Preface

The carefully chosen title of this book should alert the reader to a multiplicity of purposes to be served.

The *first purpose* is to provide an exposition of Gaussian integral operators, i.e., operators of the form

$$Sf(x) = \int_{\mathbb{R}^n} \exp\left\{\frac{1}{2}\sum_{k,l} a_{kl}x_k x_l + \sum_{k,l} b_{kl}x_k y_l + \frac{1}{2}\sum_{k,l} c_{kl}y_k y_l\right\} f(y)\,dy.$$

Such operators appear in analysis, probability theory, and mathematical physics in numerous contexts; the most classical examples are the Fourier transform, the Poisson formula for a solution of the heat equation, and the Mehler formula for the time evolution of a harmonic oscillator.

Beyond these classical integral operators, we treat "Gaussian operators" in greater generality; for instance, we discuss Gaussian operators in Fock spaces, the Segal–Bargmann transform, the Zak transform, operators with theta-kernels, Gaussian p-adic operators, the real-adelic correspondence, etc.

The *second purpose* is to present a non-orthodox introduction to the classical groups. Basically, this topic is covered in Chapters 2–3, which form an independent but highly relevant part of the book; also Chapter 10 contains a discussion of the p-adic case.

The above two purposes can be pursued independently, but they are not as different as one might think. Gaussian operators are important in the representation theory of infinite-dimensional groups; in a certain sense they replace parabolic induction which is the main tool for construction of representations of finite-dimensional groups. Infinite-dimensional groups provide an additional point of view to that of classical groups, which produces new phenomena and new problems. I completely remove infinite-dimensional groups from consideration but leave Gaussian operators, so this is some kind of view to classical groups from infinity.

On the other hand our detailed analysis of general Gaussian operators is more based on methods of classical groups than on the analytic machinery.

The *third purpose* is to present an exposition of the "Weil representation", which is closely related to Gaussian integral operators. Note that from a historical point of view, it is, actually, the "Friedrichs representation" or the "Friedrichs–Segal–Berezin–Shale–Weil" representation; it is interesting that the first four authors were motivated by Physics or Mathematical Physics.

Here I must say something about the style of the book.

Representation theory of real semisimple Lie groups and noncommutative harmonic analysis constituted an important and dynamic branch of mathematics during the 1940–70s. Nowadays we have an obvious crisis of comprehensibility. As a result, we observe an isolation of that field of study. This is perhaps not so sad, since during the period mentioned above the theory deeply influenced other branches of mathematics (and some of these branches are still dynamic, such as the theory of special functions of several variables, integrable systems, infinite dimensional groups, etc.). It is not clear how to solve this old problem. In this book I expose a piece of the theory that preserves links with numerous branches of pure and applied mathematics and mathematical physics.

The topic and the style of this book were determined by the modern crisis mentioned above. I began with the intention of writing a monograph or a textbook[1] but ended up choosing the genre of an expository "reading-book". In particular, I tried to avoid any representation-theoretical machinery and to make the chapters maximally self-contained. In short, I have tried to produce a "democratic" book on the topic, accessible to any mathematician or mathematical physicist.

The subject allows numerous possibilities for excursions in lateral directions and I have tried not to miss any of them. In such situations, I warn the reader that this is an "excursion". On the other hand, our subject may grow without limitation from an arbitrary point. For this reason, I do not even think about being complete at any specific place. For instance, the Gaussian operators in spaces of functions of an infinite number of variables are more important in probability and mathematical physics than are "finite-dimensional" operators. Nevertheless they are not discussed at all. Also, the Howe duality is not mentioned.

The present book has obvious intersections with the books by Lion and Vergne [122] and Folland [51]. I tried to avoid intersections with my own book [145], but Chapter 5 is one such.

The reader is supposed to be familiar with standard university courses of linear algebra, functional analysis, and complex analysis. Some familiarity with Lie groups and Lie algebras would be also useful, although the latter will be avoided as far as it is possible. I am trying to keep the exposition at this level. However, in some excursions or isolated topics we shall need a slightly wider background (such as elements of differential geometry or topology). There are numerous problems in the book with varying level of difficulty; hard ones in the context of the book are marked with a star.

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[1] Several textbooks of different types and levels have been published relatively recently, see e.g., [49], [62], [73], [80], [109], [184], [185], [221].

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