Chapter 1

Introduction

1.1 History of the problem

One can trace the question of *quantum ergodicity* back to a paper [70] of Einstein dated 1917. At the time, quantum mechanics was still in its prehistorical age. Planck, around 1900, was at the origin of the idea that certain physical quantities might be *quantised*, that is, they take values in a discrete set. In his paper [130] about the spectrum of the "black body", Planck introduced the constant h; however it was for him a mathematical artefact, without physical foundation. Einstein gave this idea a physical reality when in 1905 he introduced the notion of *quantum of energy* in the exchange of energy between light and matter: the photon [69].

This idea of discreteness was transposed by Bohr [40] in 1913 to the planetary model of the atom. Trying to explain the discrete emission/absorption spectrum of the hydrogen, he used the Rutherford model, where the electron gravitates around the nucleus submitted to Coulomb attraction, and postulated the quantisation of the kinetic momentum: it must be an integer multiple of h. This in turn implies that the energy can only take a discrete set of values, that explains perfectly well the discrete experimental spectrum. However, setting up quantisation rules for larger atoms turned out to be an inextricable task [47].

The aforementioned paper [70] by Einstein is a theoretical paper, that aims at extending the quantisation rules to systems with higher degrees of freedom. Although not his most celebrated paper, it contains deep ideas, and a remark that can be considered to be the starting point of a whole field of research[1]: he first corrects the quantisation rules given earlier by Epstein and Sommerfeld, but he notes that his new rules only make sense if (using modern vocabulary) the system is completely integrable, that is to say, if there exists some action/angle canonical coordinates, where the actions are invariants of motion.[2] He calls such systems *"Type a)"*, and at the end of the paper, he notes: *"on the other hand, classical statistical mechanics is essentially only concerned with Type b)* [i.e. non-integrable systems], *for in this case the microcanonical average is the same as the time average"*. The equivalence of time average with the average over phase space is the property called *ergodicity*. Einstein's point is thus the following:

[1] ... although the physicists who revived the subject in the 80s probably had other motivations than Einstein's question.

[2] Indeed, Einstein's rule is that the values taken by the action variables have to be integer multiples of h.

If a classical dynamical system is ergodic, the quantisation rules do not apply, so how can we describe its spectrum?

Facing the failure to find quantisation rules even for an atom as small as the helium, Heisenberg set up in 1925 entirely new rules of mechanics [89]. These are based on the idea that the "observable quantities" in physics should be represented by matrices (operators), subject to certain commutation rules. The non-commutativity of the algebra of observables is a fundamentally new idea. The basic rule is that the momentum observable p and the position observable q must satisfy

$$qp - pq = i\hbar I,$$

where $\hbar$ is the reduced Planck constant, $\hbar = h/2\pi$. Time evolution is governed by the energy observable H (Heisenberg gives a recipe to build the operator H starting from the classical expression of energy). Any other observable A evolves in time according to the linear equation

$$i\hbar \frac{\mathrm{d}A}{\mathrm{d}t} = [A, H],$$

where $[\bullet, \bullet]$ stands for the commutator of two operators. The physical spectrum of the system (emitted or absorbed energies) is given by the differences $E_n - E_m$, where (E_n) are the eigenvalues of H. The notions of spectrum and of eigenvalues thus merged.

At the same time, a concurrent theory emerged. In 1923, De Broglie had formulated the idea of *wave mechanics*: in the same way as light, traditionally considered to be a wave, was discovered to have a discrete behaviour embodied by the photons, the particles composing matter could, in the reverse direction, also be considered to be waves.

Schrödinger [136,137] proposed in 1926 an evolution equation for a wave/particle of mass m evolving in a force field coming from a potential V:

$$i\hbar \frac{\partial \psi}{\partial t}(t, x) = \Big(-\frac{\hbar^2}{2m}\Delta + V \Big)\psi(t, x) \tag{1.1}$$

where t is time, $x \in \mathbb{R}^3$ is the position of the particle, Δ is the Laplacian, and $\psi = \psi(t, x)$ is a function called the "wave function". Equation (1.1) replaces Newton's law of motion in classical mechanics, $m\ddot{x} = -V'(x)$.

The linear partial differential equation (1.1) can be solved by diagonalising the differential operator

$$H = -\frac{\hbar^2}{2m}\Delta + V.$$

Assume, say, that we can find an orthonormal basis of the Hilbert space $L^2(\mathbb{R}^3)$ consisting of functions ϕ_n satisfying $H\phi_n = E_n\phi_n$ with $E_n \in \mathbb{R}$. Then the general

solution of (1.1) is

$$\psi(t,x) = \sum_n c_n \phi_n(x) \mathrm{e}^{-itE_n/\hbar}$$

where the coefficients $c_n \in \mathbb{C}$ are given by the initial condition at $t = 0$. The physical spectrum is again given by the differences $E_n - E_m$.

Both the Heisenberg and the Schrödinger theories yielded correct results for the hydrogen atom, but also for larger ones. In fact, they can be shown to be mathematically equivalent. However, as Schrödinger wrote [138], mathematical equivalence is not the same as physical equivalence. The "wave function" ψ is absent from Heisenberg's theory, and its physical meaning was at the heart of a tense debate. Born gave a probabilistic interpretation (called the Copenhagen interpretation) of the function ψ: $|\psi(x,t)|^2$ represents the probability, in a measurement, to find a particle at position x, at time t. This was in complete disagreement with Schrödinger's views, but this is the interpretation that has been retained.

After 1926, Einstein's original question may be reformulated as follows:

If a classical Hamiltonian system is ergodic, and if H is the energy operator associated to the system by the rules of quantum mechanics, how can we describe the eigenvalues of the operator H?

One may broaden the question by asking about the properties of the wave functions, that is, the eigenfunctions of H, or more generally the solutions $\psi(x,t)$ of the time-dependent solutions of (1.1):

How are the probability densities $|\psi|^2$ localised in space?

In the mid-fifties, Wigner introduce Random Matrix Theory to deal with the scattering spectrum of heavy nuclei. In this case, although there is no doubt about the validity of the Schrödinger equation, it seems impossible to effectively work with it, in view of the high number of degrees of freedom of such systems. Wigner's hypothesis was that the spectrum of heavy nuclei resembles, statistically, that of certain *ensembles* (in the sense of statistical mechanics, meaning a probabilistic model) of large random matrices: the *Gaussian Orthogonal Ensemble* or the *Gaussian Unitary Ensemble*. This turned out to fit the experimental data extraordinarily well: see illustrations in Bohigas' paper [38].

1.2 Conjectures, scope of the book

In the 80s, numerical simulations started to reveal something unexpected: the spectral statistics of Random Matrix Theory also seem to fit extremely well with the spectra of certain Schrödinger operators with very few degrees of freedom; for instance, the

hydrogen atom in a strong magnetic field, as well as some 2-dimensional billiards (in the latter case, the classical system is just a free particle bouncing on the walls of a closed domain, and the Schrödinger operator is the Laplacian with Dirichlet boundary condition). See [61] for illustrations. The common point of all these examples is that the underlying classical dynamical system is *ergodic*, or even *chaotic* – meaning a very strong sensitivity to initial conditions. Thus, it seems that the answer to Einstein's question could be the following:

Random matrix conjecture (Bohigas–Giannoni–Schmit [39]). *If the classical dynamics is ergodic and sufficiently chaotic, then the spectrum of the corresponding Schrödinger operator looks like that of a large random matrix, in one of the ensembles introduced by Wigner.*

However, this statement must be qualified:

- There is to this day no mathematical proof of this fact; the question may be considered fully open, except for the heuristic arguments given by Sieber and Richter [140], that seem impossible to turn into a mathematical proof.
- There are some counter-examples to this assertion, given by Luo and Sarnak [120]; and they come from very strongly chaotic classical dynamics, so the source of the problem does not lie there.

The counter-examples are Laplacians on *arithmetic* hyperbolic surfaces (such as the modular surface); these systems are special in many ways, and thus one may conjecture that the assertions above hold for "generic" systems, whatever that means. But even in such a weakened form, the question is fully open.

Proving anything in the direction of the Bohigas–Giannoni–Schmit conjecture seems out of reach today. Maybe surprisingly, it seems more tractable to say something about the wave functions: are they localised, that is to say, confined in a small region, or are they *delocalised*, meaning that they occupy all space? This will be the main topic of this book, with a focus on delocalisation phenomena.

1.2.1 High frequency delocalisation on billiards and Riemannian manifolds

In Chapters 2 and 3, we will let $(M, \langle \cdot\,, \cdot \rangle)$ be a compact smooth Riemannian manifold of dimension d, and Δ be the Laplace–Beltrami operator on M. It is a self-adjoint operator on the Hilbert space $L^2(M, \mathrm{Vol})$, where 'Vol' is the Riemannian volume measure. We may diagonalise Δ: it is known that there is a non-decreasing sequence

$$\lambda_0 = 0 < \lambda_1 \leq \lambda_2 \leq \cdots \to +\infty,$$

and an orthonormal basis $(\phi_k)_{k\in\mathbb{N}}$ of $L^2(M, \mathrm{Vol})$, such that

$$\Delta \phi_k = -\lambda_k \phi_k. \tag{1.2}$$

If M has a boundary, we should impose a boundary condition, for instance the Dirichlet condition (i.e. we ask that ϕ_k vanishes on ∂M).

We shall look at notions of delocalisation defined in the high-frequency limit $\lambda_k \to +\infty$. The connection with the Schrödinger equation (1.1) is that (1.2) may be rewritten as $-\hbar^2 \Delta \phi = E\phi$, with $\lambda_k = E\hbar^{-2}$. We recognise equation (1.1) with a vanishing potential V. Many statements given in this books could be generalised to more general operators $-\hbar^2\Delta + V$, but we will only treat the case $V = 0$, that already contains all the difficulties and interesting features. If we impose that E stays away from 0, the limit $\lambda_k \to +\infty$ is equivalent to $\hbar \to 0$; in this régime of small wavelength, it is expected that quantum mechanics should converge to classical mechanics in a certain sense. This was actually a requirement of Schrödinger [137] when he introduced his equation.

The operator $-\hbar^2\Delta$ corresponds in the Schrödinger description to a particle moving on M in absence of any external force. In classical mechanics, this corresponds to the motion along geodesics (the geodesic flow), in other words, the motion with zero acceleration. In the presence of a smooth boundary, the geodesics are reflected on the boundary in the usual way. If the manifold is an open subset of euclidean $\mathbb{R}^d$, the particle simply moves in straight lines with constant velocity, and is reflected on the boundary: this dynamical system is called a billiard. The fact that the classical motion may be chaotic comes from the curvature of M (negatively curved manifolds are typical examples of manifolds possessing a chaotic geodesic flow) and/or from the curvature of the boundary (piecewise concave billiards are typically chaotic).

Figure 1.1 represents twelve eigenfunctions of the Laplacian, associated with twelve consecutive eigenvalues, in a stadium-shaped domain, with Dirichlet boundary conditions. We see various patterns, in particular, some eigenfunctions are localised in the inner rectangle, whereas others seem to be spread out rather uniformly in the whole domain. We want to understand which patterns persist in the limit $\lambda_k \to +\infty$, possibly by relating them to some specific properties exhibited by the classical dynamics (here the billiard flow in a stadium-shaped domain).

One of the most natural questions that comes to mind at the sight of Figure 1.1 is: *how large can the eigenfunctions be, how strongly can they be peaked, and at what points?* This may be measured by studying the growth of the L^p-norms of the eigenfunctions for $p > 2$ (especially $p = \infty$); but the results surveyed in Chapter 2 will reveal that we are technically far from being able to relate delocalisation with chaos this way. We will quickly turn to the more adapted notion of *quantum ergodicity* which will be the main subject of this book.

In physics, the probability measure $|\phi(x)|^2 \mathrm{dVol}(x)$ gives the probability to find the particle at x in a measurement, when the system is in the state described by the wave function ϕ. For chaotic systems, it is expected that this probability is close to the uniform probability on M if ϕ is a stationary solution (eigenfunction) of

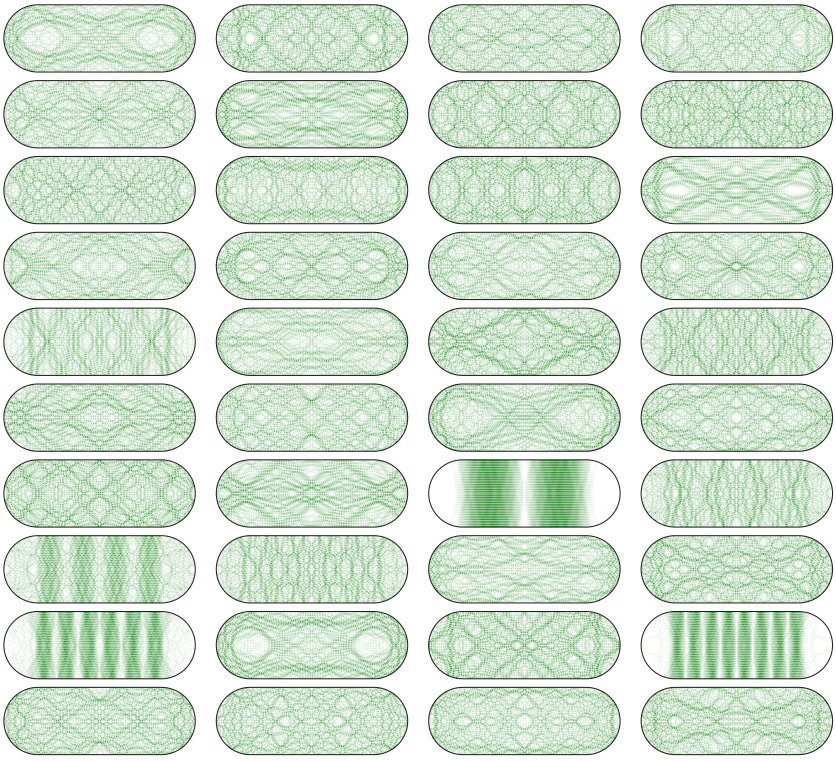

Figure 1.1. Plot of $|\phi_n(x,y)|^2$ for the stadium billiard with odd-odd symmetry, for consecutive states starting from $n = 319$. Darker shades correspond to large values of the eigenfunctions. Most eigenfunctions look "delocalised", meaning that they occupy all space uniformly. But a few of them look "localised" inside the inner rectangle. Courtesy A. Bäcker

the Schrödinger equation; less ambitiously we could try to exhibit delocalisation by showing that this measure cannot be large on "small" sets (sets of small dimension for instance). The Quantum Ergodicity theorem gives a first and almost complete answer in case the geodesic flow is ergodic, with respect to the Liouville measure:

Theorem 1.1 (Shnirelman [142], Zelditch [152], Colin de Verdière [56]). *Let M be a compact Riemannian manifold without boundary, with the metric normalised so that* $\mathrm{Vol}(M) = 1$. *Call Δ the Laplace–Beltrami operator on M. Assume that the geodesic flow of M is* ergodic *with respect to the Liouville measure. Let $(\phi_k)_{k\in\mathbb{N}}$ be an orthonormal basis of $L^2(M,g)$ made of eigenfunctions of the Laplacian:*

$$\Delta\phi_k = -\lambda_k\phi_k, \qquad \lambda_k \leq \lambda_{k+1} \to +\infty.$$

Then there exists a subset $S \subset \mathbb{N}$ of density 1 *such that the sequence of measures $(|\phi_k(x)|^2\,\mathrm{dVol}(x))_{n\in S}$ converges weakly to the uniform measure* $\mathrm{dVol}(x)$.

The subset $S \subset \mathbb{N}$ being of density 1 means that

$$\frac{\sharp\,(S \cap [0,n])}{n} \xrightarrow[n\to+\infty]{} 1,$$

for this reason it is sometimes said that Theorem 1.1 concerns "almost all" the eigenfunctions.

Actually, the full statement of the theorem says that there exists a subset $S \subset \mathbb{N}$ of density 1 such that

$$\langle \phi_k \,, A\phi_k \rangle \xrightarrow[\substack{n\to+\infty \\ n\in S}]{} \int_{S^*M} \sigma^0(A)\,\mathrm{d}\mathcal{L}_1 \tag{1.3}$$

for every pseudo-differential operator A of order 0 on M.

On the right-hand side, $\sigma^0(A)$ is the principal symbol of A, and $\mathcal{L}_1$ is the Liouville (Lebesgue) measure on the classical phase space S^*M. These notions and a proof of Theorem 1.1 will be developed in Chapter 2.

The theorem has subsequently been extended to manifolds with boundary [83, 154]. It applies, in particular, to the stadium billiard in Figure 1.1, since this billiard flow has been proven by Bunimovich to be ergodic. The observation of large samples of eigenfunctions reveals indeed that most eigenfunctions are uniformly distributed over the stadium. However, as is already seen on the small sample of Figure 1.1, some of them look very localised inside the rectangle, and some of them also exhibit a mild enhancement in the neighbourhood of unstable periodic orbits, a phenomenon called "scarring" by physicists (Heller [91]). One may wonder whether the full sequence converges in (1.3), without having to extract the subsequence S. Figure 1.1 (or larger samples of eigenfunctions) suggests that this is not the case for the stadium billiard, because we see a sparse sequence of eigenfunctions that are not at all equidistributed: they stay inside the inner rectangle. The existence of a sparse sequence of eigenfunctions that does not equidistribute was proven by Hassell in 2008 [87], by a non-constructive method that works for "almost all" stadium billiards (meaning, for Lebesgue-almost-all lengths of the stadium).

On the other hand, Rudnick and Sarnak's Quantum Unique Ergodicity (QUE) conjecture [133] predicts the following:

(QUE) Quantum Unique Ergodicity Conjecture. If M is a compact boundaryless manifold with negative sectional curvature, then one has convergence of the full sequence in (1.3). In other words the whole sequence of eigenfunctions becomes equidistributed as $\lambda \to +\infty$.

The conjecture is open, but significant progress has been made in the last twenty years and will be described in Chapter 3. In particular, the following is known:

Theorem 1.2 (see [8, 13] and [64, 65]).

(i) *Let M be a compact Riemannian manifold with negative sectional curvature. Let (ϕ_{n_k}) be a sequence of eigenfunctions of frequency going to infinity. Assume that for every pseudo-differential operator A of order 0 on M, for some probability measure μ,*

$$\langle \phi_{n_k}, A\phi_{n_k} \rangle \xrightarrow[n_k \to +\infty]{} \int_{S^*M} \sigma^0(A) \mathrm{d}\mu.$$

Then μ has positive entropy.

(ii) *Let M be a compact Riemannian manifold with negative sectional curvature and dimension $d = 2$. Let (ϕ_{n_k}) be a sequence of eigenfunctions of frequency going to infinity. Assume that for every pseudo-differential operator A of order 0 on M, for some probability measure μ,*

$$\langle \phi_{n_k}, A\phi_{n_k} \rangle \xrightarrow[n_k \to +\infty]{} \int_{S^*M} \sigma^0(A) \mathrm{d}\mu.$$

Then μ has full support, i.e. $\mu(\Omega) > 0$ for any non-empty open subset $\Omega \subset S^*M$.

Both results imply that μ cannot be a singular measure supported on a closed geodesic. Note that the (QUE) conjecture would consist in proving that μ is necessarily the Liouville measure $\mathcal{L}_1$.

Besides the Random Matrix conjecture and the Quantum Unique Ergodicity conjecture, another central question is the following:

Random Wave Conjecture (Berry [33]). If the classical dynamics is chaotic, then the values of $\phi_\lambda(x)$, when x belongs to a ball $B(x_0, R\lambda^{-1/2})$ of radius $R\lambda^{-1/2}$, follow approximately the distribution of a Gaussian process, for R fixed arbitrary and in the limit $\lambda \to +\infty$.

This conjecture is fully open, and its mathematical formulation itself is a subject of debate. Here we propose an interpretation where the eigenfunctions are submitted to a local zoom of magnitude $\lambda^{1/2}$, in order to rescale the wavelength from $\lambda^{-1/2}$ to 1. After the recent work of Backhausz–Szegedy for the discrete Laplacian on regular graphs [22], it was suggested in [2, 93] that one should choose the point x_0 uniformly at random in the Berry conjecture. One may consider the work of Backhausz–Szegedy to prove the Berry conjecture for large regular graphs chosen at random; but this again can be debated, since in their work there is extra randomness in the geometry itself. The reader can learn more about this in Chapter 7; we think this result is a sign among many others that it is instructive to turn to discrete graph models to gain some insight into the conjectures of quantum chaos.

1.2.2 Delocalisation on large graphs

Understanding the validity and the universality of the conjectures above is one amongst many reasons for studying toy models. Toy models are supposed to be simpler, either because explicit calculations of the spectrum and eigenfunctions may be derived, or because numerical simulations are easier. Toy models often have some built-in discreteness, for instance they can be Schrödinger operators defined on a finite-dimensional Hilbert space, and the semiclassical limit $\hbar \to 0$ means that the dimension of this Hilbert space tends to $+\infty$.

Schrödinger operators defined on graphs are natural toy models. Originally, models of *quantum graphs* were studied. By this, we mean 1-dimensional CW-complexes with $\Delta = \mathrm{d}^2/\mathrm{d}x^2$ on the edges and suitable matching conditions on the vertices. See [30, 32, 57, 84, 99, 100, 108, 109]. However, in this book we focus on discrete graphs and the eigenfunctions of their adjacency operators. We consider finite graphs of bounded valency, in the limit where the number of vertices go to infinity (large scale limit). For such toy models, we deal only with (bounded) operators on finite dimensional spaces, which avoids any difficulty from functional analysis.

Recently and in various contexts, the question of delocalisation of eigenfunctions of the adjacency matrix of large graphs, or more generally, of large matrices has been a subject of intense activity. Let us mention several ways of testing delocalisation that have been used. Let M_N be a large symmetric matrix of size $N \times N$, and let $(\phi_j)_{j=1}^N$ be an orthonormal basis of eigenfunctions.

The eigenfunction ϕ_j defines on the set $\{1, \dots, N\}$ a probability measure

$$\sum_{x=1}^{N} \big|\, \phi_j(x) \,\big|^2 \delta_x .$$

The goal is to compare this probability measure with the uniform measure, which puts mass $1/N$ on each point. The question is considered in the asymptotic régime $N \to +\infty$. We will say that the eigenfunction ϕ_j is delocalised if the associated probability measure is close to being uniform, localised otherwise. We may use various criteria:

- ℓ^∞ *norms* – Can we have a pointwise upper bound on $|\phi_j(x)|$, in other words, is $\|\phi_j\|_\infty$ small, and how small compared with $1/\sqrt{N}$?
- ℓ^p *norms* – Can we compare $\|\phi_j\|_p$ with $N^{1/p-1/2}$? In [60], a state ϕ_j is called *non-ergodic* (and *multi-fractal*) if $\|\phi_j\|_p$ behaves like $N^{f(p)}$ with $f(p) \neq \frac{1}{p} - \frac{1}{2}$.
- *Scarring* (We borrow the term "scarring" from the term coined by Heller [91] in the theory of quantum chaos.) – Can we have full concentration ($\sum_{x\in\Lambda} |\phi_j(x)|^2 \geq 1 - \varepsilon$) or partial concentration ($\sum_{x\in\Lambda} |\phi_j(x)|^2 \geq \varepsilon$) on a set $\Lambda \subset \{1 \dots, N\}$ of "small" cardinality?
- *Quantum ergodicity* – Given a function $a : \{1, \dots, N\} \to \mathbb{C}$, can we compare $\sum_x a(x)|\phi_j(x)|^2$ with $\frac{1}{N}\sum_x a(x)$? This criterion is borrowed again from

quantum chaos, it is inspired from the Shnirelman Theorem 1.1. It was applied to discrete regular graphs in [9,10]. Quantum ergodicity means that the two averages are close for *most* j. If they are close for *all* j, one may speak of *quantum unique ergodicity*, in analogy with the QUE conjecture.

We will be considering the case where the matrix M_N in question is the adjacency matrix of a graph with N vertices. We will consider a sequence of graphs

$$(G_N) = (V_N, E_N)$$

with $|V_N| = N$, denote its adjacency matrix by $\mathcal{A}_N$, and we shall be interested in its eigenvectors $(\phi_1^{(N)}, \dots, \phi_N^{(N)})$. Chapter 4 will be dedicated to proving the following theorem, reminiscent of Theorem 1.1. There we focus on regular graphs of fixed degrees (i.e. for which each point has the same number $(q+1)$ of neighbours), but will also extend the result to non-regular graphs in Chapter 6.

Theorem 1.3 (Anantharaman–Le Masson [10]). *Let*

$$(G_N) = (V_N, E_N)$$

be a sequence of $(q+1)$-regular graphs with $|V_N| = N$. Denote by $\mathcal{A}_N$ the adjacency matrix of G_N. Assume that (G_N) has few short cycles and forms an expander family. Let $(\phi_1^{(N)}, \dots, \phi_N^{(N)})$ be an orthonormal basis of eigenfunctions of $\mathcal{A}_N$ in $\ell^2(V_N)$. Let $a_N : V_N \to \mathbb{C}$ be a sequence of functions such that

$$\sup_N \sup_{x \in V_N} \left| a_N(x) \right| \leq 1.$$

Define

$$\langle a_N \rangle = \frac{1}{N} \sum_{x \in V_N} a_N(x).$$

Then for any $\varepsilon > 0$,

$$\frac{1}{N} \sharp \left\{ j \in [1, N] \, ; \, \left| \left\langle \phi_j^{(N)}, a_N \phi_j^{(N)} \right\rangle_{\ell^2(V_N)} - \langle a_N \rangle \right| > \varepsilon \right\} \xrightarrow[N \to +\infty]{} 0. \tag{1.4}$$

Note that

$$\langle \phi_j^{(N)}, a_N \phi_j^{(N)} \rangle_{\ell^2(V_N)} = \sum_{x \in V_N} a_N(x) |\phi_j^{(N)}(x)|^2.$$

The interpretation of Theorem 1.3 is that we are trying to measure the distance between the two probability measures on V_N,

$$\sum_{x \in V_N} \left| \phi_j^{(N)}(x) \right|^2 \delta_x \quad \text{and} \quad \frac{1}{N} \sum_{x \in V_N} \delta_x \quad \text{(uniform measure)}$$

in a rather weak sense, namely, by testing only one function a_N against both. What (1.4) tells us is that for large N and for most indices j, this distance is small.

The following gives a delocalisation result valid for all eigenfunctions, under slightly different assumptions (see Theorem 4.4 for a precise statement):

Theorem 1.4 (Brooks–Lindenstrauss [54]). *Let*

$$(G_N) = (V_N, E_N)$$

be a sequence of $(q+1)$*-regular graphs with* $|V_N| = N$*. Denote by* $\mathcal{A}_N$ *the adjacency matrix of* G_N*. Assume* G_N *has few cycles of length* $\leq c \log N$*. Let* $\varepsilon > 0$*. Then, for any* ℓ^2*-normalised eigenfunction* ϕ *on* G_N*, for any* $B \subset V_N$*, we have*

$$\sum_{x \in B} \left|\phi(x)\right|^2 \geq \varepsilon \implies \sharp B \geq N^{\delta}$$

with $\delta > 0$*, explicit in terms of* c *and* ε*.*

A proof of those two theorems is sketched in Chapter 4. An extension to non-regular graphs is discussed in Chapter 6, with a quite different proof. In Chapter 5, we describe a paper by Brooks–Le Masson–Lindenstrauss that uses ideas coming from graphs to study Arithmetic Quantum Ergodicity on the sphere.

1.2.3 Delocalisation on random graphs

Theorem 1.3 is about deterministic sequences of graphs satisfying certain geometric assumptions (expansion and few short cycles, see assumptions (EXP) and (BST) in Section 4.1). In the past years, there has been tremendous interest in spectral statistics and delocalisation of eigenfunctions of *random* sequences of graphs and Schrödinger operators. Many papers consider random regular graphs, with degree going slowly to infinity [23, 25, 63, 150] or fixed [24, 82], sometimes adding a random i.i.d. potential [82]. As was demonstrated in a recent series of papers by Yau, Erdős, Schlein, Knowles, Bourgade, Bauerschmidt, Yin, Huang, adding some randomness may allow to understand completely the delocalisation of eigenfunctions, at the expense of replacing results valid for *all* graphs by results holding for *almost all* graphs. These authors proved *almost sure* optimal ℓ^∞-bounds and quantum unique ergodicity for various models of *random* matrices and *random* graphs [23–25, 48, 73–75].

Here again we focus on regular graphs. A (labelled) random regular graph on N vertices is produced as follows: given the vertex set $\{1, \dots, N\}$, consider all the ways to draw edges between those vertices, that produce a $(q+1)$-regular graph (without self-loops and multiple edges); note that $(q+1)N$ has to be an even integer. Then pick a graph at random for the uniform probability measure on all possible configurations.

The recent papers [23–25] show *probabilistic quantum unique ergodicity* for the adjacency matrix of random regular graphs, in the following sense: given an

observable

$$a_N : \{1, \dots, N\} \to \mathbb{R},$$

for most $(q+1)$-regular graphs on the vertices $\{1, \dots, N\}$ we have that

$$\sum_{x=1}^{N} a_N(x) \left| \phi_j^{(N)}(x) \right|^2$$

is close to $\langle a_N \rangle$ for *all* indices j, with an excellent control of the remainder term:

Theorem 1.5 (Bauerschmidt–Huang–Yau [24]). *Let ω be such that*

$$\sqrt{q} \geq (\omega + 1) 2^{2\omega + 45}.$$

(i) *With probability $\geq 1 - o(N^{-\omega+8})$ on the choice of the graph,*

$$\|\phi_j\|_\infty \leq \frac{(\log N)^{121}}{\sqrt{N}}$$

for all eigenfunctions associated to eigenvalues such that $|\lambda_j \pm 2\sqrt{q}\,| > (\log N)^{-3/2}$.

(ii) *(Quantum Unique Ergodicity for random regular graphs) – Given an observable $a_N : \{1, \dots, N\} \to \mathbb{R}$, we have, with probability $\geq 1 - o(N^{-\omega+8})$ on the choice of the graph, for N large enough,*

$$\left| \sum_{x=1}^{N} a_N(x) \cdot \left| \phi_j^{(N)}(x) \right|^2 - \langle a_N \rangle \right| \leq \frac{(\log N)^{250}}{N} \Big(\sum_x \left| a_N(x) \right|^2 \Big)^{1/2}, \quad (1.5)$$

*for all eigenfunctions associated to eigenvalues $\lambda_j \in (-2\sqrt{q} + \varepsilon, 2\sqrt{q} - \varepsilon)$ (*bulk eigenvalues*). In particular, if $a_N = \mathbb{1}_{\Lambda_N}$ where $\Lambda_N \subset \{1, \dots, N\}$, we find*

$$\left| \sum_{x \in \Lambda_N} \left| \phi_j^{(N)}(x) \right|^2 - \frac{\sharp\, \Lambda_N}{N} \right| \leq \frac{(\log N)^{250}}{N} \sqrt{\sharp\, \Lambda_N}.$$

Note by the way that the condition $\omega > 8$, necessary for the remainder $o(N^{-\omega+8})$ to be small, implies that $q > 2^{128}$, although the result is expected to hold as soon as the valency $q+1$ is greater than 3.

The conclusion (1.5) holds for *all* the bulk eigenvalues – hence the name (QUE) – but for *almost all* graphs. This is to be compared to Theorem 1.3, that gave information for *all* graphs but only *almost all* eigenvalues. If we are given a deterministic sequence of regular graphs (for instance, say, the Lubotzky–Phillips–Sarnak Ramanujan graphs [119]), we do not know if Theorem 1.5 applies to it, as it is an *almost sure* conclusion.

Note that we emphasised Theorem 1.5 from [24] because our main concern is the delocalisation of eigenfunctions. The main focus of [23, 24] is however on the universality of the local spectral statistics for random regular graphs. If the valency q grows mildly with N, [23] proves that the spacing between eigenvalues obeys the same statistics as Wigner matrices; this is believed to be true also for fixed q (see numerical experiments in [94, 110, 126, 149]).

The recent paper [22] by Backhausz and Szegedy may be interpreted as almost completely settling the Berry conjecture transposed to the case of random regular graphs. Their result says that for almost all random regular graphs G_N on N vertices, and *all* eigenvectors $\phi_j^{(N)}$s, the value distribution of $\sqrt{N}\,\phi_j(x)$ as x runs over $\{1, \dots, N\}$ is close to some Gaussian $\mathcal{N}(0, \sigma_j^2)$ with $0 \leq \sigma_j \leq 1$. Proving that $\sigma_j = 1$, or even just that $\sigma_j \neq 0$, is still a challenge; it would amount to proving that eigenfunctions cannot put a fixed positive mass on any set of vertices of size $o(N)$. Such a statement is unfortunately not provided by Theorems 1.3 nor 1.5. This result by Backhausz and Szegedy is the object of our last Chapter 7. I see the results of Chapter 7 as an invitation to introduce, in more diverse contexts, some randomness in the model to make progress on the relation between geometry and eigenfunction (de)localisation.