Appendix F

Division lemmas and normal forms

We have discussed in Section 1.6 normal forms for an equation of the form

$$\big(D_t - p(D_x)\big)u = N(u),$$

where $p(\xi) = \sqrt{1+\xi^2}$ and $N(u)$ is some polynomial in $u, \bar{u}$. We distinguish among the monomials of u the characteristic ones, that are those of the form

$$u^{p+1}\bar{u}^p = |u|^{2p}u$$

and the non-characteristics ones, of the form $u^p\bar{u}^q$ with $p - q \neq 1$. We have seen that if $L_+ = x + tp'(D_x)$, a characteristic monomial will satisfy essentially an equality of the form

$$L_+(|u|^{2p}u) = (p+1)(L_+u)|u|^{2p} - pu^{p+1}\bar{u}^{p-1}\overline{L_+u} + \text{remainder}, \qquad \text{(F.1)}$$

that allows one to obtain for the L^2 norm of the left side a bound in $\|u\|_{L^\infty}^{2p}\|L_+u\|_{L^2}$.

Our first goal in this appendix is to give a proof of inequalities of that form for more general characteristic nonlinearities, given in terms of the kind of non-local multilinear operators that we have to use in the proof of the main theorem of the book. Section F.2 below is devoted to that, except that we put ourselves in the semiclassical framework that is very convenient for the proofs.

For non-characteristic nonlinearities, (F.1) non-longer works, and as explained in Section 1.6, one has then to eliminate such nonlinearities by space-time normal forms. We perform in section (F.4) these space-time normal forms in the semiclassical framework, for general non-characteristic nonlinearities given by the multilinear pseudo-differential operators introduced in Appendix B. The method is the one outlined in Section 1.6, extended to these general multilinear expressions. We make also normal forms for quadratic contributions given in terms of symbols with space decaying symbols, along the lines of the end of Section 2.7.

F.1 Division lemmas

We establish in this section some division lemmas, which are variants of similar results obtained in [20].

Definition F.1.1. For n in $\mathbb{N}^*$, denote by Γ_n the set of multi-indices $I = (i_1, \dots, i_n)$ with $i_j = \pm 1$ for $j = 1, \dots, n$. Denote by Γ_n^{ch} the subset of Γ_n made by those $I = (i_1, \dots, i_n)$ such that $\sum_{j=1}^n i_j = 1$ and $\Gamma_n^{\text{nch}} = \Gamma_n - \Gamma_n^{\text{ch}}$.

Let us fix some notation. If $I = (i_1, \dots, i_n)$ is in Γ_n and as above

$$p(\xi) = \sqrt{1+\xi^2},$$

we define

$$g_I(\xi_1,\dots,\xi_n) = -p\big(\xi_1+\cdots+\xi_n\big) + \sum_{j=1}^{n} i_j\, p(\xi_j). \tag{F.2}$$

Set also $\varphi(x) = \sqrt{1-x^2}$ for $|x|<1$, so that by [20, Lemma 1.8], if $\gamma \in C_0^\infty(\mathbb{R})$ has small enough support

$$\begin{aligned} a_\pm(x,\xi) &= \frac{x \pm p'(\xi)}{\xi \mp d\varphi(x)}\gamma\big(\langle\xi\rangle^2(x\pm p'(\xi))\big),\\ b_\pm(x,\xi) &= \frac{\xi \mp d\varphi(x)}{x \pm p'(\xi)}\gamma\big(\langle\xi\rangle^2(x\pm p'(\xi))\big) \end{aligned} \tag{F.3}$$

satisfy estimates

$$\begin{aligned} |\partial_x^\alpha \partial_\xi^\beta a_\pm(x,\xi)| &\le C_{\alpha\beta}\langle\xi\rangle^{-3+2|\alpha|-|\beta|},\\ |\partial_x^\alpha \partial_\xi^\beta b_\pm(x,\xi)| &\le C_{\alpha\beta}\langle\xi\rangle^{3+2|\alpha|-|\beta|}. \end{aligned} \tag{F.4}$$

Proposition F.1.2. *Recall notation (B.10) for the function $M_0(\xi_1,\dots,\xi_n)$ and the class of symbols introduced in Definition B.1.2 for $\beta \ge 0$, $\kappa \ge 0$. Let $\nu \ge 0$.*

(i) *Let I be a multi-index in $(i_1,\dots,i_\ell)$ be in Γ_n and let m_I be a symbol in $S_{1,\beta}(\prod_{j=1}^n \langle\xi_j\rangle^{-1} M_0(\xi)^\nu, n)$. Then we may find symbols*

$$m_{I,\ell} \in S_{4,\beta}\Bigg(\prod_{j=1}^{n}\langle\xi_j\rangle^{-1}M_0(\xi)^{4+\nu}\langle x\rangle^{-1}, n\Bigg), \quad \ell = 1,\dots,n, \tag{F.5}$$

such that if γ is in $C_0^\infty(\mathbb{R})$ and has small enough support, one may write

$$\begin{aligned} &m_I(y,x,\xi_1,\dots,\xi_n)\\ &\quad = m_I(y,x,\xi_1,\dots,\xi_n)\prod_{\ell=1}^{n}\gamma\big(M_0(\xi)^4(x+i_\ell p'(\xi_\ell))\big)\\ &\qquad + \sum_{\ell=1}^{n}(x+i_\ell p'(\xi_\ell))m_{I,\ell}(y,x,\xi_1,\dots,\xi_n). \end{aligned} \tag{F.6}$$

(ii) *Assume that I is in Γ_n^{nch}. Then we may find a symbol*

$$a_I \in S_{4,\beta}\Bigg(\prod_{j=1}^{n}\langle\xi_j\rangle^{-1}M_0(\xi)^{\nu}\langle x\rangle^{-\infty}, n\Bigg) \tag{F.7}$$

and symbols $m_{I,j}$ as in (F.5) such that

$$\begin{aligned} m_I(y,x,\xi_1,\dots,\xi_n) &= g_I(\xi_1,\dots,\xi_n)a_I(y,x,\xi_1,\dots,\xi_n)\\ &\quad + \sum_{\ell=1}^{n}(x+i_\ell p'(\xi_\ell))m_{I,\ell}(y,x,\xi_1,\dots,\xi_n). \end{aligned} \tag{F.8}$$

Proof. Define

$$m_{I,1}(y,x,\xi_1,\dots,\xi_n) = m_I(y,x,\xi_1,\dots,\xi_n)\frac{(1-\gamma)(M_0(\xi)^4(x+i_1p'(\xi_1)))}{x+i_1p'(\xi_1)},$$
$$m_I^{(1)}(y,x,\xi_1,\dots,\xi_n) = m_I(y,x,\xi_1,\dots,\xi_n)\gamma\big(M_0(\xi)^4(x+i_1p'(\xi_1))\big)$$

and write

$$m_I(y,x,\xi_1,\dots,\xi_n) = m_1^{(1)}(y,x,\xi_1,\dots,\xi_n) + m_{I,1}(y,x,\xi_1,\dots,\xi_n)(x+i_1p'(\xi_1)).$$

Then $m_{I,1}$ satisfies (F.5), and repeating the process with m_I replaced by $m_{I,1}$, successively with respect to $\xi_2,\dots,\xi_n$, we get (F.6).

(ii) Equality (F.8) is obtained from (F.6) defining

$$a_I = m_I g_I^{-1}\prod_{j=1}^n \gamma\big(M_0(\xi)^4(x+i_\ell p'(\xi_\ell))\big) \tag{F.9}$$

and showing that a_I belongs to $S_{4,\beta}\big(\prod_{j=1}^n\langle\xi_j\rangle^{-1}M_0(\xi)^{\nu+1}\langle x\rangle^{-\infty},n\big)$. This is done in [20, proof of (i) of Proposition 2.2] (with the parameter κ in that reference set to 2). ■

F.2 Commutation results

We study now the action of the operator $\mathcal{L}_+ = \frac{1}{h}\mathrm{Op}_h(x+p'(\xi))$ introduced in (D.8) on characteristic terms.

Proposition F.2.1. *Let I be in Γ_n^{ch} for some (odd) $n\geq 3$ and let ν be nonnegative. Let m_I be an element of $S_{1,\beta}(\prod_{j=1}^n\langle\xi_j\rangle^{-1}M_0(\xi)^\nu,n)$ with $\beta>0$. Then, for some new value of ν, there are symbols $m_{I,j}$ in $S_{4,\beta}(\prod_{j=1}^n\langle\xi_j\rangle^{-1}M_0^\nu,n)$, $j=1,\dots,n$, r in $S_{4,\beta}(\prod_{j=1}^n\langle\xi_j\rangle^{-1}M_0^\nu,n)$, r' in $S'_{4,\beta}(\prod_{j=1}^n\langle\xi_j\rangle^{-1}M_0^\nu,n)$, such that for any functions $\underline{v}_1,\dots,\underline{v}_n$,*

$$\begin{aligned}\mathcal{L}_+\mathrm{Op}_h(m_I)(\underline{v}_1,\dots,\underline{v}_n) &= \sum_{j=1}^n \mathrm{Op}_h(m_{I,j})(\underline{v}_1,\dots,\mathcal{L}_{i_j}\underline{v}_j,\dots,\underline{v}_n)\\ &\quad + \mathrm{Op}_h(r)(\underline{v}_1,\dots,\underline{v}_n)\\ &\quad + \frac{1}{h}\mathrm{Op}_h(r')(\underline{v}_1,\dots,\underline{v}_n).\end{aligned} \tag{F.10}$$

Proof. We write decomposition (F.6) of m_I, denoting the first term on the right-hand side by $m_I^{(1)}$. This is an element of $S_{4,\beta}(\prod_{j=1}^n\langle\xi_j\rangle^{-1}M_0^\nu,n)$ supported in

$$\bigcap_{\ell=1}^n\{(y,x,\xi_1,\dots,\xi_n) : |x+i_\ell p'(\xi_\ell)| < \alpha M_0(\xi_1,\dots,\xi_n)^{-4}\} \tag{F.11}$$

for some small $\alpha > 0$. It is proved in the proof of [20, Proposition 2.2] that on domain (F.11), one has $|\xi_\ell| \le CM_0(\xi)$ for any $\ell = 1, \dots, n$ and that $\langle d\varphi(x)\rangle \sim M_0(\xi)$ (see [20, formulas (2.10)–(2.13), and the lines following them as well as Lemma 1.8]). Let us show that

$$\begin{aligned} m_I^{(1)}(y, x, \xi_1, \dots, \xi_n)\Big(p'(\xi_1 + \dots + \xi_n) - \sum_{j=1}^n p'(\xi_j)\Big) \\ = \sum_{j=1}^n m_{I,j}(y, x, \xi_1, \dots, \xi_n)(x + i_j p'(\xi_j)) \end{aligned} \tag{F.12}$$

for symbols $m_{I,j}$ in $S_{4,\beta}(\prod_{j=1}^n \langle \xi_j\rangle^{-1} M_0(\xi)^{3+\nu}\langle x\rangle^{-\infty}, n)$. Actually, expanding the bracket in the left hand side of (F.12) on $\xi_j = i_j d\varphi(x)$, $j = 1, \dots, n$ and using $\sum_{j=1}^n i_j = 1$, one may write the left-hand side of (F.12) as

$$\sum_{j=1}^n m_I^{(1)}(y, x, \xi_1, \dots, \xi_n)(\xi_j - i_j d\varphi(x))\tilde{e}_j(x, \xi) \tag{F.13}$$

with

$$\begin{aligned} \tilde{e}_j(x, \xi) = \int_0^1 \Big(p''\big((1-\mu)d\varphi(x) + \mu(\xi_1 + \dots + \xi_n)\big) \\ - \sum_{j=1}^n p''\big((1-\mu)i_j d\varphi(x) + \mu\xi_j\big)\Big)\, d\mu. \end{aligned} \tag{F.14}$$

Notice that on the set (F.11) containing the support of $m_I^{(1)}$, x stays for any ξ in a compact subset of $]-1, 1[$ and that for any α in $\mathbb{N}^*$,

$$\langle \partial^\alpha d\varphi(x)\rangle = O\big(\langle d\varphi(x)\rangle^{1+2\alpha}\big) = O\big(M_0(\xi)^{1+2\alpha}\big) = O\big(M_0(\xi)^{3\alpha}\big),$$

so that each ∂_x^α-derivative of $\tilde{e}_j(x, \xi)$ is $O(M_0(\xi)^{3\alpha})$ on that support. Moreover, we may write using (F.3)

$$(\xi_j - i_j d\varphi(x))\tilde{e}_j(x, \xi) = (x + i_j p'(\xi_j))b_+(x, \xi_j)\tilde{e}_j(x, \xi)$$

if (x, ξ) stays in (F.11) and the function γ in (F.3) is conveniently chosen. Plugging this in (F.13) and defining

$$m_{I,j}(y, x, \xi_1, \dots, \xi_n) = m_I^{(1)}(y, x, \xi_1, \dots, \xi_n)b_+(x, \xi_j)\tilde{e}_j(x, \xi),$$

we get (F.12), with a symbol $m_{I,j}$ in the wanted class because of (F.4) and of the fact that $|\xi_j| = O(M_0(\xi))$ on (F.11). We use now Proposition B.2.1 to write

$$\begin{aligned} &\mathrm{Op}_h(p'(\xi)) \circ \mathrm{Op}_h\big(m_I^{(1)}(y, x, \xi_1, \dots, \xi_n)\big) \\ &\quad = \mathrm{Op}_h\big(p'(\xi_1 + \dots + \xi_n)m_I^{(1)}(y, x, \xi_1, \dots, \xi_n)\big) \\ &\qquad + h\mathrm{Op}_h\big(r_1(y, x, \xi_1, \dots, \xi_n)\big) + \mathrm{Op}_h\big(r_1'(y, x, \xi_1, \dots, \xi_n)\big) \end{aligned} \tag{F.15}$$

with r_1 in $S_{4,\beta}(\prod_{j=1}^n \langle \xi_j \rangle^{-1} M_0^\nu, n)$, and r_1' in $S_{4,\beta}'(\prod_{j=1}^n \langle \xi_j \rangle^{-1} M_0^\nu, n)$ for some ν. Using (F.12), we may rewrite the first term on the right-hand side as

$$\sum_{j=1}^n \mathrm{Op}_h\big(m_I^{(1)}(y,x,\xi_1,\dots,\xi_n) p'(\xi_j)\big) + \sum_{j=1}^n \mathrm{Op}_h\big(m_{I,j}(y,x,\xi_1,\dots,\xi_n)(x + i_j p'(\xi_j))\big). \tag{F.16}$$

Using that $\sum_{j=1}^n i_j = 1$, and that $\mathcal{L}_+ = \frac{1}{h}\mathrm{Op}_h(x + p'(\xi))$, it follows from (F.6), (F.15), (F.16) and Proposition B.2.1 that $\mathcal{L}_+ \mathrm{Op}_h(m_I)$ is the sum of terms of the following form:

$$\begin{aligned}
&\frac{i_j}{h}\mathrm{Op}_h\big(m_I^{(1)}(y,x,\xi_1,\dots,\xi_n)(x + i_j p'(\xi_j))\big), \quad j = 1,\dots,n,\\
&\frac{1}{h}\mathrm{Op}_h\big(m_{I,j}(y,x,\xi_1,\dots,\xi_n)(x + i_j p'(\xi_j))\big), \quad j = 1,\dots,n,\\
&\mathrm{Op}_h\big(r_1(y,x,\xi_1,\dots,\xi_n)\big) + \frac{1}{h}\mathrm{Op}_h\big(r_1'(y,x,\xi_1,\dots,\xi_n)\big)
\end{aligned} \tag{F.17}$$

with $m_{I,j}$ in $S_{4,\beta}\big(\prod_{j=1}^n \langle \xi_j \rangle^{-1} M_0(\xi)^\nu \langle x \rangle^{-1}, n\big)$ coming from (F.6) or (F.16). To conclude the proof, we just have to apply again Proposition B.2.1 to the first two lines of (F.17), in order to rewrite them as the sum on the right-hand side of (F.10), up to new contributions to the remainders. ∎

In the non-characteristic case, we cannot expect an equality of the form (F.10). Instead, we shall have:

Corollary F.2.2. *Let I be in Γ_n^{nch}. Then there are symbols $m_{I,j}$, r, r' as in the statement of Proposition F.2.1 and a symbol r_1 in $S_{4,\beta}(\prod_{j=1}^n \langle \xi_j \rangle^{-1} M_0^\nu, n)$ for some ν, such that*

$$\begin{aligned}
\mathcal{L}_+ \mathrm{Op}_h(m_I)(\underline{v}_1,\dots,\underline{v}_n) = \sum_{j=1}^n &\mathrm{Op}_h(m_{I,j})(\underline{v}_1,\dots,\mathcal{L}_{i_j}\underline{v}_j,\dots,\underline{v}_n)\\
&+ \mathrm{Op}_h(r)(\underline{v}_1,\dots,\underline{v}_n)\\
&+ \frac{1}{h}\mathrm{Op}_h(r')(\underline{v}_1,\dots,\underline{v}_n)\\
&+ \frac{x}{h}\mathrm{Op}_h(r_1)(\underline{v}_1,\dots,\underline{v}_n).
\end{aligned} \tag{F.18}$$

Proof. We may reproduce the proof of Proposition F.2.1, except that, when Taylor expanding the bracket on the left-hand side of (F.12) on $\xi_j = i_j d\varphi(x)$, we shall get the right-hand side of this equality and the extra term

$$m_I^{(1)}(y,x,\xi_1,\dots,\xi_n)\Big(p'\Big(\sum_{j=1}^n i_j d\varphi(x)\Big) - \sum_{j=1}^n p'\big(i_j d\varphi(x)\big)\Big) \tag{F.19}$$

which does not vanish if $\sum_{j=1}^{n} i_j \neq 1$. Since

$$p'(\xi) = \frac{\xi}{\langle \xi \rangle} \quad \text{and} \quad d\varphi(x) = -x\langle d\varphi(x)\rangle,$$

with $\langle d\varphi(x)\rangle = O(M_0(\xi))$ on the support of $m_I^{(1)}$, we see that (F.19) may be written as xr_1 for some r_1 as in the statement. This gives the last contribution to (F.18), the preceding ones being those furnished by the proof of Proposition F.2.1. ■

The last term in (F.18) does not enjoy nice estimates. Because of that, non-characteristic terms have to be eliminated by normal forms. We describe such normal forms in next section.

F.3 Normal forms for non-characteristic terms

Proposition F.3.1. *With the notation and under the assumptions of* (ii) *of Proposition* F.1.2, *one may write for any* $\underline{v}_1, \dots, \underline{v}_n$,

$$\begin{aligned}\Big(D_t - \mathrm{Op}_h\Big(x\xi + p(\xi) - in\frac{h}{2}\Big)\Big)&\mathrm{Op}_h(a_I)(\underline{v}_1, \dots, \underline{v}_n)\\ = \mathrm{Op}_h(m_I)&(\underline{v}_1, \dots, \underline{v}_n)\\ &+ \sum_{j=1}^{n} \mathrm{Op}_h(a_I)[\underline{v}_1, \dots, (D_t - \mathrm{Op}_h(\lambda_{i_j}))\underline{v}_j, \dots, \underline{v}_n]\\ &+ \underline{R}(\underline{v}_1, \dots, \underline{v}_n),\end{aligned} \tag{F.20}$$

where $\lambda_{i_j}(x,\xi) = x\xi + i_j p(\xi) - \frac{i}{2}h$, *and where* $\underline{R}$ *is the sum of terms of the following form*

$$\begin{aligned}&h\mathrm{Op}_h(m_{I,j})(\underline{v}_1, \dots, \mathcal{L}_{i_j}\underline{v}_j, \dots, \underline{v}_n), \quad 1 \leq j \leq n,\\ &\mathrm{Op}_h(r_I')(\underline{v}_1, \dots, \underline{v}_n),\\ &h\mathrm{Op}_h(r_I)(\underline{v}_1, \dots, \underline{v}_n),\end{aligned} \tag{F.21}$$

where $m_{I,j}$ *is a symbol in* $S_{4,\beta}(\prod_{j=1}^{n}\langle\xi_j\rangle^{-1}M_0^{\nu}\langle x\rangle^{-1}, n)$, r_I *(resp.* r_I' *) belongs to the class* $S_{4,\beta}(\prod_{j=1}^{n}\langle\xi_j\rangle^{-1}M_0^{\nu}\langle x\rangle^{-\infty}, n)$ *(resp.* $S'_{4,\beta}(\prod_{j=1}^{n}\langle\xi_j\rangle^{-1}M_0^{\nu}, n)$*) for some* ν*. The first line in* (F.21) *may also be written as*

$$\mathrm{Op}_h(r_I^1)(\underline{v}_1, \dots, \underline{v}_n) \tag{F.22}$$

for a symbol r_I^1 *in* $S_{4,\beta}(\prod_{j=1}^{n}\langle\xi_j\rangle^{-1}M_0^{\nu}, n)$.

Proof. Notice first that by the definition (B.14) of Op_h and the fact that $h = \frac{1}{t}$, one has

$$\begin{aligned}\big(D_t - \mathrm{Op}_h(x\xi)\big)&\mathrm{Op}_h(a_I)(\underline{v}_1, \dots, \underline{v}_n)\\ &= \sum_{j=1}^{n} \mathrm{Op}_h(a_I)\big(\underline{v}_1, \dots, (D_t - \mathrm{Op}_h(x\xi))\underline{v}_j, \dots, \underline{v}_n\big)\\ &\qquad + ih\mathrm{Op}_h\big((x\partial_x a_I)(y,x,\xi)\big)(\underline{v}_1, \dots, \underline{v}_n).\end{aligned} \tag{F.23}$$

Moreover, by Proposition B.2.1 and the definition (F.2) of g_I,

$$\begin{aligned}
-\mathrm{Op}_h(p(\xi))\mathrm{Op}_h(a_I)(\underline{v}_1,\dots,\underline{v}_n) \\
= \mathrm{Op}_h(a_I g_I)(\underline{v}_1,\dots,\underline{v}_n) \\
- \sum_{j=1}^{n} i_j \mathrm{Op}_h(a_I)\big(\underline{v}_1,\dots,\mathrm{Op}_h(p(\xi))\underline{v}_j,\dots,\underline{v}_n\big) \\
+ h\mathrm{Op}_h(r_I)(\underline{v}_1,\dots,\underline{v}_n) + \mathrm{Op}_h(r'_I)(\underline{v}_1,\dots,\underline{v}_n),
\end{aligned} \tag{F.24}$$

where r_I is in $S_{4,\beta}(\prod_{j=1}^n \langle\xi_j\rangle^{-1} M_0^\nu \langle x\rangle^{-\infty}, n)$ and r'_I in $S'_{4,\beta}(\prod_{j=1}^n \langle\xi_j\rangle^{-1} M_0^\nu, n)$. Notice that $p(\xi)$ is in $S_{\kappa,\beta}(\langle\xi\rangle, 1)$ (for any κ,β since, this symbol depending only on one variable ξ, $M_0(\xi) = 1$), so that, to get from Proposition B.2.1 symbols r_I, r'_I in the indicated classes, we would need that a_I be in $S_{4,\beta}(M_0^\nu \prod_{j=1}^n \langle\xi_j\rangle^{-2}\langle x\rangle^{-\infty}, n)$ instead of (F.7). But by (F.9), a_I is supported in (F.11), and we have seen just after this formula that this implies that $|\xi_\ell| \le CM_0(\xi)$ for any ℓ. Consequently, the above property for a_I does hold, for large enough ν. If we make the sum of (F.23) and (F.24), we get that the left-hand side of (F.20) is given by the sum on the right-hand side of (F.20), contributions to $\underline{R}$ of the form of the last two lines in (F.21) and the term $\mathrm{Op}_h(a_I g_I)(\underline{v}_1,\dots,\underline{v}_n)$. By (F.8), we thus get the first term on the right-hand side of (F.20) and expressions

$$-\mathrm{Op}_h\big(m_{I,\ell}(y,x,\xi_1,\dots,\xi_n)(x + i_\ell p'(\xi_\ell))\big)(\underline{v}_1,\dots,\underline{v}_n).$$

Using again Proposition B.2.1, we write these terms as contributions to $\underline{R}$ given by (F.21). This concludes the proof. ∎

F.4 Quadratic normal forms for space decaying symbols

In Section 3.2 we have performed an easy quadratic normal form, that allowed us to get rid of the quadratic term on the right-hand side of (3.11), given by $\mathrm{Op}_h(m_{0,I})[u_I]$, with $|I| = 2$ and $m_{0,I}$ in $\tilde{S}_{0,0}(\prod_{j=1}^2 \langle\xi_j\rangle^{-1}, 2)$. This procedure made appear a new quadratic term $\mathrm{Op}_h(m'_{0,I})[u_I]$ on the right-hand side of equation (3.13), given in terms of a symbol $m'_{0,I}$ in $\tilde{S}'_{0,0}(\prod_{j=1}^2 \langle\xi_j\rangle^{-1}, 2)$. We shall have to perform also a normal form to eliminate such terms. We define a new class of operators.

Definition F.4.1. Let $\omega \in [0,1]$, and $i = (i_1, i_2, i_3)$ in $\{-1,1\}^3$. We denote by $\mathcal{K}_{\kappa,\omega}$, resp. $\mathcal{K}'_{\kappa,\omega}(i)$, the space of operators of the form

$$\begin{aligned}
(f_1, f_2) \mapsto \frac{1}{2\pi}\int_{-1}^{1}\int_{-1}^{1}\int e^{ix\xi_0} k(t,\xi_0,\xi_1,\xi_2,\mu_1,\mu_2) \\
\times \hat{f}(\xi_1)\hat{f}(\xi_2)\, d\xi_0\, d\xi_1\, d\xi_2\, d\mu_1\, d\mu_2,
\end{aligned} \tag{F.25}$$

where k is a smooth function of $(t,\xi_0,\xi_1,\xi_2,\mu_1,\mu_2)$ that satisfies for some ν in $\mathbb{N}$, any $N, \gamma_0, \gamma_1, \gamma_2, \mu_1, \mu_2, j$ in $\mathbb{N}$,

$$\begin{aligned}&|\partial_t^j \partial_{\xi_0}^{\gamma_0}\partial_{\xi_1}^{\gamma_1}\partial_{\xi_2}^{\gamma_2} k(t,\xi_0,\xi_1,\xi_2,\mu_1,\mu_2)|\\ &\quad\le CM_0(\xi_1,\xi_2)^{\nu+(\gamma_0+\gamma_1+\gamma_2)\kappa}\langle\xi_0-\mu_1\xi_1-\mu_2\xi_2\rangle^{-N} t^{\omega(\gamma_0+\gamma_1+\gamma_2)-j},\end{aligned} \tag{F.26}$$

resp. that satisfies

$$\begin{aligned}&|\partial_t^j \partial_{\xi_0}^{\gamma_0}\partial_{\xi_1}^{\gamma_1}\partial_{\xi_2}^{\gamma_2} k(t,\xi_0,\xi_1,\xi_2,\mu_1,\mu_2)|\\ &\quad\le CM_0(\xi_1,\xi_2)^{\nu+(\gamma_0+\gamma_1+\gamma_2)\kappa}\langle\xi_0-\mu_1\xi_1-\mu_2\xi_2\rangle^{-N} t^{\omega(\gamma_0+\gamma_1+\gamma_2)-j}\\ &\qquad\times\langle t^\omega(i_0\langle\xi_0\rangle-i_1\langle\xi_1\rangle-i_2\langle\xi_2\rangle)\rangle^{-1}\end{aligned} \tag{F.27}$$

in the case of $\mathcal{K}'_{\kappa,\omega}(i)$), where $M_0(\xi_1,\xi_2)$ still denoted the second largest among $\langle\xi_1\rangle$ and $\langle\xi_2\rangle$.

If k satisfies

$$k(t,-\xi_0,-\xi_1,-\xi_2) = -k(t,\xi_0,\xi_1,\xi_2), \tag{F.28}$$

then (F.25) sends a couple of two odd functions or two even functions to an odd function. If k satisfies

$$k(t,-\xi_0,-\xi_1,-\xi_2) = k(t,\xi_0,\xi_1,\xi_2), \tag{F.29}$$

then (F.25) sends a couple (f_1, f_2) with f_1 odd, f_2 even or f_1 even, f_2 odd to an odd function.

Let us check first that we may express operators of the form $\mathrm{Op}(m')(v_1,v_2)$ with m' in $\tilde{S}'_{1,0}(M_0(\xi_1,\xi_2)\prod_{j=1}^2\langle\xi_j\rangle^{-1},2)$ in terms of operators $\mathcal{K}_{\kappa,\omega}$.

Lemma F.4.2. *Let m' be in $\tilde{S}'_{1,0}(M_0\prod_{j=1}^2\langle\xi_j\rangle^{-1},2)$. Let $i_1,i_2\in\{-1,1\}^2$ be any choice of signs. Then if $L_\pm$ is defined by (C.5), one may find operators K_{ℓ_1,ℓ_2} in $\mathcal{K}_{1,0}$, $0\le\ell_1,\ell_2\le 1$, such that the action of* $\mathrm{Op}(m')$ *on any couple of odd functions (v_1,v_2) (as defined in (3.6)) may be written as*

$$t^{-2}\sum_{\ell_1=0}^{1}\sum_{\ell_2=0}^{1} K_{\ell_1,\ell_2}(L_{i_1}^{\ell_1}v_1, L_{i_2}^{\ell_1}v_2). \tag{F.30}$$

Moreover, if m satisfies (3.7), then K_{ℓ_1,ℓ_2} is given by a symbol k satisfying (F.28) if $\ell_1+\ell_2=0$ or 2 and (F.29) if $\ell_1+\ell_1=1$.

Proof. We may rewrite

$$\mathrm{Op}(m')(v_1,v_2) = \mathrm{Op}(m'_1)(\langle D_x\rangle^{-1}v_1,\langle D_x\rangle^{-1}v_2)$$

with m'_1 in $\tilde{S}_{1,0}(M_0,2)$. Using the oddness of v_j, we write

$$\begin{aligned}\langle D_x\rangle^{-1}v_j &= \frac{i}{2}x\int_{-1}^{1}(D_x\langle D_x\rangle^{-1}v_j)(\mu_j x)\,d\mu_j\\ &= \frac{i}{2}\frac{x}{t}i_j\int_{-1}^{1}\big((L_{i_j}v_j)(\mu_j x)-\mu_j x v_j(\mu_j x)\big)\,d\mu_j\end{aligned} \tag{F.31}$$

for any choice of the signs $i_j = \pm$. By definition (3.6) of the quantization and inequalities (3.4) satisfied by elements of the class S', one may rewrite expressions like $\mathrm{Op}(m'_1)(xf_1, f_2)$ as sums of expressions of the form $\mathrm{Op}(\tilde{m}'_1)(f_1, f_2)$, for new symbols $\tilde{m}'_1$ in $\tilde{S}_{1,0}(M_0^\nu, 2)$ for some ν. Using (F.31), we thus see that $\mathrm{Op}(m')(v_1, v_2)$ may be rewritten as a sum of terms

$$t^{-2}\int_{-1}^{1}\int_{-1}^{1}\mu_1^{1-\ell_1}\mu_2^{1-\ell_2}\mathrm{Op}(\tilde{m}')\big[(L_{i_1}^{\ell_1}v_1)(\mu_1\cdot),(L_{i_2}^{\ell_2}v_2)(\mu_2\cdot)\big]\,d\mu_1\,d\mu_2$$

for some symbols $\tilde{m}'$ in $S'_{1,0}(M_0^\nu, 2)$. By (3.6), we have

$$\begin{aligned}&\mathrm{Op}(\tilde{m}')[f_1(\mu_1\cdot), f_2(\mu_2\cdot)]\\&\quad=\frac{1}{(2\pi)^2}\int e^{ix(\mu_1\xi_1+\mu_2\xi_2)}m'(x,\mu_1\xi_1,\mu_2\xi_2)\hat{f}_1(\xi_1)\hat{f}_2(\xi_2)\,d\xi_1\,d\xi_2\\&\quad=\frac{1}{2\pi}\int e^{ix\xi_0}k(\xi_0,\xi_1,\xi_2,\mu_1,\mu_2)\hat{f}_1(\xi_1)\hat{f}_2(\xi_2)\,d\xi_1\,d\xi_2\end{aligned}$$

with

$$k(\xi_0,\xi_1,\xi_2,\mu_1,\mu_2)=\frac{1}{(2\pi)^2}\hat{m}'(\xi_0-\mu_1\xi_1-\mu_2\xi_2,\mu_1\xi_1,\mu_2\xi_2).$$

It follows from estimates (3.4) that hold for any α, α'_0, that inequalities (F.26) are true for some ν, $\kappa = 1$, $\omega = 0$, which implies the conclusion as the last statement follows from the transfer of property (3.7) to k by inspection. ■

Proposition F.4.3. *Let K be in $\mathcal{K}_{\kappa,0}$. Let $i = (i_0, i_1, i_2) \in \{-,+\}^3$. One may find operators K_L, K_H in $\mathcal{K}'_{\kappa,\frac{1}{2}}(i)$ such that for any f_1, f_2,*

$$\begin{aligned}&(D_t - i_0p(D_x))(\sqrt{t}K_H(f_1,f_2))\\&\quad=K(f_1,f_2)+\sqrt{t}K_H\big((D_t-i_1p(D_x))f_1,f_2\big)\\&\qquad+\sqrt{t}K_H\big(f_1,(D_t-i_2p(D_x))f_2\big)+K_L(f_1,f_2).\end{aligned}\tag{F.32}$$

If K satisfies (F.28) (resp. (F.29)), so do K_H, K_L.

Proof. Take χ in $C_0^\infty(\mathbb{R})$ equal to one close to zero and set $\chi_1(z) = \frac{1-\chi(z)}{z}$. Define from the function k associated to K by (F.25) a new function

$$\begin{aligned}k_H(t,\xi_0,\xi_1,\xi_2,\mu_1,\mu_2)&=k(\xi_0,\xi_1,\xi_2,\mu_1,\mu_2)\\&\quad\times\chi_1\big(\sqrt{t}(-i_0\langle\xi_0\rangle+i_1\langle\xi_1\rangle+i_2\langle\xi_2\rangle)\big).\end{aligned}\tag{F.33}$$

Then k_H satisfies (F.27) with $\omega = \frac{1}{2}$. Call K_H the associated operator. If we make act $D_t - i_0p(D_x)$ on $\sqrt{t}K_H(f_1, f_2)$, we get the second and third terms on the right-hand side of (F.32), an operator associated to the function

$$k(\xi_0,\xi_1,\xi_2,\mu_1,\mu_2)(1-\chi)\big(\sqrt{t}(-i_0\langle\xi_0\rangle+i_1\langle\xi_1\rangle+i_2\langle\xi_2\rangle)\big)\tag{F.34}$$

and contributions coming from the action of D_t on k_H, that may be written as contributions to K_L in (F.32) (with even an extra factor $t^{-1/2}$). Finally, we see that (F.34) provides K on the right-hand side of (F.32), modulo another contribution to K_L. This concludes the proof as the last statement follows from (F.34). ∎

Corollary F.4.4. *Let m' be in $S'_{1,0}(\prod_{j=1}^{2}\langle\xi_j\rangle^{-1}, 2)$. One may find for any i_1, i_2 in $\{-,+\}$, any ℓ_1, ℓ_2 in $\{0,1\}$ operators*

$$K^{\ell_1,\ell_2}_{H,i_1,i_2}, \quad K^{\ell_1,\ell_2}_{L,i_1,i_2}$$

in the class $\mathcal{K}'_{1,\frac{1}{2}}(1, i_1, i_2)$ such that for any odd functions v_1, v_2, if one sets

$$Q_{i_1,i_2}(v_1, v_2) = t^{-\frac{3}{2}} \sum_{\ell_1=0}^{1}\sum_{\ell_2=0}^{1} K^{\ell_1,\ell_2}_{H,i_1,i_2}\big(L^{\ell_1}_{i_1} v_1, L^{\ell_2}_{i_2} v_2\big), \tag{F.35}$$

then

$$\begin{aligned}\big(D_t - p(D_x)\big) Q_{i_1,i_2}(v_1, v_2) \\ = \mathrm{Op}(m')(v_1, v_2) + Q_{i_1,i_2}\big((D_t - i_1 p(D_x))v_1, v_2\big) \\ + Q_{i_1,i_2}\big(v_1, (D_t - i_2 p(D_x))v_2\big) + R_{i_1,i_2}(v_1, v_2),\end{aligned} \tag{F.36}$$

where

$$\begin{aligned}R_{i_1,i_2}(v_1, v_2) = t^{-2} \sum_{\ell_1=0}^{1}\sum_{\ell_2=0}^{1} K^{\ell_1,\ell_2}_{L,i_1,i_2}\big(L^{\ell_1}_{i_1} v_1, L^{\ell_2}_{i_2} v_2\big) \\ + 2it^{-\frac{5}{2}} \sum_{\ell_1=0}^{1}\sum_{\ell_2=0}^{1} K^{\ell_1,\ell_2}_{H,i_1,i_2}\big(L^{\ell_1}_{i_1} v_1, L^{\ell_2}_{i_2} v_2\big).\end{aligned} \tag{F.37}$$

Moreover, if m' satisfies (3.7), $K^{\ell_1,\ell_2}_{H,i_1,i_2}, K^{\ell_1,\ell_2}_{L,i_1,i_2}$ satisfy (F.28) if $\ell_1 + \ell_2 = 0$ or 2 and (F.29) if $\ell_1 + \ell_2 = 1$. In particular, Q_{i_1,i_2} sends a couple of odd functions to an odd function.

Proof. By Lemma F.4.2, we may write $\mathrm{Op}(m')(v_1, v_2)$ under the form (F.30). We apply to each K_{ℓ_1,ℓ_2} in (F.30) Proposition F.4.3. If we define $K^{\ell_1,\ell_2}_{H,i_1,i_2}$ (resp. $K^{\ell_1,\ell_2}_{L,i_1,i_2}$) from the operator K_H (resp. K_L) in equation (F.32), and use that L_{i_ℓ} commutes to $D_t - i_\ell p'(D_x)$, we obtain (F.36) for the Q_{i_1,i_2} defined in equation (F.35). The last statement of the corollary follows from the last statement in Proposition F.4.3 and Lemma F.4.2. ∎

F.5 Sobolev estimates

We shall prove Sobolev estimates for operators introduced in Definition F.4.1.

Proposition F.5.1. *Let $\omega \in [0,1]$, $\kappa \geq 0$, let K be an operator in the class $\mathcal{K}'_{\kappa,\omega}(i)$ (for a triple $i = (i_1, i_2, i_3) \in \{-,+\}^3$). Assume moreover that the function k in (F.25) is supported for $|\xi_2| \leq 2\langle \xi_1 \rangle$. There exists $\sigma_0 \in \mathbb{R}_+$ (depending on the exponent ν in (F.27)) such that the following estimates hold true for any s in $\mathbb{R}_+$, any test functions f_1, f_2:*

$$\|K(f_1, f_2)\|_{H^s} \leq C t^{-\frac{\omega}{2}} \|f_2\|_{H^{\sigma_0}} \|f_1\|_{H^s}, \tag{F.38}$$

$$\begin{aligned} &\|K(f_1, xf_2)\|_{H^s} + \|K(xf_1, f_2)\|_{H^s} + \|xK(f_1, f_2)\|_{H^s} \\ &\quad \leq C t^{\frac{\omega}{2}} \|f_2\|_{H^{\sigma_0}} \|f_1\|_{H^s}, \end{aligned} \tag{F.39}$$

$$\|K(xf_1, xf_2)\|_{H^s} \leq C t^{\frac{3\omega}{2}} \|f_2\|_{H^{\sigma_0}} \|f_1\|_{H^s}. \tag{F.40}$$

Proof. By (F.25), we have to prove, in order to establish (F.38), that the operator

$$\begin{aligned} (g_1, g_2) \mapsto \int_{-1}^{1} \int_{-1}^{1} \int &\langle \xi_0 \rangle^s k(t, \xi_0, \xi_1, \xi_2, \mu_1, \mu_2) \langle \xi_1 \rangle^{-s} \langle \xi_2 \rangle^{-\sigma_0} \\ &\times g_1(\xi_1) g_2(\xi_2)\, d\xi_1\, d\xi_2\, d\mu_1\, d\mu_2 \end{aligned} \tag{F.41}$$

is bounded from $L^2 \times L^2$ to L^2, with operator norm $O(t^{-\frac{\omega}{2}})$. Because of our support assumptions, $M_0(\xi_1, \xi_2) \leq C\langle \xi_2 \rangle$, so that we may control the factor $M_0(\xi_1, \xi_2)$ in (F.27) by $C\langle \xi_2 \rangle$, i.e. M_0^ν will be bounded using $\langle \xi_2 \rangle^{-\sigma_0}$ if σ_0 is taken large enough. Moreover, as $s \geq 0$, $\langle \xi_0 \rangle^s \langle \xi_0 - \mu_1 \xi_1 - \mu_2 \xi_2 \rangle^{-N} \langle \xi_1 \rangle^{-s} = O(1)$ when $|\xi_2| \leq 2\langle \xi_1 \rangle$ if N is large enough relatively to s. The proof of (F.38) is thus reduced to the proof that operators of the form

$$(g_1, g_2) \mapsto \int_{-1}^{1} \int_{-1}^{1} \int \tilde{k}(t, \xi_0, \xi_1, \xi_2, \mu_1, \mu_2) g_1(\xi_1) g_2(\xi_2)\, d\xi_1\, d\xi_2\, d\mu_1\, d\mu_2 \tag{F.42}$$

are bounded from $L^2 \times L^2$ to L^2, with operator norm $O(t^{-\frac{\omega}{2}})$, if $\tilde{k}$ satisfies

$$\begin{aligned} |\tilde{k}(t, \xi_0, \xi_1, \xi_2, \mu_1, \mu_2)| \leq C &\langle \xi_0 - \mu_1 \xi_1 - \mu_2 \xi_2 \rangle^{-1} \langle \xi_2 \rangle^{-2} \\ &\times \langle t^\omega (i_0 \langle \xi_0 \rangle - i_1 \langle \xi_1 \rangle - i_2 \langle \xi_2 \rangle) \rangle^{-1}. \end{aligned} \tag{F.43}$$

The operator norm of (F.42) is bounded from above by

$$\begin{aligned} C \int_{-1}^{1} \int_{-1}^{1} &\left(\sup_{\xi_0} \int |\tilde{k}(t, \xi_0, \xi_1, \xi_2, \mu_1, \mu_2)|\, d\xi_1\, d\xi_2 \right)^{\frac{1}{2}} \\ &\times \left(\sup_{\xi_1, \xi_2} \int |\tilde{k}(t, \xi_0, \xi_1, \xi_2, \mu_1, \mu_2)|\, d\xi_0 \right)^{\frac{1}{2}} d\mu_1\, d\mu_2. \end{aligned} \tag{F.44}$$

Notice that there is $C > 0$ such that for any α, β in $\mathbb{R}$, any $\mu \in [-1,1]$,

$$\int \langle t^\omega (\alpha + \langle \xi \rangle) \rangle^{-1} \langle \beta + \mu \xi \rangle^{-1}\, d\xi \leq C |\mu|^{-\frac{1}{2}} t^{-\frac{\omega}{2}} \tag{F.45}$$

uniformly in α, β. Actually, if we integrate for $|\xi| \geq 1$, we bound (F.45) by

$$C|\mu|^{-\frac{1}{2}}\left(\int_{|\xi|>1} \langle t^{\omega}(\alpha+\langle\xi\rangle)\rangle^{-2}\, d\xi\right)^{\frac{1}{2}}.$$

If one takes in the above integral computed either on domain $\xi > 1$ or $\xi < -1$, $\eta = \langle \xi \rangle$ as a new variable of integration, we get a bound by the right-hand side of (F.45). If one integrates for $|\xi| < 1$ on the left-hand side of (F.45), we bound the corresponding quantity by

$$\int_{|\xi|<1} \left\langle t^{\omega}(\alpha+\sqrt{1+\xi^2})\right\rangle^{-1} d\xi \leq C\int \langle \alpha' + t^{\omega}\zeta^2\rangle^{-1}\, d\zeta \leq Ct^{-\frac{\omega}{2}}$$

which is better than the bound we want. We use (F.43) and (F.45) with $\xi = \xi_0$ to estimate the second factor in (F.44) by $t^{-\frac{\omega}{4}}$ and (F.45) with $\xi = \xi_1$ to estimate the first integral factor by $t^{-\frac{\omega}{2}}|\mu_1|^{-\frac{1}{2}}$. We obtain that (F.44) is $O(t^{-\frac{\omega}{2}})$ from which (F.38) follows.

To get estimate (F.39), we notice that, by (F.25), $K(xf_1, f_2)$ (resp. $K(f_1, xf_2)$, resp. $xK(f_1, f_2)$) may be written as $K_1(f_1, f_2)$ for an operator K_1 of the form (F.25), obtained replacing k by $D_{\xi_1}k$ (resp. $D_{\xi_2}k$, resp. $-D_{\xi_0}k$). Since by (F.27) these D_{ξ_j}-derivatives make lose t^{ω} (and change the value of the exponent ν), we get (F.39) from (F.38) (with a new value of σ_0).

One obtains (F.40) in a same way. ∎

Corollary F.5.2. *Let K be an element of $\mathcal{K}'_{\kappa,\omega}(i)$ for $\omega \in [0,1]$, $\kappa \geq 0$, $i \in \{-,+\}^3$. The following estimates hold true for any $s \geq 0$ and some σ_0 independent of s:*

$$\|K(f_1, f_2)\|_{H^s} \leq Ct^{-\frac{\omega}{2}}\left(\|f_1\|_{H^{\sigma_0}}\|f_2\|_{H^s} + \|f_1\|_{H^s}\|f_2\|_{H^{\sigma_0}}\right), \tag{F.46}$$

$$\begin{aligned} &\|K(f_1, f_2)\|_{L^2} \leq Ct^{-\frac{\omega}{2}}\|f_1\|_{L^2}\|f_2\|_{H^{\sigma_0}},\\ &\|K(f_1, f_2)\|_{L^2} \leq Ct^{-\frac{\omega}{2}}\|f_1\|_{H^{\sigma_0}}\|f_2\|_{L^2}, \end{aligned} \tag{F.47}$$

$$\begin{aligned} &\|K(xf_1, f_2)\|_{L^2} + \|K(f_1, xf_2)\|_{L^2} + \|xK(f_1, f_2)\|_{L^2}\\ &\quad \leq Ct^{\frac{\omega}{2}}\|f_1\|_{L^2}\|f_2\|_{H^{\sigma_0}},\\ &\|K(xf_1, f_2)\|_{L^2} + \|K(f_1, xf_2)\|_{L^2} + \|xK(f_1, f_2)\|_{L^2}\\ &\quad \leq Ct^{\frac{\omega}{2}}\|f_1\|_{H^{\sigma_0}}\|f_2\|_{L^2}, \end{aligned} \tag{F.48}$$

$$\begin{aligned} &\|K(xf_1, f_2)\|_{H^s} + \|K(f_1, xf_2)\|_{H^s} + \|xK(f_1, f_2)\|_{H^s}\\ &\quad \leq Ct^{\frac{\omega}{2}}\left(\|f_1\|_{H^{\sigma_0}}\|f_2\|_{H^s} + \|f_1\|_{H^s}\|f_2\|_{H^{\sigma_0}}\right). \end{aligned} \tag{F.49}$$

Proof. We may split $K = K_< + K_>$, where $K_>$ (resp. $K_<$) is given by an expression of the form (F.25) with k supported for $|\xi_2| \leq 2\langle\xi_1\rangle$ (resp. $|\xi_1| \leq 2\langle\xi_2\rangle$). If we apply (F.38) to $K_>$ and the symmetric inequality to $K_<$, we obtain (F.46).

Let us prove (F.47). It suffices to show that the two estimates hold for $K_>$ for instance. The first one follows from (F.38) with $s = 0$. To get the second one, we notice that it is enough to establish the $L^2 \times L^2 \to L^2$ boundedness of

$$(g_1, g_2) \mapsto \int_{-1}^{1} \int_{-1}^{1} k(t, \xi_0, \xi_1, \xi_2, \mu_1, \mu_2) \langle \xi_1 \rangle^{-\sigma_0} g_1(\xi_1) g_2(\xi_2)\, d\xi_1\, d\xi_2\, d\mu_1\, d\mu_2$$

with operator norm $O(t^{-\frac{\omega}{2}})$. Since $|\xi_2| \leq 2\langle \xi_1 \rangle$ on the support, if σ_0 has been taken large enough, we see that we may rewrite this under the form (F.42), with some $\tilde{k}$ fulfilling (F.43) so that the conclusion follows.

Finally, estimates (F.48) follow from (F.47), noticing that, as in the proof of (F.39), we may reduce ourselves to operator $K_1(f_1, f_2)$ satisfying the same assumptions as K, up to the loss of a factor t^{ω}. This concludes the proof, as (F.49) follows from (F.39) and the above decomposition $K = K_< + K_>$. ∎

Corollary F.5.3. *Let $\beta > 0$, K, σ_0 as in Corollary F.5.2 and take s large enough so that $(s - \sigma_0)\beta \geq 1$. Then*

$$\|K(L_\pm f_1, f_2)\|_{L^2} \leq C t^{-\frac{\omega}{2}} \big(t^{\beta\sigma_0} \|L_\pm f_1\|_{L^2} + \|f_1\|_{H^s}\big) \|f_2\|_{L^2}, , \tag{F.50}$$

$$\|K(f_1, L_\pm f_2)\|_{L^2} \leq C t^{-\frac{\omega}{2}} \|f_1\|_{L^2} \big(t^{\beta\sigma_0} \|L_\pm f_2\|_{L^2} + \|f_2\|_{H^s}\big). \tag{F.51}$$

Proof. Let χ be in $C_0^\infty(\mathbb{R})$, $\chi \equiv 1$ close to zero. Decompose

$$L_\pm f_1 = \chi(t^{-\beta} D_x)(L_\pm f_1) + (1 - \chi)(t^{-\beta} D_x)(L_\pm f_1).$$

Write

$$\begin{aligned}(1 - \chi)(t^{-\beta} D_x)(L_\pm f_1) &= x(1 - \chi)(t^{-\beta} D_x) f_1 + i t^{-\beta} \chi'(t^{-\beta} D_x) f_1 \\ &\quad \pm t(1 - \chi)(t^{-\beta} D_x) \frac{D_x}{\langle D_x \rangle} f_1.\end{aligned}$$

If one applies the second estimate in (F.47) and (F.48), one gets then

$$\begin{aligned}&\|K\big((1 - \chi)(t^{-\beta} D_x) L_\pm f_1, f_2\big)\|_{L^2} \\ &\quad \leq C\Big(t^{\frac{\omega}{2}} \|(1 - \chi)(t^{-\beta} D_x) f_1\|_{H^{\sigma_0}} \\ &\qquad + t^{-\frac{\omega}{2}} \big(\|\chi'(t^{-\beta} D_x) f_1\|_{H^{\sigma_0}} + t\|(1 - \chi)(t^{-\beta} D_x) f_1\|_{H^{\sigma_0}}\big)\Big) \|f_2\|_{L^2}.\end{aligned}$$

Since $(s - \sigma_0)\beta \geq 1$, this is bounded by $C t^{-\frac{\omega}{2}} \|f_1\|_{H^s} \|f_2\|_{L^2}$.

On the other hand, by the second estimate (F.47)

$$\begin{aligned}\|K\big(\chi(t^{-\beta} D_x) L_\pm f_1, f_2\big)\|_{L^2} &\leq C t^{-\frac{\omega}{2}} \|\chi(t^{-\beta} D_x) L_\pm f_1\|_{H^{\sigma_0}} \|f_2\|_{L^2} \\ &\leq C t^{-\frac{\omega}{2} + \beta\sigma_0} \|L_\pm f_1\|_{L^2} \|f_2\|_{L^2}.\end{aligned}$$

This concludes the proof of (F.50), and thus of the corollary since (F.51) is just the symmetric estimate. ∎

Let us get next some Sobolev estimates for $K(L_\pm f_1, L_\pm f_2)$.

Corollary F.5.4. *Let K be in the class $\mathcal{K}'_{\kappa,\omega}(i)$. Assume moreover that k in (F.25) is supported for $|\xi_1| \leq 2\langle\xi_2\rangle$. Let s, σ_0, β be as in Corollary F.5.3. Then, if $(s-\sigma_0)\beta \geq 1$,*

$$\|K(L_\pm f_1, L_\pm f_2)\|_{H^s} \leq Ct^{1-\frac{\omega}{2}}\|f_2\|_{H^s}\big(t^{\beta\sigma_0}\|L_\pm f_1\|_{L^2} + \|f_2\|_{H^s}\big), \tag{F.52}$$

$$\|K(L_\pm f_1, f_2)\|_{H^s} + \|K(f_1, L_\pm f_2)\|_{H^s} \leq Ct^{1-\frac{\omega}{2}}\|f_1\|_{H^s}\|f_2\|_{H^s}, \tag{F.53}$$

$$\begin{aligned}\|K(xf_1, f_2)\|_{H^s} + \|K(f_1, xf_2)\|_{H^s} &\leq Ct^{\frac{\omega}{2}}\|f_1\|_{H^s}\|f_2\|_{H^s},\\ \|K(xf_1, xf_2)\|_{H^s} &\leq Ct^{3\frac{\omega}{2}}\|f_1\|_{H^s}\|f_2\|_{H^s}.\end{aligned} \tag{F.54}$$

Proof. Take χ in $C_0^\infty(\mathbb{R})$, equal to one close to zero and write $K(L_\pm f_1, L_\pm f_2)$ as a linear combination of the four terms

$$\begin{aligned}
I &= tK\Big(\chi(t^{-\beta}D_x)L_\pm f_1, \frac{D_x}{\langle D_x\rangle}f_2\Big),\\
II &= tK\Big((1-\chi)(t^{-\beta}D_x)L_\pm f_1, \frac{D_x}{\langle D_x\rangle}f_2\Big),\\
III &= K\big(\chi(t^{-\beta}D_x)L_\pm f_1, xf_2\big),\\
IV &= K\big((1-\chi)(t^{-\beta}D_x)L_\pm f_1, xf_2\big).
\end{aligned} \tag{F.55}$$

We apply (F.38) (with f_1 and f_2 exchanged since we assume here $|\xi_1| \leq 2\langle\xi_2\rangle$ on the support instead of $|\xi_2| \leq 2\langle\xi_1\rangle$) in order to estimate the H^s norm of I by

$$Ct^{1-\frac{\omega}{2}}\|\chi(t^{-\beta}D_x)L_\pm f_1\|_{H^{\sigma_0}}\|f_2\|_{H^s} \leq Ct^{1-\frac{\omega}{2}+\beta\sigma_0}\|L_\pm f_1\|_{L^2}\|f_2\|_{H^s} \tag{F.56}$$

which is bounded by the right-hand side of (F.52).

To study II, we write it as a combination of terms

$$\begin{aligned}
&t^2K\Big((1-\chi)(t^{-\beta}D_x)\frac{D_x}{\langle D_x\rangle}f_1, \frac{D_x}{\langle D_x\rangle}f_2\Big),\\
&tK\Big(x(1-\chi)(t^{-\beta}D_x)f_1, \frac{D_x}{\langle D_x\rangle}f_2\Big),\\
&it^{1-\beta}K\Big(\chi'(t^{-\beta}D_x)f_1, \frac{D_x}{\langle D_x\rangle}f_2\Big).
\end{aligned}$$

We estimate their H^s norm using (F.38) and (F.39) (with f_1 and f_2 interchanged) by

$$\begin{aligned}
&Ct^{2-\frac{\omega}{2}}\|f_2\|_{H^s}\big(\|(1-\chi)(t^{-\beta}D_x)f_1\|_{H^{\sigma_0}} + \|\chi'(t^{-\beta}D_x)f_1\|_{H^{\sigma_0}}\big)\\
&\quad\leq Ct^{2-(s-\sigma_0)\beta-\frac{\omega}{2}}\|f_1\|_{H^s}\|f_2\|_{H^s}.
\end{aligned}$$

This implies a bound by the right-hand side of (F.52) since $(s-\sigma_0)\beta \geq 1$.

By (F.39) (with f_1 and f_2 exchanged), we estimate the H^s norm of III by

$$Ct^{\frac{\omega}{2}}\|\chi(t^{-\beta}D_x)L_\pm f_1\|_{H^{\sigma_0}}\|f_2\|_{H^s}$$

that we bound by the right-hand side of (F.52) as in (F.56) since $\omega \leq 1$.

We write IV as a combination of terms

$$
\begin{aligned}
&tK\Big((1-\chi)(t^{-\beta}D_x)\frac{D_x}{\langle D_x\rangle}f_1, xf_2\Big),\\
&K\big(x(1-\chi)(t^{-\beta}D_x)f_1, xf_2\big),\\
&it^{-\beta}K\big(\chi'(t^{-\beta}D_x)f_1, xf_2\big).
\end{aligned}
$$

We estimate the H^s norm of these quantities using (F.39) and (F.40) with f_1 and f_2 interchanged. We get

$$
\begin{aligned}
C\big(t^{1+\frac{\omega}{2}} + t^{3\frac{\omega}{2}}\big)&\|(1-\chi)(t^{-\beta}D_x)f_1\|_{H^{\sigma_0}}\|f_2\|_{H^s}\\
&+ Ct^{-\beta+\frac{\omega}{2}}\|\chi'(t^{-\beta}D_x)f_1\|_{H^{\sigma_0}}\|f_2\|_{H^s}.
\end{aligned}
$$

As $(s-\sigma_0)\beta \geq \omega$, this implies a bound by the right-hand side of (F.52). This concludes the proof of (F.52)

To prove (F.53), we decompose $K(L_\pm f_1, f_2)$ (resp. $K(f_1, L_\pm f_2)$) as the sum of $\pm tK(\frac{D_x}{\langle D_x\rangle}f_1, f_2)$ (resp. $\pm tK(f_1, \frac{D_x}{\langle D_x\rangle}f_2)$) and of $K(xf_1, f_2)$ (resp. $K(f_1, xf_2)$) and we apply (F.38) and (F.39) to get the conclusion.

Finally, (F.54) is just a consequence of (F.39) and (F.40). ■

We translate finally the preceding corollary when one does not make any assumption of support on the frequencies.

Corollary F.5.5. *Let K be in the class $\mathcal{K}'_{\kappa,\omega}(i)$. With the notation of Corollary F.5.4, one has the following inequalities:*

$$
\begin{aligned}
\|K(L_\pm f_1, L_\pm f_2)\|_{H^s} \leq Ct^{1-\frac{\omega}{2}}\Big(&t^{\beta\sigma_0}\big(\|L_\pm f_1\|_{L^2}\|f_2\|_{H^s}\\
&+ \|f_1\|_{H^s}\|L_\pm f_2\|_{L^2}\big) + \|f_1\|_{H^s}\|f_2\|_{H^s}\Big)
\end{aligned}
\tag{F.57}
$$

and

$$
\|K(f_1, L_\pm f_2)\|_{H^s} + \|K(L_\pm f_1, f_2)\|_{H^s} \leq Ct^{1-\frac{\omega}{2}}\|f_1\|_{H^s}\|f_2\|_{H^s}, \tag{F.58}
$$

(with any choice of the signs $\pm$ in the left and right-hand side of these inequalities).

Proof. One decomposes $K = K_< + K_>$ as in the proof of Corollary F.5.2 and applies (F.52) and (F.53). ■