (E.29) holds trivially, as a consequence of (E.27), with factor t^{-1} instead of $t^{-\mu(b+a)}$. If we integrate for $|\phi_\pm(\xi,\zeta,\lambda)| < 1$, the contribution to q_3 is bounded from above by

$$CK_1K_2\langle|\xi|-|\eta|\rangle^{-N}\int_{|\phi_\pm(\xi,\zeta,\lambda)|<1}\langle t\phi_\pm(\xi,\zeta,\lambda)\rangle^{-1}\Big(\frac{|\zeta|}{\langle\zeta\rangle}\Big)^{a+b}\,d\zeta.$$

Applying Lemma E.2.1, we get (E.29).

(ii) Since $N \ge s+2$, the $\mathcal{L}(H^s)$ estimate is reduced to an $\mathcal{L}(L^2)$ one for $N \ge 2$ using the decay in $\langle|\xi|-|\eta|\rangle$ in (E.30). If the kernel of the operator K_q is cut-off for $|\phi_\pm(\xi,\eta,\lambda)| \ge 1$, then Schur's lemma shows that estimate (E.31) holds with t^{-1} instead of $t^{-\frac{3}{4}+\delta'}$. We have thus to study

$$f \mapsto \int q(\xi,\eta)\langle t\phi_\pm(\xi,\eta,\lambda)\rangle^{-1}\mathbb{1}_{|\phi_\pm(\xi,\eta,\lambda)|<1}f(\eta)\,d\eta.$$

By Schur's lemma and (E.30), the $\mathcal{L}(L^2)$ norm of this operator is bounded from above by

$$CK\Big(\sup_\xi\int\langle|\xi|-|\eta|\rangle^{-N}\langle t\phi_\pm(\xi,\eta,\lambda)\rangle^{-1}\frac{|\eta|}{\langle\eta\rangle}\,d\eta\Big)^{\frac{1}{2}} \times\Big(\sup_\eta\int\langle|\xi|-|\eta|\rangle^{-N}\langle t\phi_\pm(\xi,\eta,\lambda)\rangle^{-1}\,d\xi\Big)^{\frac{1}{2}} \tag{E.34}$$

and by the symmetric quantity. Using (E.25) and (E.26), we get (E.31).

(iii) We make the same reasoning as above, except that (E.34) is now replaced by

$$CK\Big(\sup_\xi \int \langle |\xi|-|\eta|\rangle^{-N} \langle t\phi_\pm(\xi,\eta,\lambda)\rangle^{-1} \frac{|\eta|}{\langle\eta\rangle}\, d\eta\Big)^{\frac{1}{2}} \times \Big(\sup_\eta \int \langle |\xi|-|\eta|\rangle^{-N} \langle t\phi_\pm(\xi,\eta,\lambda)\rangle^{-1} \frac{|\xi|}{\langle\xi\rangle}\, d\xi\Big)^{\frac{1}{2}}.$$

We conclude by (E.26). ∎

Let us define a class that will contain functions obtained from those of Definition E.1.1 by introduction of an extra variable.

Definition E.2.3. We denote by $\widetilde{\Sigma}_{0,0}^{\iota,m,m_0}$ the space of functions

$$(t,v,\xi,\eta) \mapsto q(t,v,\xi,\eta),$$

defined for $t \geq 1$, $v \geq 0$, ξ,η in $\mathbb{R}$, that are Lipschitz and compactly supported in v and satisfy for any N and $j=0,1$,

$$|\partial_v^j q(t,v,\xi,\eta)| \leq C_N \varepsilon^\iota t^{1-m}(1+v)^{-m_0-j} \langle |\xi|-|\eta|\rangle^{-N}. \tag{E.35}$$

For a,b in $\mathbb{N}$, we denote by $\widetilde{\Sigma}_{a,b}^{\iota,m,m_O}$ the space of functions that may be written

$$\Big(\frac{\xi}{\langle\xi\rangle}\Big)^a \Big(\frac{\eta}{\langle\eta\rangle}\Big)^b q_1$$

with q_1 in $\widetilde{\Sigma}_{0,0}^{\iota,m,m_0}$.

We shall also allow q to depend on extra parameters, estimates (E.35) being uniform in these parameters.

Notice that if q belongs to the class $\Sigma_{a,b}^{\iota,m}$ of Definition E.1.1 and is compactly supported in time, then $\tilde{q}(t,v,\xi,\eta) = tq(t(1+v),\xi,\eta)$ is in $\widetilde{\Sigma}_{a,b}^{\iota,m,m_0}$ if $m \geq m_0$.

We shall discuss some operators constructed from functions in $\widetilde{\Sigma}_{a,b}^{\iota,m,m_0}$. In the following discussion, we shall identify operators and their kernels.

Let Q be in $\widetilde{\Sigma}_{a,b}^{\iota,m,m_0} \otimes \mathcal{M}_2(\mathbb{R})$ (i.e. a 2×2 matrix of elements of $\widetilde{\Sigma}_{a,b}^{\iota,m,m_0}$). If λ is in $\mathbb{R}$, we consider the operator from $L^2(\mathbb{R})$ to $L^2(\mathbb{R})$ given at fixed t,v by the kernel in (ξ,η)

$$S(t,v,Q,\lambda) = e^{-itvP_0(\xi)} Q(t,v,\xi,\eta) e^{itv(P_0(\eta)+\lambda)}. \tag{E.36}$$

If we decompose

$$Q(t,v,\xi,\eta) = \sum_{j=1}^{2}\sum_{k=1}^{2} q_{jk}(t,v,\xi,\eta) E_{jk},$$

where

$$E_{jk} = (\delta_j^{j'}\delta_k^{k'})_{1\leq j',k'\leq 2}, \tag{E.37}$$

we may write

$$S(t,v,Q,\lambda)=\sum_{j=1}^{2}\sum_{k=1}^{2}S_{jk}(t,v,Q,\lambda) \tag{E.38}$$

with

$$S_{jk}(t,v,Q,\lambda)=q_{jk}(t,v,\xi,\eta)e^{itv\phi_{jk}(\xi,\eta,\lambda)}E_{jk}, \tag{E.39}$$

where

$$\phi_{jk}(\xi,\eta,\lambda)=(-1)^j p(\xi)-(-1)^k p(\eta)+\lambda. \tag{E.40}$$

We assume given functions Q^ℓ in $\widetilde{\Sigma}^{\iota^\ell,m^\ell,m_0^\ell}_{a^\ell,b^\ell}\otimes\mathcal{M}_2(\mathbb{R})$ and real numbers λ_ℓ for ℓ in $\mathbb{N}^*$. We set

$$\underline{Q}_n=(Q^n,\dots,Q^1),\quad \underline{\lambda}=(\lambda^n,\dots,\lambda^1). \tag{E.41}$$

We define inductively a sequence of operators by their kernels, starting with

$$M_1(t,u,\underline{Q}_1,\underline{\lambda}_1)=\int_u^{+\infty}S(t,v,Q^1,\lambda^1)\,dv \tag{E.42}$$

and for $n\geq 1$,

$$\begin{aligned}M_{n+1}&(t,u,\underline{Q}_{n+1},\underline{\lambda}_{n+1})\\&=\int_u^{+\infty}S(t,v,Q^{n+1},\lambda^{n+1})\circ M_n(t,v,\underline{Q}_n,\underline{\lambda}_n)\,dv.\end{aligned} \tag{E.43}$$

Notice that the above integrals converge since S is compactly supported in v. According to our convention of identification between kernels and operators, we shall set for a function f

$$M_n(t,v,\underline{Q}_n,\underline{\lambda}_n)f(\xi)=\int M_n(t,v,\underline{Q}_n,\underline{\lambda}_n)(\xi,\eta)f(\eta)\,d\eta. \tag{E.44}$$

We shall prove the following estimates:

Lemma E.2.4. *Let m,m_0^n,m_0',ι,a,b satisfy*

$$m_0^n,m_0'>\frac{1}{4},\quad a,b\in\mathbb{N},\ a+b\geq 1,\quad \iota>0,\quad m>0. \tag{E.45}$$

Let Q be in $\widetilde{\Sigma}^{\iota,m,m_0'}_{a,b}\otimes\mathcal{M}_2(\mathbb{R})$, λ in $\mathbb{R}$, and let K_N be the best constant C_N in (E.35) for the entries of Q. In the same way, denote by $K_{N,\ell}$ the best constant in (E.35) for the entries of Q_ℓ, $\ell=1,\dots,n$. There are for any $N\geq 2$, any $\delta'>0$, a constant C_N that does not depend on K_N, $K_{N,\ell}$ and a symbol $\tilde{Q}$ in

$$\widetilde{\Sigma}^{\iota+\iota_n,m+m^n-\frac{1}{2},m_0^n+m_0'-\frac{1}{2}}_{a,b^n}\otimes\mathcal{M}_2(\mathbb{R})$$

if $a^n + b = 0$*, and in*

$$\widetilde{\Sigma}^{\iota+\iota_n, m+m^n-\delta', m_0^n+m_0'-\delta'}_{a,b^n} \otimes \mathcal{M}_2(\mathbb{R})$$

if $a^n + b \geq 1$*, whose* N*-th semi-norm is bounded from above by* $C_N K_N K_{N,n}$*, such that if* $n \geq 1$*,*

$$\begin{aligned}\int_u^{+\infty} & S(t,v,Q,\lambda)\circ M_n(t,v,\underline{Q}_n,\underline{\lambda}_n)\,dv \\ &= \int_u^{+\infty} S(t,v,\tilde{Q},\tilde{\lambda})\circ M_{n-1}(t,v,\underline{Q}_{n-1},\underline{\lambda}_{n-1})\,dv + R_n(t,u),\end{aligned} \tag{E.46}$$

where $\tilde{\lambda} = \lambda + \lambda_n$ *and* R_n *satisfies for any* f *in* $L^2(\mathbb{R})$ *and any* $\delta' > 0$*,*

$$\|\sup_u |R_n(t,u)f|\|_{L^2} \leq CK_2\varepsilon^\iota t^{-m+\frac{1}{4}+\delta'}\|\sup_u |M_n(t,u,\underline{Q}_n,\underline{\lambda}_n)f|\|_{L^2}. \tag{E.47}$$

If $n = 0$*, then* (E.46) *holds as well without the integral term on the right-hand side.*

Proof. In the left-hand side of (E.46) we plug (E.38). Then the kernel of that operator is the sum in j,k, $1 \leq j,k \leq 2$, of

$$\int_u^{+\infty}\int S_{jk}(t,v,Q,\lambda)(\xi,\zeta)M_n(t,v,\underline{Q}_n,\underline{\lambda}_n)(\zeta,\eta)\,d\zeta\,dv. \tag{E.48}$$

Let us define for $1 \leq j,k \leq 2$ the operator

$$\begin{aligned}L_{jk\lambda}(\xi,\zeta) = {}& \langle t(1+v)\phi_{jk}(\xi,\zeta,\lambda)\rangle^{-2} \\ & \times \big(1 + t(1+v)\phi_{jk}(\xi,\zeta,\lambda)(1+v)D_v\big),\end{aligned} \tag{E.49}$$

where we used notation (E.40). Then, by (E.39),

$$\begin{aligned}L_{jk\lambda}S_{jk}(\xi,\zeta) = S_{jk}(\xi,\zeta) + {}& \frac{t(1+v)\phi_{jk}(\xi,\zeta,\lambda)}{\langle t(1+v)\phi_{jk}(\xi,\zeta,\lambda)\rangle^2}(1+v) \\ & \times D_v q_{jk}(t,v,\xi,\zeta,\lambda)e^{itv\phi_{jk}(\xi,\zeta,\lambda)}E_{jk}.\end{aligned} \tag{E.50}$$

We plug the expression of S_{jk} deduced from (E.50) inside (E.48). We obtain on the one hand

$$\begin{aligned}-\int_u^{+\infty}\int & \frac{t(1+v)\phi_{jk}(\xi,\zeta,\lambda)}{\langle t(1+v)\phi_{jk}(\xi,\zeta,\lambda)\rangle^2}(1+v)D_v q_{jk}(t,v,\xi,\zeta,\lambda) \\ & \times e^{itv\phi_{jk}(\xi,\zeta,\lambda)}E_{jk}M_n(t,v,\underline{Q}_n,\underline{\lambda}_n)(\zeta,\eta)\,d\zeta\,dv\end{aligned} \tag{E.51}$$

and on the other hand

$$\int_u^{+\infty}\int L_{jk\lambda}S_{jk}(t,v,Q,\lambda)(\xi,\zeta)M_n(t,v,\underline{Q}_n,\underline{\lambda}_n)(\zeta,\eta)\,d\zeta\,dv. \tag{E.52}$$

Using the expression (E.49) of $L_{jk\lambda}$, we perform in (E.52) one integration by parts in v. We get the following contributions:

$$\int_u^{+\infty} \int \Big(\Big\langle t(1+v)\phi_{jk}(\xi,\zeta,\lambda) \Big\rangle^{-2} - D_v \Big((1+v) \frac{t(1+v)\phi_{jk}(\xi,\zeta,\lambda)}{\langle t(1+v)\phi_{jk}(\xi,\zeta,\lambda)\rangle^2} \Big) \Big) \times S_{jk}(t,v,Q,\lambda)(\xi,\zeta) M_n(t,v,\underline{Q}_n,\underline{\lambda}_n)(\zeta,\eta)\, d\zeta\, dv, \tag{E.53}$$

$$-\int_u^{+\infty} \int \frac{t(1+v)\phi_{jk}(\xi,\zeta,\lambda)}{\langle t(1+v)\phi_{jk}(\xi,\zeta,\lambda)\rangle^2} S_{jk}(t,v,Q,\lambda)(\xi,\zeta) \times (1+v) D_v M_n(t,v,\underline{Q}_n,\underline{\lambda}_n)(\zeta,\eta)\, d\zeta\, dv, \tag{E.54}$$

$$-\frac{1}{i} \int \frac{t(1+u)^2\phi_{jk}(\xi,\zeta,\lambda)}{\langle t(1+u)\phi_{jk}(\xi,\zeta,\lambda)\rangle^2} S_{jk}(t,u,Q,\lambda)(\xi,\zeta) \times M_n(t,u,\underline{Q}_n,\underline{\lambda}_n)(\zeta,\eta)\, d\zeta. \tag{E.55}$$

Let us show that (E.51), (E.53), (E.54), (E.55) may be written as contributions to the right-hand side of (E.46).

Contributions of (E.51) *and* (E.53). We make act (E.51) and (E.53) on a function f. We shall get an expression

$$\int_u^{+\infty} \int K(v,\xi,\zeta) \big(M_n(t,v,\underline{Q}_n,\underline{\lambda}_n) f \big)(\zeta)\, d\zeta\, dv, \tag{E.56}$$

where, by the fact that q_{jk} in (E.39) is in $\widetilde{\Sigma}^{\iota,m,m_0'}_{a,b}$ and (E.35), the kernel K satisfies the bound

$$|K(v,\xi,\zeta)| \leq CK_2 \langle t(1+v)\phi_{jk}(\xi,\zeta,\lambda)\rangle^{-1} \Big(\frac{|\xi|}{\langle\xi\rangle}\Big)^a \Big(\frac{|\zeta|}{\langle\zeta\rangle}\Big)^b \times \varepsilon^\iota t^{1-m} (1+v)^{-m_0'} \langle |\xi| - |\eta| \rangle^{-2}. \tag{E.57}$$

We bound the modulus of (E.56) by

$$\int_0^{+\infty} \int |K(v,\xi,\zeta)| \big(\sup_w |M_n(t,w,\underline{Q}_n,\underline{\lambda}_n) f(\zeta)| \big)\, d\zeta\, dv.$$

Then the L^2 norm in ξ of the supremum in u of (E.56) is bounded from above by

$$\int_0^{+\infty} \Big\| \int |K(v,\xi,\zeta)| \big(\sup_w |M_n(t,w,\underline{Q}_n,\underline{\lambda}_n) f(\zeta)| \big)\, d\zeta \Big\|_{L^2(d\xi)} dv. \tag{E.58}$$

As $a+b \geq 1$, (E.57) shows that we may apply to the $d\zeta$-integral, which is of the form of the right-hand side of (E.30), estimate (E.31), with t replaced by $t(1+v)$. We obtain that (E.58) is smaller than

$$CK_2 \int_0^{+\infty} \varepsilon^\iota t^{\frac{1}{4}-m+\delta'} (1+v)^{-m_0'-\frac{3}{4}+\delta'}\, dv \big\| \sup_w |M_n(t,w,\underline{Q}_n,\underline{\lambda}_n) f| \big\|_{L^2}$$

with $\delta' > 0$ as small as we want. Since by assumption $m_0' > \frac{1}{4}$, we obtain a bound of the form (E.47), that shows that (E.51) and (E.53) contribute to R_n in (E.46).

Contribution of (E.55). This is an expression similar to (E.53), except that we no not have a dv integral and have a factor $(1+u)^2$ instead of $(1+v)$. Consequently, for the L^2 norm of that operator acting on f, we get a bound of the form (E.58) but without dv-integration and an extra factor $(1+u)$, and with K estimated at u instead of v. This implies again that we obtain a contribution to R_n.

Contribution of (E.54). By (E.43) at order $n-1$,

$$D_v M_n(t, v, \underline{Q}_n, \underline{\lambda}_n) = iS(t, v, Q^n, \lambda^n) \circ M_{n-1}(t, v, \underline{Q}_{n-1}, \underline{\lambda}_{n-1}).$$

Plugging this in (E.54), we get the expression

$$\begin{aligned} -i \int_u^{+\infty} \iint & \frac{t(1+v)\phi_{jk}(\xi,\zeta,\lambda)}{\langle t(1+v)\phi_{jk}(\xi,\zeta,\lambda)\rangle^2} S_{jk}(t,v,Q,\lambda)(\xi,\zeta) \\ & \times (1+v)S(t,v,Q^n,\lambda^n)(\zeta,\eta')M_{n-1}(t,v,\underline{Q}_{n-1},\underline{\lambda}_{n-1})(\eta',\eta)\,d\zeta\,d\eta'\,dv. \end{aligned} \tag{E.59}$$

We write by (E.38)

$$S(t, v, Q^n, \lambda^n) = \sum_{k'=1}^{2}\sum_{\ell=1}^{2} S_{k'\ell}(t, v, Q^n, \lambda^n).$$

By (E.39) and the fact that $E_{jk}E_{k'\ell} = \delta_k^{k'} E_{j\ell}$, we have

$$\begin{aligned} & \sum_{k'=1}^{2} S_{jk}(t,v,Q,\lambda)(\xi,\zeta)S_{k',\ell}(t,v,Q^n,\lambda^n)(\zeta,\eta') \\ & \quad = q_{jk}(t,v,\xi,\zeta)q^n_{k\ell}(t,v,\zeta,\eta')e^{itv\phi_{jk}(\xi,\zeta,\lambda)+itv\phi_{k\ell}(\zeta,\eta',\lambda_n)}E_{j\ell}, \end{aligned} \tag{E.60}$$

where $q^n_{k\ell}$ denote the entries of matrix Q^n. By (E.40), the phase in the exponential is $\phi_{j\ell}(\xi,\eta',\lambda+\lambda^n)$. Define

$$\begin{aligned} \tilde{q}_{j\ell}(t,v,\xi,\eta',\lambda) = -i(1+v)\int \sum_{k=1}^{2} & q_{jk}(t,v,\xi,\zeta)q^n_{k\ell}(t,v,\zeta,\eta') \\ & \times t(1+v)\phi_{jk}(\xi,\zeta,\lambda)\langle t(1+v)\phi_{jk}(\xi,\eta,\lambda)\rangle^{-2}\,d\zeta. \end{aligned} \tag{E.61}$$

Since q_{jk} is in $\widetilde{\Sigma}^{\iota,m,m'_0}_{a,b}$, estimate (E.35) shows that we may write this function as $(\frac{\xi}{\langle\xi\rangle})^a$ multiplied by a function that will satisfy the first estimate (E.27), with K_1 bounded by $\varepsilon^\iota t^{1-m}(1+v)^{-m'_0}$. In the same way, since $q^n_{k\ell}$ is in $\widetilde{\Sigma}^{\iota^n,m^n,m^n_0}_{a^n,b^n}$, it may be written as $(\frac{\eta'}{\langle\eta'\rangle})^{b^n}$ times a function satisfying the second estimate (E.27), with a replaced by a^n and K_2 bounded by $\varepsilon^{\iota^n} t^{1-m_n}(1+v)^{-m^n_0}$. By (i) of Lemma E.2.2, applied with t replaced by $t(1+v)$, we see that (E.61) may be written as a product of $(\frac{\xi}{\langle\xi\rangle})^a(\frac{\eta'}{\langle\eta'\rangle})^{b^n}$ times a quantity bounded from above by

$$CK_N K_{N,n}\varepsilon^{\iota+\iota^n} t^{\frac{3}{2}-m-m^n}(1+v)^{\frac{1}{2}-m^n_0-m'_0}\langle|\xi|-|\eta'|\rangle^{-N}$$

if $b + a^n = 0$ and by

$$CK_N K_{N,n} \varepsilon^{\iota+\iota^n} t^{1-m-m^n+\delta'} (1+v)^{-m_0^n - m_0' + \delta'} \langle |\xi| - |\eta'| \rangle^{-N}$$

for any $\delta' > 0$ if $b + a^n \geq 1$, according to (E.29).

If one takes a ∂_v-derivative of (E.61), one gains an extra decay factor in $(1+v)^{-1}$. Consequently, equation (E.61) defines a symbol in the class $\widetilde{\Sigma}_{a,b^n}^{\iota+\iota^n, m+m^n-\frac{1}{2}, m_0^n+m_0'-\frac{1}{2}}$ (resp. in the class $\widetilde{\Sigma}_{a,b^n}^{\iota+\iota^n, m+m^n-\delta', m_0^n+m_0'-\delta'}$) if $b + a^n = 0$ (resp. $b + a^n \geq 1$). Since the phases in equation (E.60) satisfy

$$\phi_{jk}(\xi, \zeta, \lambda) + \phi_{k\ell}(\zeta, \eta', \lambda^n) = \phi_{j\ell}(\xi, \eta', \lambda + \lambda^n),$$

this shows that (E.59) may be written under the form of the first integral on the right-hand side of (E.46), with a matrix function $\tilde{Q}$, depending on λ, but with estimates uniform in λ, whose entries are respectively in the classes of the statement of the lemma. This concludes the proof as, in the case $n = 0$, one has just to estimate terms of the form (E.51), (E.53), (E.55). ■

Our next goal will be to obtain bounds for (E.43) iterating (E.46). We introduce some notation.

Let p, n be in $\mathbb{N}^*$. Assume given for each (n, p) a sequence $(X_{(n,p)}^j)_{1 \leq j \leq n}$, where $X_{(n,p)}^j$ is an element

$$X_{(n,p)}^j = \left(\iota_{(n,p)}^j, m_{(n,p)}^j, m_{(n,p),0}^j, a_{(n,p)}^j, b_{(n,p)}^j\right) \tag{E.62}$$

of $]0, +\infty[\, \times \,]\frac{1}{4}, +\infty[\, \times \,]\frac{1}{4}, +\infty[\, \times \mathbb{N} \times \mathbb{N}$ satisfying the following conditions:

$$\begin{aligned} &\text{If } p \leq n, \text{ then } m_{(n,p),0}^j > \frac{3}{8}, \ j = 1, \dots, n. \\ &\text{If } p \geq n+1, \text{ then } m_{(n,p),0}^j > \frac{3}{8}, \ j = 1, \dots, n-1, \text{ and } m_{(n,p),0}^n > \frac{1}{4}. \end{aligned} \tag{E.63}$$

$$\begin{aligned} &\text{For } 1 \leq j', j'' \leq n, a_{(n,p)}^{j'} + b_{(n,p)}^{j''} \geq 1 \text{ except eventually if} \\ &j' < j'' = p \text{ (this exception being void if } p > n \text{ or } p = 1). \end{aligned} \tag{E.64}$$

For any $X_{(n,p)}^j$ of the form (E.62), we denote for short by $\widetilde{\Sigma}(X_{(n,p)}^j)$ the class

$$\widetilde{\Sigma}(X_{(n,p)}^j) = \widetilde{\Sigma}_{a_{(n,p)}^j, b_{(n,p)}^j}^{\iota_{(n,p)}^j, m_{(n,p)}^j, m_{(n,p),0}^j}$$

of Definition E.2.3.

If $(X_{(n+1,p)}^j)_{1 \leq j \leq n+1}$ is a sequence of the form (E.62), we define from it the concatenated sequence $(X_{(n,p)}^{j,\mathrm{C}})_{1 \leq j \leq n}$ and the truncated sequence $(X_{(n,p)}^{j,\mathrm{T}})_{1 \leq j \leq n}$ in the following way: We just set

$$X_{(n,p)}^{j,\mathrm{T}} = X_{(n+1,p)}^j, \quad j = 1, \dots, n, \tag{E.65}$$

while we denote

$$X_{(n,p)}^{j,\mathrm{C}} = \left(\iota_{(n,p)}^{j,\mathrm{C}}, m_{(n,p)}^{j,\mathrm{C}}, m_{(n,p),0}^{j,\mathrm{C}}, a_{(n,p)}^{j,\mathrm{C}}, b_{(n,p)}^{j,\mathrm{C}}\right),$$

where the components of the preceding vector are defined in the following way:

$$\iota^{n,C}_{(n,p)} = \iota^{n+1}_{(n+1,p)} + \iota^{n}_{(n+1,p)}, \quad \iota^{j,C}_{(n,p)} = \iota^{j}_{(n+1,p)}, \quad j = 1, \dots, n-1. \tag{E.66}$$

If $n \neq p-1$, we set for $j = 1, \dots, n-1$,

$$\begin{aligned} m^{n,C}_{(n,p)} &= m^{n+1}_{(n+1,p)} + m^{n}_{(n+1,p)} - \delta', & m^{j,C}_{(n,p)} &= m^{j}_{(n+1,p)}, \\ m^{n,C}_{(n,p),0} &= m^{n+1}_{(n+1,p),0} + m^{n}_{(n+1,p),0} - \delta', & m^{j,C}_{(n,p),0} &= m^{j}_{(n+1,p),0}, \end{aligned} \tag{E.67}$$

where $\delta' > 0$ is as small as wanted (in particular, δ' will be small enough so that the lower bound (E.63) still holds with $m^{j}_{(n,p),0}$ replaced by $m^{j}_{(n,p),0} - \delta'$).

If $n = p-1$, we define instead of (E.67), for $j = 1, \dots, p-2$,

$$\begin{aligned} m^{p-1,C}_{(p-1,p)} &= m^{p}_{(p,p)} + m^{p-1}_{(p,p)} - \frac{1}{2}, & m^{j,C}_{(p-1,p)} &= m^{j}_{(p,p)}, \\ m^{p-1,C}_{(p-1,p),0} &= m^{p}_{(p,p),0} + m^{p-1}_{(p,p),0} - \frac{1}{2}, & m^{j,C}_{(p-1,p),0} &= m^{j}_{(p,p),0}. \end{aligned} \tag{E.68}$$

Finally, we set for all (n, p),

$$\begin{aligned} a^{n,C}_{(n,p)} &= a^{n+1}_{(n+1,p)}, & b^{n,C}_{(n,p)} &= b^{n}_{(n+1,p)}, \\ a^{j,C}_{(n,p)} &= a^{j}_{(n+1,p)}, & b^{j,C}_{(n,p)} &= b^{j}_{(n+1,p)}, \quad j = 1, \dots, n-1. \end{aligned} \tag{E.69}$$

Let us check that if the sequence $(X^{j}_{(n+1,p)})_{1 \le j \le n+1}$ satisfies (E.63)–(E.64) (with n replaced by $n+1$), then $(X^{j,C}_{(n,p)})_{1 \le j \le n}$ satisfies also (E.63)–(E.64).

Verification of condition (E.63).

Case $p \le n$. As $n \neq p-1$, (E.67) applies and shows that

$$m^{j,C}_{(n,p),0} = m^{j}_{(n+1,p),0}$$

for $j = 1, \dots, n-1$. On the other hand, by (E.63) with n replaced by $n+1$,

$$m^{j}_{(n+1,p),0} > \frac{3}{8},$$

so that the first condition (E.63) holds for $m^{j,C}_{(n,p),0}$ if $j = 1, \dots, n-1$. To get it for $m^{n,C}_{(n,p),0}$, we write by (E.67) that

$$m^{n,C}_{(n,p),0} = m^{n+1}_{(n+1,p),0} + m^{n}_{(n+1,p),0} - \delta' > \frac{3}{8} + \frac{3}{8} - \delta' > \frac{3}{8}$$

using the first line in (E.63) with n replaced by $n+1$.

Case $p = n+1$. By (E.68), we have

$$m^{j,C}_{(p-1,p),0} = m^{j}_{(p,p),0}$$

for $j = 1, \dots, p-2$, and by the first line in (E.63) (with n replaced by $n+1 = p$), this is strictly larger than $\frac{3}{8}$, so that the second line of (E.63) holds for $m^{j,\mathrm{C}}_{(p-1,p),0}$, $j = 1, \dots, p-2$. On the other hand, still by (E.68)

$$m^{p-1,\mathrm{C}}_{(p-1,p),0} = m^{p}_{(p,p),0} + m^{p-1}_{(p,p),0} - \frac{1}{2} > \frac{3}{8} + \frac{3}{8} - \frac{1}{2} = \frac{1}{4}$$

so that the last condition (E.63) holds for $m^{p-1,\mathrm{C}}_{(p-1,p),0}$. We thus got (E.63) for $m^{j,\mathrm{C}}_{(n,p),0}$ when $n = p-1$.

Case $p \geq n+2$. Again, we may apply (E.67) to write for $j = 1, \dots, n-1$,

$$m^{j,\mathrm{C}}_{(n,p),0} = m^{j}_{(n+1,p),0} > \frac{3}{8}$$

by the second condition of (E.63) with n replaced by $n+1$. On the other hand, still by (E.67)

$$m^{n,\mathrm{C}}_{(n,p),0} = m^{n+1}_{(n+1,p),0} + m^{n}_{(n+1,p),0} - \delta' > \frac{1}{4} + \frac{3}{8} - \delta' > \frac{3}{8}$$

using (E.63) with n replaced by $n+1$. This is better than what we need to ensure the last condition (E.63) for $m^{n,\mathrm{C}}_{(n,p),0}$. This concludes the verification.

Verification of (E.64). We assume that (E.64) holds at rank $n+1$, i.e.

> For $1 \leq j', j'' \leq n+1$, $a^{j'}_{(n+1,p)} + b^{j''}_{(n+1,p)} \geq 1$ except eventually if $j' < j'' = p$.

Let us check (E.64) for $a^{j',\mathrm{C}}_{(n,p)}$, $b^{j'',\mathrm{C}}_{(n,p)}$. If both j' and j'' are strictly smaller than n, then (E.69) shows that the wanted property holds. On the other hand, if $j'' \leq n$, $j' < n$, then

$$a^{j',\mathrm{C}}_{(n,p)} + b^{j'',\mathrm{C}}_{(n,p)} = a^{j'}_{(n+1,p)} + b^{j''}_{(n+1,p)}$$

by (E.69), and this expression is larger than or equal to one, except eventually if $j' < j'' = p$, whence again (E.64). It remains to study the case $j' = n$. We have then

$$a^{n,\mathrm{C}}_{(n,p)} + b^{j'',\mathrm{C}}_{(n,p)} = a^{n+1}_{(n+1,p)} + b^{j''}_{(n+1,p)}.$$

The inequality $n+1 < j'' = p$ cannot hold, so that the above quantity is always larger than or equal to one. This shows that (E.64) is satisfied by $(X^{j,\mathrm{C}}_{(n,p)})_{1 \leq j \leq n}$.

We may state our main proposition.

Proposition E.2.5. *Let n be in $\mathbb{N}$, p in $\mathbb{N}^*$. Assume a sequence $(X^{j}_{(n+1,p)})_{1 \leq j \leq n+1}$ of the form (E.62) is given, satisfying (E.63) and (E.64), with n replaced by $n+1$. For $j = 1, \dots, n+1$, let $Q^{j}_{(n+1,p)}$ be an element of $\widetilde{\Sigma}(X^{j}_{(n+1,p)}) \otimes \mathcal{M}_2(\mathbb{R})$. Denote by $K^{j}_{(n+1,p)}$ the semi-norm provided by the best constant in estimate (E.35), in the special case $N = 2$. Set as in (E.41),*

$$\underline{Q}_{n+1} = (Q^{n+1}_{(n+1,p)}, \dots, Q^{1}_{(n+1,p)}).$$

Then there exists a universal constant C_0 *such that, for any function* f *in* L^2*, any* $\underline{\lambda}_{n+1} = (\lambda^{n+1}, \dots, \lambda^1)$ *in* $\mathbb{R}^{n+1}$*, one has when* $p > n+1$ *or* $p = 1$ *the bounds*

$$\begin{aligned} &\big\| \sup_{u>0} |M_{n+1}(t,u,\underline{Q}_{n+1},\underline{\lambda}_{n+1}) f| \big\|_{L^2} \\ &\quad \leq C_0^{n+1} \underline{K}_{(n+1,p)} \varepsilon^{\underline{\iota}_{(n+1,p)}} t^{-\underline{m}_{(n+1,p)}} \|f\|_{L^2}, \end{aligned} \tag{E.70}$$

where

$$\begin{aligned} \underline{\iota}_{(n+1,p)} &= \sum_{j=1}^{n+1} \iota^j_{(n+1,p)}, \\ \underline{m}_{(n+1,p)} &= \sum_{j=1}^{n+1} m^j_{(n+1,p)} - (n+1)\Big(\delta' + \frac{1}{4}\Big), \\ \underline{K}_{(n+1,p)} &= K^1_{(n+1,p)} \cdots K^{n+1}_{(n+1,p)}, \end{aligned} \tag{E.71}$$

while if $2 \leq p \leq n+1$*, one gets instead*

$$\begin{aligned} &\big\| \sup_{u>0} |M_{n+1}(t,u,\underline{Q}_{n+1},\underline{\lambda}_{n+1}) f| \big\|_{L^2} \\ &\quad \leq C_0^{n+1} \underline{K}_{(n+1,p)} \varepsilon^{\underline{\iota}_{(n+1,p)}} t^{-\underline{m}_{(n+1,p)} + \frac{1}{2} - (\delta' + \frac{1}{4})} \|f\|_{L^2}. \end{aligned} \tag{E.72}$$

The proposition will be deduced from the following lemma.

Lemma E.2.6. *Let* $\underline{Q}_{n+1}$ *be as in the statement of Proposition E.2.5. There are* $C > 0$*, a sequence*

$$\underline{Q}^{\mathrm{T}}_n = (Q^{j,\mathrm{T}}_{(n,p)})_{1 \leq j \leq n},$$

with $Q^{j,\mathrm{T}}_{(n,p)}$ *in* $\widetilde{\Sigma}(X^{j,\mathrm{T}}_{(n,p)}) \otimes \mathcal{M}_2(\mathbb{R})$ *with semi-norms* $K^{j,\mathrm{T}}_{(n,p)}$ *satisfying*

$$K^{j,\mathrm{T}}_{(n,p)} \leq K^j_{(n+1,p)}, \tag{E.73}$$

a sequence

$$\underline{Q}^{\mathrm{C}}_n = (Q^{j,\mathrm{C}}_{(n,p)})_{1 \leq j \leq n},,$$

with $Q^{j,\mathrm{C}}_{(n,p)}$ *in* $\widetilde{\Sigma}(X^{j,\mathrm{C}}_{(n,p)}) \otimes \mathcal{M}_2(\mathbb{R})$ *and semi-norms* $K^{j,\mathrm{C}}_{(n,p)}$ *satisfying*

$$\begin{aligned} K^{j,\mathrm{C}}_{(n,p)} &\leq K^j_{(n+1,p)}, \quad j = 1, \dots, n-1, \\ K^{n,\mathrm{C}}_{(n,p)} &\leq C K^n_{(n+1,p)} K^{n+1}_{(n+1,p)}, \end{aligned} \tag{E.74}$$

such that

$$\begin{aligned} &\big\| \sup_{u>0} |M_{n+1}(t,u,\underline{Q}_{n+1},\underline{\lambda}_{n+1}) f| \big\|_{L^2} \\ &\quad \leq \big\| \sup_{u>0} |M_n(t,u,\underline{Q}^{\mathrm{C}}_n,\underline{\lambda}^{\mathrm{C}}_n) f| \big\|_{L^2} \\ &\qquad + C t^{-m^{n+1}_{(n+1,p)} + \frac{1}{4} + \delta'} \varepsilon^{\iota^{n+1}_{(n+1,p)}} K^{n+1}_{(n+1,p)} \\ &\qquad\quad \times \big\| \sup_{u>0} |M_n(t,u,\underline{Q}^{\mathrm{T}}_n,\underline{\lambda}^{\mathrm{T}}_n) f| \big\|_{L^2} \end{aligned} \tag{E.75}$$

for other sequences of real numbers $\underline{\lambda}^{\mathrm{C}}_n, \underline{\lambda}^{\mathrm{T}}_n$.

Proof. We apply Lemma E.2.4 with

$$\underline{Q}_n = (Q^n_{(n+1,p)}, \ldots, Q^1_{(n+1,p)}),$$
$$Q = Q^{n+1}_{(n+1,p)},$$
$$\underline{Q}_{n-1} = (Q^{n-1}_{(n+1,p)}, \ldots, Q^1_{(n+1,p)}).$$

The left-hand side of equation (E.46) is then, according to equation (E.43), equal to $M_{n+1}(t, u, \underline{Q}_{n+1}, \underline{\lambda}_{n+1})$. Let us check that condition (E.45) holds. By (E.63) with n replaced by $n+1$, we have

$$m^{n+1}_{(n+1,p),0} > \frac{1}{4}, \quad m^{n}_{(n+1,p),0} > \frac{1}{4}.$$

We have to check that

$$a^{n+1}_{(n+1,p)} + b^{n+1}_{(n+1,p)} \geq 1,$$

that follows from (E.64) at order $n+1$. Let us check that the first term on the right-hand side of (E.46) may be written as $M_n(t, u, \underline{Q}^{\mathrm{C}}_n, \underline{\lambda}^{\mathrm{C}}_n)$, so that it will provide the first term on the right-hand side of (E.75). We shall define the sequence $\underline{Q}^{\mathrm{C}}_n$ by

$$Q^{n,\mathrm{C}}_{(n,p)} = \tilde{Q}, \; Q^{j,\mathrm{C}}_{(n,p)} = Q^{j}_{(n+1,p)}, \quad j = 1, \ldots, n-1, \tag{E.76}$$

where $\tilde{Q}$ is introduced in the statement of Lemma E.2.4. Let us check that we get for the elements of the sequence $(X^{j,\mathrm{C}}_{(n,p)})_{1 \leq j \leq n}$ the expressions in (E.66)–(E.69). For $j = 1, \ldots, n-1$, this follows from the definition of $Q^{j,\mathrm{C}}_{(n,p)}$ in (E.76). Consider now $\tilde{Q}$. The class to which it belongs depends on the fact that

$$b^{n+1}_{(n+1,p)} + a^{n}_{(n+1,p)} \geq 1 \tag{E.77}$$

or not. By (E.64) at order $n+1$, (E.77) holds except if $n+1 = p > n$. Consequently, when $n \neq p-1$, we have according to Lemma E.2.4 that $\iota^{n,\mathrm{C}}_{(n,p)}, m^{n,\mathrm{C}}_{(n,p)}, m^{n,\mathrm{C}}_{(n,p),0}$ are given by (E.66)–(E.67) and $a^{n,\mathrm{C}}_{(n,p)}, b^{n,\mathrm{C}}_{(n,p)}$ by (E.69). If $n = p-1$, then we know only that

$$b^{n+1}_{(n+1,p)} + a^{n}_{(n+1,p)} \geq 0,$$

and in this case, the lemma shows that $m^{n,\mathrm{C}}_{(n,p)}$ and $m^{n,\mathrm{C}}_{(n,p),0}$ are given by the expressions in equation (E.68). We thus obtain that the first term on the right-hand side of equation (E.46) is $M_n(t, u, \underline{Q}^{\mathrm{C}}_n, \underline{\lambda}^{\mathrm{C}}_n)$ for a convenient sequence $\underline{\lambda}^{\mathrm{C}}_n$. Moreover, again by Lemma E.2.4, the semi-norm of $\tilde{Q} = Q^{n,\mathrm{C}}_{(n,p)}$ (corresponding to $N = 2$ in (E.35)) is controlled according to the last inequality in (E.74), the case of the semi-norms of

$$Q^{j,\mathrm{C}}_{(n,p)} = Q^{j}_{(n+1,p)}, \quad j = 1, \ldots, n-1,$$

being trivial.

We have next to check that the remainder R_n in (E.46) provides the last contribution to (E.75). This follows from (E.47) and the fact that, by definition, $\underline{Q}^{\mathrm{T}}_n$ is the truncated sequence $(Q^n_{(n+1,p)}, \ldots, Q^1_{(n,p)})$. This concludes the proof. ∎

Proof of Proposition E.2.5. We proceed by induction on n. If $n = 0$, the last statement in Lemma E.2.4 shows that we get (E.70). We assume from now on that $n \geq 1$. Assume that (E.70) and (E.72) have been proved at order n instead of $n+1$.

Case $p \geq n+2$. We apply inequality (E.75). On its right-hand side, we may apply the induction hypothesis to $M_n(t, u, \underline{Q}_n^{\mathrm{C}}, \underline{\lambda}_n^{\mathrm{C}})$ and $M_n(t, u, \underline{Q}_n^{\mathrm{T}}, \underline{\lambda}_n^{\mathrm{T}})$. Since $p > n$, it follows that estimate (E.70) (with $n+1$ replaced by n) for $M_n(t, u, \underline{Q}_n^{\mathrm{C}}, \underline{\lambda}_n^{\mathrm{C}})$ will hold, with $\underline{\iota}_{(n+1,p)}, \underline{m}_{(n+1,p)}, \underline{K}_{(n+1,p)}$ replaced by

$$\begin{aligned}
\underline{\iota}^{\mathrm{C}}_{(n,p)} &= \sum_{j=1}^{n} \iota^{j,\mathrm{C}}_{(n,p)}, \\
\underline{m}^{\mathrm{C}}_{(n,p)} &= \sum_{j=1}^{n} m^{j,\mathrm{C}}_{(n,p)} - n\Big(\delta' + \frac{1}{4}\Big), \\
\underline{K}^{\mathrm{C}}_{(n,p)} &= \prod_{j=1}^{n} K^{j,\mathrm{C}}_{(n,p)},
\end{aligned}$$

respectively. Using (E.66), (E.67), (E.74), we get a bound of the first term on the right-hand side of (E.75) by

$$C_0^n C \prod_{j=1}^{n+1} K^j_{(n+1,p)} \varepsilon^{\underline{\iota}_{(n+1,p)}} t^{-\underline{m}_{(n+1,p)}} \|f\|_{L^2}. \tag{E.78}$$

On the other hand, if we apply inequality (E.70) (with $n+1$ replaced by n) to $M_n(t, u, \underline{Q}_n^{\mathrm{T}}, \underline{\lambda}_n^{\mathrm{T}})$ and use (E.73), we bound the last term in (E.75) by

$$C t^{-m^{n+1}_{(n+1,p)} + \frac{1}{4} + \delta'} \varepsilon^{\iota^{n+1}_{(n+1,p)}} K^{n+1}_{(n+1,p)} C_0^n \underline{K}^{\mathrm{T}}_{(n,p)} \varepsilon^{\underline{\iota}^{\mathrm{T}}_{(n,p)}} t^{-\underline{m}^{\mathrm{T}}_{(n,p)}} \|f\|_{L^2}, \tag{E.79}$$

where we denoted

$$\begin{aligned}
\underline{\iota}^{\mathrm{T}}_{(n,p)} &= \sum_{j=1}^{n} \iota^{j,\mathrm{T}}_{(n,p)} = \sum_{j=1}^{n} \iota^{j}_{(n+1,p)}, \\
\underline{m}^{\mathrm{T}}_{(n,p)} &= \sum_{j=1}^{n} m^{j,\mathrm{T}}_{(n,p)} - n\Big(\frac{1}{4} + \delta'\Big) = \sum_{j=1}^{n} m^{j}_{(n+1,p)} - n\Big(\frac{1}{4} + \delta'\Big), \\
\underline{K}^{\mathrm{T}} &= \prod_{j=1}^{n} K^{j,\mathrm{T}}_{(n,p)} = \prod_{j=1}^{n} K^{j}_{(n+1,p)}
\end{aligned}$$

according to the definition of $X^{j,\mathrm{T}}_{(n,p)}$ in (E.65). Taking (E.71) into account, we bound again (E.79) by (E.78).

Case $p = n+1$. We apply again (E.75). On the right-hand side, the first term may be estimated again from (E.70) with $n+1$ replaced by $n = p-1$, since we have

$p > p - 1$. The exponent $\underline{m}^{\mathrm{C}}_{(n,p)}$ of t on the right-hand side will be here

$$\underline{m}^{\mathrm{C}}_{(p-1,p)} = \sum_{j=1}^{p-1} m^{j,\mathrm{C}}_{(p-1,p)} - (p-1)\Big(\delta' + \frac{1}{4}\Big)$$
$$= \sum_{j=1}^{p} m^{j}_{(p,p)} - (p-1)\Big(\delta' + \frac{1}{4}\Big) - \frac{1}{2}$$

according to (E.68). On the other hand, the last term in (E.75) will be estimated by (E.70) at order n instead of $n+1$, and thus by (E.79). We thus get a bound of the form (E.72).

Case $2 \le p \le n$. We apply again (E.75). The first term on the right-hand side may be estimated from the induction hypothesis (E.72), applied with $n+1$ replaced by n, to $M_n(t, u, \underline{Q}^{\mathrm{C}}_n, \underline{\lambda}^{\mathrm{C}}_n)$. Since $n \neq p-1$, the exponent $m^{j,\mathrm{C}}_{(n,p)}$ are given by (E.67), so that

$$\underline{m}^{\mathrm{C}}_{(n,p)} = \sum_{j=1}^{n} m^{j,\mathrm{C}}_{(n,p)} - n\Big(\delta' + \frac{1}{4}\Big) \ge \underline{m}_{(n+1,p)} + \frac{1}{4}$$

which largely allows to bound the first term by

$$C_0^n C \underline{K}_{(n+1,p)} \varepsilon^{\underline{\iota}(n+1,p)} t^{-\underline{m}(n+1,p) + \frac{1}{2} - (\delta' + \frac{1}{4})} \|f\|_{L^2}. \tag{E.80}$$

The second term on the right-hand side of (E.75) is estimated using the induction assumption for $M_n(t, u, \underline{Q}^{\mathrm{T}}_n, \underline{\lambda}^{\mathrm{T}}_n)$, i.e. writing for this expression (E.72) with $n+1$ replaced by n. One gets again a bound of the form (E.80).

Case $p = 1$. In this case, we proceed as when $p > n+1$: We prove (E.70) by induction, using at each step (E.75), and the fact that the condition $n \neq p - 1 = 0$ holding for all $n \ge 1$, we may use at each step (E.67). This concludes the proof. ∎

E.3 Proof of Proposition E.1.3

We shall prove first Sobolev estimates.

Lemma E.3.1. *Let $B_n(t)$ (resp. $C_n(t)$) be given by (E.10) (resp. (E.13)) with $\mathcal{V}(\cdot)$ of the form (E.5), Q_j being in $\Sigma^{\iota,m}_{1,1}$ for some $\iota > 0$, some $m \in\,]0, \frac{1}{2}[$ close to $\frac{1}{2}$ (as in the example following Definition E.1.1). There are $K > 0$, $\delta' > 0$ small, such that for any n in $\mathbb{N}^*$,*

$$\|B_n(t)\|_{\mathcal{L}(H^s)} \le \Big(K\varepsilon^{\iota} t^{-(m-\delta'-\frac{1}{4})}\Big)^n,$$
$$\|C_n(t)\|_{\mathcal{L}(H^s)} \le \Big(K\varepsilon^{\iota} t^{-(m-\delta'-\frac{1}{4})}\Big)^n. \tag{E.81}$$

The same conclusion holds true if Q_j is in $\Sigma^{\iota,m}_{2,0}$ for all j or Q_j is in $\Sigma^{\iota,m}_{0,2}$ for all j.

Proof. We shall estimate $\|\langle D_x\rangle^s B_n(t)\langle D_x\rangle^{-s}\|_{\mathcal{L}(L^2)}$. By (E.10),

$$\langle D_x\rangle^s B_n(t)\langle D_x\rangle^{-s} = \int \prod_{j=1}^n e^{-i\tau_j P_0}\langle D_x\rangle^s(-i)\mathcal{V}(t+\tau_j)\langle D_x\rangle^{-s} e^{i\tau_j P_0} \times \mathbb{1}_{0<\tau_1<\cdots<\tau_n}\, d\tau_1\cdots\tau_n. \tag{E.82}$$

By (E.5), this may be written as a sum of 5^n terms

$$\sum_{i_1=-2}^{2}\cdots\sum_{i_n=-2}^{2}\int\prod_{j=1}^n(-i)e^{-i\tau_j P_0}\langle D_x\rangle^s K_{Q_{i_{n+1-j}}(t+\tau_j)} \times e^{i\tau_j P_0+i(t+\tau_j)\lambda_{i_{n+1-j}}}\langle D_x\rangle^{-s}\mathbb{1}_{0<\tau_1<\cdots<\tau_n}\, d\tau_1\cdots d\tau_n, \tag{E.83}$$

where by assumption Q_{i_j} is an element of $\Sigma^{\iota,m}_{1,1}$ (resp. $\Sigma^{\iota,m}_{2,0}$, resp. $\Sigma^{\iota,m}_{0,2}$) for all j. We shall set $(a,b)=(1,1)$ (resp. $(2,0)$, resp. $(0,2)$). Composing (E.83) by Fourier transform on the left and inverse Fourier transform on the right, as in (E.6), we reduce ourselves to the $\mathcal{L}(L^2)$ boundedness of an operator that may be written, setting $\tau_j = v_j t$ in the integral, as the sum in $i_1,\dots,i_n$ of

$$\int\prod_{j=1}^n S(t,v_j,\tilde{Q}_{i_{n+1-j}},\lambda_{i_{n+1-j}})\mathbb{1}_{0<v_1<\cdots<v_n}\, dv_1\cdots dv_n, \tag{E.84}$$

where $\tilde{Q}_{i_{n+1-j}}$ is defined from $Q_{i_{n+1-j}}$ by

$$\tilde{Q}_{i_{n+1-j}}(t,v_j,\xi,\eta) = e^{it\lambda_{i_{n+1-j}}t}\langle\xi\rangle^s Q_{i_{n+1-j}}(t(1+v_j),\xi,\eta)\langle\eta\rangle^{-s} \tag{E.85}$$

and $S(t,v_j,\tilde{Q}_{i_{n+1-j}},\lambda_{i_{n+1-j}})$ is defined in (E.36). Since $Q_{i_{n+1-j}}$ belongs to the class $\Sigma^{\iota,m}_{a,b}$ of Definition E.1.1, $\tilde{Q}_{i_{n+1-j}}$ is in the class $\widetilde{\Sigma}^{\iota,m,m_0}_{a,b}$ of Definition E.2.3, taking for m_0 any number $m_0\le m$. Since m is taken close to $\frac{1}{2}$, we may assume that $m_0>\frac{3}{8}$. In other words, the integral in (E.84) is of the form $M_n(t,0,\underline{\tilde{Q}}^n,\underline{\lambda}^n)$, with notation (E.43) with $\underline{\tilde{Q}}=(\tilde{Q}_{i_n},\dots,\tilde{Q}_{i_1})$.

We shall apply Proposition E.2.5 with $n+1$ replaced by n and $p=n+1$. This is possible since, if in condition (E.64), $a_j=b_j=1$ for all j, or $a_j=2, b_j=0$ for all j, or $a_j=0, b_j=2$ for all j, inequality $a_{n'}+b_{n''}\ge 1$ is always satisfied. We deduce from (E.70) that the $\mathcal{L}(L^2)$ norm of (E.84) is bounded from above by

$$\big(\tilde{K}\varepsilon^\iota t^{-(m-\delta'-\frac{1}{4})}\big)^n$$

for some $\tilde{K}>0$. Since we have 5^n terms in the sum (E.83), (E.81) follows for $B_n(t)$. Since according to (E.13), $C_n(t)$ may be written as $B_n(t)^*$ for some $B_n(t)$ of the form (E.10), we get also the second estimate of (E.81).

This concludes the proof. ■

We want next to obtain $\mathcal{L}(L^2)$ bounds for $L\circ C_n(t)$, where L is defined in equation (E.17). We compute first the composition between L and an operator of the form $e^{-i\tau P_0}\mathcal{V}(t+\tau)e^{i\tau P_0}$, where $\mathcal{V}$ is of the form (E.5).

Lemma E.3.2. *Let Q be a 2×2 matrix of functions in the class $\Sigma^{\iota,m}_{1,1}$ of Definition E.1.1. Let λ be in $\mathbb{R}$ and set $\mathcal{V}_Q(t) = e^{i\lambda t} K_Q$ according to notation (E.5)–(E.6). Then one may find 2×2 matrices Q' (resp. Q'') with entries in $\Sigma^{\iota,m}_{2,0}$ (resp. $\Sigma^{\iota,m}_{2,0}$ or $\Sigma^{\iota,m}_{0,1}$) such that*

$$\begin{aligned} &L \circ \left(e^{-i\tau P_0} \mathcal{V}_Q(t+\tau) e^{i\tau P_0}\right) \\ &\quad = \left(e^{-i\tau P_0} \mathcal{V}_{Q'}(t+\tau) e^{i\tau P_0}\right) \circ L + \left(e^{-i\tau P_0} \mathcal{V}_{Q''}(t+\tau) e^{i\tau P_0}\right). \end{aligned} \tag{E.86}$$

Proof. Using notation (E.37), we write

$$Q(t,\xi,\eta) = \sum_{j=1}^{2}\sum_{k=1}^{2} q_{jk}(t,\xi,\eta) E_{jk} \frac{\xi}{\langle\xi\rangle}\frac{\eta}{\langle\eta\rangle}$$

with q_{jk} in $\Sigma^{\iota,m}_{0,0}$. We have to compute the action of L on the operator with kernel

$$\begin{aligned} &\sum_{1\le j,k\le 2} \frac{e^{i\lambda(t+\tau)}}{2\pi} \int e^{i(x\xi - y\eta) + i\tau((-1)^j p(\xi) - (-1)^k p(\eta))} E_{jk} \\ &\qquad \times \frac{\xi}{\langle\xi\rangle}\frac{\eta}{\langle\eta\rangle} q_{jk}(t+\tau,\xi,\eta)\, d\xi\, d\eta. \end{aligned} \tag{E.87}$$

One gets, using expression (E.17) of L,

$$\begin{aligned} &\sum_{1\le j,k\le 2} \frac{e^{i\lambda(t+\tau)}}{2\pi} \int e^{i(x\xi - y\eta) + i\tau((-1)^j p(\xi) - (-1)^k p(\eta))} E_{jk} \\ &\qquad \times \left(x + (-1)^{j+1} t p'(\xi)\right) \frac{\xi}{\langle\xi\rangle}\frac{\eta}{\langle\eta\rangle} q_{jk}(t+\tau,\xi,\eta)\, d\xi d\eta. \end{aligned} \tag{E.88}$$

As $p'(\xi) = \frac{\xi}{\langle\xi\rangle}$, we have

$$\begin{aligned} \left(x + (-1)^{j+1} t p'(\xi)\right) \frac{\xi}{\langle\xi\rangle}\frac{\eta}{\langle\eta\rangle} &= (-1)^j \frac{\xi}{\langle\xi\rangle}\left(x \frac{\eta}{\langle\eta\rangle}(-1)^j - y \frac{\xi}{\langle\xi\rangle}(-1)^k\right) \\ &\quad + (-1)^{j+k} \frac{\xi^2}{\langle\xi\rangle^2}\left(y + (-1)^{k+1} t p'(\eta)\right). \end{aligned} \tag{E.89}$$

We plug (E.89) in (E.88). The last term in (E.89) gives an expression of the form of the first term on the right-hand side of (E.86), where the operator $e^{-i\tau P_0}\mathcal{V}_{Q'}(t+\tau)e^{i\tau P_0}$ is given by an expression of the form (E.87), with $\frac{\xi}{\langle\xi\rangle}\frac{\eta}{\langle\eta\rangle} q_{jk}$ replaced by

$$(-1)^{j+k} \frac{\xi^2}{\langle\xi\rangle^2} q_{jk},$$

i.e. Q' is given by

$$Q'(t,\xi,\eta) = \sum_{j=1}^{2}\sum_{k=1}^{2} q_{jk}(t,\xi,\eta)(-1)^{j+k} E_{jk} \frac{\xi^2}{\langle\xi\rangle^2}.$$

This is an element of $\Sigma^{\iota,m}_{2,0}$ as wanted.

On the other hand, if we plug the first term of the right-hand side of (E.89) in (E.88) and perform one integration by parts, we get

$$(-1)^{j+1}\sum_{j=1}^{2}\sum_{k=1}^{2}\frac{e^{i\lambda(t+\tau)}}{2\pi}\int e^{i(x\xi-y\eta)+i\tau((-1)^j p(\xi)-(-1)^k p(\eta))} \times\Big((-1)^j\frac{\eta}{\langle\eta\rangle}D_\xi+(-1)^k\frac{\xi}{\langle\xi\rangle}D_\eta\Big)\Big(\frac{\xi}{\langle\xi\rangle}q_{jk}(t+\tau,\xi,\eta)\Big)\,d\xi d\eta.$$

We get an operator of the form of the last term in (E.86), with a symbol Q'' that may be written as the sum of an element in $\Sigma^{\iota,m}_{2,0}$ and an element in $\Sigma^{\iota,m}_{0,1}$. This concludes the proof of the lemma. ∎

We may prove now the following statement.

Lemma E.3.3. *For any n in $\mathbb{N}^*$, one may find operators $C_n^p(t)$, $0\le p\le n$, such that*

$$L\circ C_n(t)=C_n^0(t)\circ L+\sum_{p=1}^{n}C_n^p(t) \tag{E.90}$$

which have the following structure: Operator $C_n^0(t)$ is of the form

$$\int\prod_{j=1}^{n}e^{-i\tau_j P_0}i\mathcal{V}'(t+\tau_j)e^{i\tau_j P_0}\mathbb{1}_{0<\tau_n<\dots<\tau_1}\,d\tau_1\cdots d\tau_n, \tag{E.91}$$

where $\mathcal{V}'(t)=\sum_{\ell=-2}^{2}e^{i\lambda_\ell t}K_{Q'_\ell}$, with Q'_ℓ matrices with entries in $\Sigma^{\iota,m}_{2,0}$. Operator $C_n^p(t)$ for $1\le p\le n$ has the structure

$$\int\prod_{j=1}^{p-1}e^{-i\tau_j P_0}i\mathcal{V}'(t+\tau_j)e^{i\tau_j P_0}\times e^{-i\tau_p P_0}i\mathcal{V}''(t+\tau_p)e^{i\tau_p P_0} \times\prod_{j=p+1}^{n}e^{-i\tau_j P_0}i\mathcal{V}(t+\tau_j)e^{i\tau_j P_0}\mathbb{1}_{0<\tau_n<\dots<\tau_1}\,d\tau_1\cdots d\tau_n, \tag{E.92}$$

where $\mathcal{V}$ is as in (E.5), $\mathcal{V}'$ is as above and $\mathcal{V}''$ is a sum $\mathcal{V}''(t)=\sum_{\ell=-2}^{2}e^{i\lambda_\ell t}K_{Q''_\ell}$, with Q''_ℓ matrices with entries in $\Sigma^{\iota,m}_{2,0}$ or $\Sigma^{\iota,m}_{0,1}$. Moreover, one has the estimates

$$\|C_n^0(t)\|_{\mathcal{L}(L^2)}\le\big(\tilde{K}\varepsilon^{\iota}t^{\delta'+\frac14-m}\big)^n, \tag{E.93}$$

$$\|C_n^p(t)\|_{\mathcal{L}(L^2)}\le\big(\tilde{K}\varepsilon^{\iota}t^{\delta'+\frac14-m}\big)^n t^{\frac12-\left(\delta'+\frac14\right)},\ 1\le p\le n. \tag{E.94}$$

Proof. We start from expression (E.13) of $C_n(t)$. If we compose at the left with L and use (E.86), we obtain the sum of an expression of the form (E.92) with $p=1$ and a quantity of the form (E.13), with the product replaced by

$$e^{-i\tau_1 P_0}i\mathcal{V}'(t+\tau_1)e^{i\tau_1 P_0}\circ L\circ\prod_{j=2}^{n}e^{-i\tau_j P_0}i\mathcal{V}(t+\tau_j)e^{i\tau_j P_0}. \tag{E.95}$$

If we iterate, we obtain $C_n^0(t) \circ L$ with $C_n^0(t)$ given by (E.91) and the sum for p going from 1 to n of (E.92).

We have next to obtain (E.93) and (E.94). By duality, we may replace (E.91) by

$$(-1)^n \int \prod_{j=1}^{n} e^{-i\tau_j P_0} i \mathcal{V}'(t+\tau_j)^* e^{i\tau_j P_0} \mathbb{1}_{0<\tau_1<\dots<\tau_n} \, d\tau_1 \cdots d\tau_n \tag{E.96}$$

and (E.92) by

$$\begin{aligned}(-1)^n \int \prod_{j=1}^{n-p} & e^{-i\tau_j P_0} i \mathcal{V}(t+\tau_j)^* e^{i\tau_j P_0} \\ & \times e^{-i\tau_{n+1-p} P_0} i \mathcal{V}''(t+\tau_{n+1-p})^* e^{i\tau_{n+1-p} P_0} \\ & \times \prod_{j=n+2-p}^{n} e^{-i\tau_j P_0} i \mathcal{V}'(t+\tau_j)^* e^{i\tau_j P_0} \mathbb{1}_{0<\tau_1<\dots<\tau_n} \, d\tau_1 \cdots d\tau_n\end{aligned} \tag{E.97}$$

for $1 \leq p \leq n$.

Consider first (E.96). We have an operator of the form (E.83) (with $s = 0$) whose $\mathcal{L}(L^2)$ boundedness reduces to the one of an expression of the form (E.84) in terms of symbols $\tilde{Q}_{i_{n+1-j}}$ given by (E.85) from symbols in the class $\Sigma_{0,2}^{\iota,m}$ because of the definition of $\mathcal{V}'(t+\tau_j)$. It follows from the last statement in Lemma E.3.1 that the same estimate as (E.81) holds, which gives a bound of the $\mathcal{L}(L^2)$ norm of (E.96) by the right-hand side of (E.93).

Let us study expression (E.97) and show that its $\mathcal{L}(L^2)$ norm is bounded from above by the right-hand side of (E.94). Operator (E.97) is of the form (E.84), with a sequence of symbols $(\tilde{Q}_{i_n}, \dots, \tilde{Q}_{i_1})$ with $\tilde{Q}_{i_j}$ belonging to the classes $\widetilde{\Sigma}_{a_j,b_j}^{\iota,m,m_0}$, where $(a_j, b_j)_{1\leq j\leq n}$ has the following form:

$$\begin{aligned}(a_n, b_n) = (1,1), \ \dots, \ (a_{p+1}, b_{p+1}) = (1,1), \ (a_p, b_p) = (0,2) \text{ or } (1,0), \\ (a_{p-1}, b_{p-1}) = (0,2), \ \dots, \ (a_1, b_1) = (0,2).\end{aligned} \tag{E.98}$$

The only couples (j', j'') such that $a_{j'} + b_{j''}$ may be eventually equal to zero are those with $j' < j'' = p$, i.e. those for which condition (E.64) is satisfied. We thus obtain that (E.97) is of the form (E.84) and has $\mathcal{L}(L^2)$ norm bounded from above by (E.70) and (E.72), so by the right-hand side of (E.94). This concludes the proof. ■

Proof of Proposition E.1.3. Since m is taken close to $\frac{1}{2}$ and δ' close to zero, the exponent of t on the right-hand side of (E.81) is negative. As $\iota > 0$, for ε small enough, we have

$$\|B_n(t)\|_{\mathcal{L}(H^s)} \leq \frac{1}{2^n}, \quad \|C_n(t)\|_{\mathcal{L}(H^s)} \leq \frac{1}{2^n}.$$

In particular, (E.11) and its counterpart for $C_n(t)$ holds, so that $B(t)$ and $C(t)$ are well defined, bounded on H^s and satisfy (E.19)

Since by (E.93), $\|C_n^0(t)\|_{\mathcal{L}(L^2)}$ satisfies the same estimate as $\|B_n(t)\|_{\mathcal{L}(H^s)}$ and $\|C_n(t)\|_{\mathcal{L}(H^s)}$, the operator

$$\tilde{C}(t) = \mathrm{Id} + \sum_{n=1}^{+\infty} C_n^0(t)$$

is well defined and satisfies (E.21). We notice next that if we set for $n \geq 1$,

$$\tilde{C}_{1,n}(t) = \sum_{p=1}^{n} C_n^p(t),$$

we have by (E.94)

$$\|\tilde{C}_{1,n}(t)\|_{\mathcal{L}(L^2)} \leq Cn(\tilde{K}\varepsilon^\iota)^n t^{(n-1)(\delta'+\frac{1}{4}-m)} t^{\frac{1}{2}-m}.$$

Since $\delta' + \frac{1}{4} - m < 0$, we get after summation estimate (E.22) for

$$\tilde{C}_1(t) = \sum_{n=1}^{+\infty} \tilde{C}_{1,n}(t).$$

We still have to check the last assertions of the proposition. To prove (E.23), it suffices to check that for any n, $N_0 B_n(t) = \overline{B_n(t)} N_0$ for any n, and the corresponding equality for $C_n(t)$. Because of (E.10) and (E.13), it is enough to show that

$$N_0 e^{-i\tau P_0} \mathcal{V}(t+\tau) e^{i\tau P_0} = -e^{i\tau P_0} \overline{\mathcal{V}(t+\tau)} e^{-i\tau P_0} N_0.$$

But this equality follows from (E.7) and the fact that $N_0 e^{i\tau P_0} = e^{-i\tau P_0} N_0$.

Moreover, if $\mathcal{V}$ preserves the space of odd functions, so do $B_n(t)$ and $C_n(t)$ because of their definition, and of the fact that P_0 preserves such spaces. This concludes the proof. ∎