Appendix G

Verification of Fermi's golden rule

The goal of this Appendix is to check that Fermi's golden rule, used in Chapter 4 (see Lemma 4.2.3 and the proof of Proposition 4.2.1) does hold. We already know that from Kowalcyk, Martel and Muñoz, who gave a numerical verification of the condition. We shall prove here that it may actually be checked analytically.

G.1 Reductions

We want to prove the following:

Proposition G.1.1. *Let Y_2 be the function defined in (4.22). Then $\hat{Y}_2(\sqrt{2}) \neq 0$.*

Let us prove here the following reduction:

Lemma G.1.2. *Define the integral*

$$I = \int_{\mathbb{R}} e^{2ix\sqrt{2}}\Big(\cosh^2 x + \frac{1}{2} + i\sqrt{2}\sinh x\cosh x\Big)\frac{\sinh^3 x}{\cosh^7 x}\,dx. \tag{G.1}$$

If $I \neq 0$, then $\hat{Y}_2(\sqrt{2}) \neq 0$.

Proof. Recall that by (4.22), Y_2 is given by

$$Y_2(x) = b(x, D_x)^*(\kappa(x)Y(x)^2), \tag{G.2}$$

where κ, Y are defined in (2.5)–(2.6) and $b(x, D_x)$ has been introduced in Proposition A.1.1. Since $b(x, D_x)^*$ preserves real-valued functions and odd functions, we see that Y_2 is real valued and odd. By Proposition A.1.1, $W_+^* = c(D_x)^* \circ b(x, D_x)^*$ (when acting on odd functions), where $c(\xi)$ has modulus one. In order to show that $\hat{Y}_2(\sqrt{2}) \neq 0$, it thus suffices, according to (G.2), to prove that

$$\widehat{W_+^*(\kappa(x)Y^2)}(\sqrt{2}) \neq 0.$$

Recall that by (A.33) and (A.34),

$$W_+w = \frac{1}{2\pi}\int \psi_+(x,\xi)\hat{w}(\xi)\,d\xi \tag{G.3}$$

with, by (A.35),

$$\psi_+(x,\xi) = \mathbb{1}_{\xi>0}T(\xi)f_1(x,\xi) + \mathbb{1}_{\xi<0}T(-\xi)f_2(x,-\xi), \tag{G.4}$$

where f_1, f_2 are the two Jost functions introduced at the beginning of Appendix A and $T(\xi)$ is defined in (A.26). We thus get

$$\begin{aligned}\widehat{W_+^*(\kappa(x)Y^2)}(\sqrt{2}) &= \int \overline{\psi_+(x,\sqrt{2})}\kappa(x)Y(x)^2\,dx \\ &= \overline{T(\sqrt{2})}\int \overline{f_1(x,\sqrt{2})}\kappa(x)Y(x)^2\,dx.\end{aligned} \tag{G.5}$$

Since the transmission coefficient $T(\sqrt{2})$ is non-zero, it remains to prove that if I given by (G.1) is different from zero, the same is true for the last integral in (G.5), or since κY^2 is real valued, that

$$\int f_1(x,\sqrt{2})\kappa(x)Y(x)^2\,dx \neq 0. \tag{G.6}$$

One checks by a direct computation that the function

$$e^{ix\sqrt{2}}\Big(1+\frac{1}{2}\cosh^{-2}\Big(\frac{x}{2}\Big)+i\sqrt{2}\tanh\frac{x}{2}\Big)(1+i\sqrt{2})^{-1}$$

solves (A.1) with $\xi=\sqrt{2}$ and is equivalent to $e^{ix\sqrt{2}}$ when x goes to $+\infty$, so that is the Jost function $f_1(x,\sqrt{2})$. If one plugs that value in (G.6) and uses the definition (2.5)–(2.6) of κ, Y, one obtains that (G.6) is just a non-zero multiple of (G.1). This concludes the proof. ■

G.2 Proof of the non-vanishing of $\hat{Y}_2(\sqrt{2})$

In order to prove Proposition G.1.1, it remains to show that I given by (G.1) is non-zero. We compute explicitly this integral by residues.

Lemma G.2.1. *One has*

$$I = -\frac{2i\pi}{\sinh(\pi\sqrt{2})}. \tag{G.7}$$

Proof. Denote

$$F(z) = e^{2iz\sqrt{2}}\Big(\cosh^2 z+\frac{1}{2}+i\sqrt{2}\sinh z\cosh z\Big)\frac{\sinh^3 z}{\cosh^7 z}. \tag{G.8}$$

This is a meromorphic function on $\mathbb{C}$ with poles $z_k = i\frac{\pi}{2}(2k+1)$, $k\in\mathbb{Z}$. Let $\mathcal{R}_k$ be the rectangle in the complex plane with vertices at $\pm k\pi$, $\pm k\pi + ik\pi$ for k in $\mathbb{N}^*$. In order to show that

$$I = 2i\pi\sum_{k=0}^{+\infty}\operatorname{Res}(F,z_k) \tag{G.9}$$

we have to check that

$$\int_0^1 |F(\pm k\pi + itk\pi)|k\,dt \to 0, \quad \int_{-1}^1 |F(tk\pi + ik\pi)|k\,dt \to 0$$

when k goes to $+\infty$. As $F(-\bar{z}) = -\overline{F(z)}$, we just have to prove

$$k \int_0^1 \big(|F(k\pi + itk\pi)| + |F(tk\pi + ik\pi)|\big)\, dt \to 0 \tag{G.10}$$

when $k \to +\infty$. As $F(z)$ is a sum of expressions of the form $e^{2iz\sqrt{2}} \frac{\sinh^p z}{\cosh^q z}$ with p, q in $\mathbb{N}$, $p < q$, and bounding

$$\left| \frac{\sinh^p z}{\cosh^q z} \right| \le e^{(p-q)\operatorname{Re} z} \left| \frac{(1 - e^{-2z})^p}{(1 + e^{-2z})^q} \right|,$$

we obtain when $0 \le t \le 1$, $k \in \mathbb{N}^*$,

$$|F(tk\pi + ik\pi)| \le e^{-2k\pi\sqrt{2} - tk\pi},$$

$$|F(k\pi + itk\pi)| \le e^{-2k\pi\sqrt{2}t - k\pi} \frac{(1 + e^{-2k\pi})^p}{(1 - e^{-2k\pi})^q}$$

from which (G.10) follows.

Using

$$\cosh(z_k + w) = i(-1)^k \sinh w \quad \text{and} \quad \sinh(z_k + w) = i(-1)^k \cosh w,$$

we may write

$$F(z_k + w) = e^{-\pi\sqrt{2}(2k+1)} G(w),$$

$$G(w) = e^{2i\sqrt{2}w} \Big(-\sinh^2 w + \frac{1}{2} - i\sqrt{2} \sinh w \cosh w \Big) \frac{\cosh^3 w}{\sinh^7 w}$$

so that $\operatorname{Res}(F, z_k) = e^{-\pi\sqrt{2}(2k+1)} \operatorname{Res}(G, 0)$. One checks by direct computation that $\operatorname{Res}(G, 0) = -2$. It follows that (G.9) is given by

$$I = -4i\pi e^{-\pi\sqrt{2}} \sum_{k=0}^{+\infty} e^{-2\pi k\sqrt{2}} = -\frac{2i\pi}{\sinh(\pi\sqrt{2})}$$

whence (G.7). ∎