Introduction

This book is based on Nachdiplom lectures at the ETH Zurich in Spring 2018. The general context for these lectures is the study of derived categories of coherent sheaves on algebraic varieties. These categories can be viewed as refining various cohomological invariants such as de Rham cohomology or Chow groups. While a lot of interesting questions can be asked viewing derived categories as triangulated categories, in many recent studies one has to pay attention to the natural *enhancement* of this structure, which can be understood as that of a dg-category or of an A_∞-category. A more specific motivation for us is to study examples in which, looking at A_∞-structures, one gets some information about the moduli spaces. One can say that in some examples, associating an A_∞-structure to a derived category can be viewed as a kind of algebraic period map.

Derived categories of algebraic varieties first appeared in Grothendieck's school in connection with duality theory. The next chapter in their study began with two discoveries due to Beilinson and Mukai: that the derived category of coherent sheaves on a projective variety can be "affine", i.e., described by the endomorphism algebra of one object; and that derived categories of nonisomorphic varieties can be equivalent. In the course of developing these ideas, it was realized early on that the axiomatics of triangulated categories is somewhat deficient, e.g., since it does not give functorial cones of morphisms. The works of Bondal–Kapranov [9] and later Toën [77] gave frameworks for working on the enhanced level, i.e., with dg-categories.

Another impetus for the theory of derived categories came from *homological mirror symmetry* proposed by Kontsevich, which states that when W and M are mirror dual CY-varieties then $D^b\,\mathrm{Coh}(W)$ should be equivalent to the Fukaya category of M. The latter category is naturally defined as an A_∞-category, so developing this picture requires looking at A_∞-structures associated with derived categories of coherent sheaves.

The formalism of A_∞-algebras (aka strong homotopy algebras) is recalled in Chapter 1. The reader is also referred to the excellent introductory paper by Keller [29]. Note that A_∞-algebras can be viewed as generalizations of dg-algebras. They also have a differential m_1 of degree 1, a double product m_2 satisfying the Leibniz rule with respect to m_1. However, m_2 is not required to be associative: instead there is a triple product m_3 of degree -1, measuring the defect of the associativity of m_2. Furthermore, one has higher products m_n, for $n \geq 3$, satisfying higher associativity constraints.

There is an important construction, the so-called homological perturbation lemma, that produces an A_∞-structure on the cohomology of a dg-algebra (say, over a field) in a way generalizing Massey products. The obtained A_∞-structures are *minimal* in the sense that the corresponding differential m_1 is zero. The downside is that the construction depends on some choices, so the resulting structure is not canonical: it is only canonical up to an equivalence of A_∞-algebras. This means that one has to study the notion of equivalences of A_∞-structures, which can be expressed in terms of an action of an infinite-dimensional gauge group. One may wonder, whether replacing dg-algebras with the corresponding minimal A_∞-structure is worth the trouble. The goal of these notes is to show that at least in some examples coming from geometry, minimal A_∞-structures seem to package the information about the equivalence class of a dg-algebra in a more succinct way.

According to a result of Bondal–van den Bergh and of Kontsevich (see [8]), the derived category of a quasi-projective variety is generated by a single object G (which can be chosen to be a vector bundle). This implies that the entire derived category is recovered from the dg-algebra of endomorphisms of G (and hence, from the corresponding A_∞-algebra structure on $\mathrm{Ext}^*(G, G)$). For example, on an irreducible curve one can take $G = \mathcal{O} \oplus L$, where L is a line bundle of positive degree, or $G = \mathcal{O} \oplus \mathcal{O}_p$, where p is a smooth point.

Given a projective scheme X, one can start with a nice generator of $D^b \operatorname{Coh}(X)$, then compute the associated A_∞-algebra, and then study the corresponding moduli space of A_∞-algebras. The hope is that there will only be a finite amount of data on which this A_∞-algebra will depend, so that we will get some affine scheme of finite type with a reductive group action. Then the corresponding geometric invariant theory (GIT) picture will provide notions of stability and modular compactifications for the moduli of X. We will implement this scheme in the case when X is a reduced projective curve. In fact, we establish an isomorphism between appropriate moduli spaces of curves and moduli spaces of minimal A_∞-structures on a given finite-dimensional associative algebra (for genus 1, such an isomorphism was first established in the work of Lekili–Perutz [34]).

Note that studying general moduli spaces of A_∞-structures is quite difficult. We make first steps in developing the theory of such moduli spaces. Namely, in Chapter 2 we prove a general representability theorem, which establishes the representability of the functor of gauge equivalence classes of minimal A_∞-structures on a fixed associative algebra A, assuming the vanishing of certain components of the Hochschild cohomology of A.

This result turns out to be sufficient for the example arising by considering certain special generators of the derived categories of projective curves. Namely, for a reduced projective curve C of arithmetic genus g, with smooth distinct points $p_1, \ldots, p_g$, such that $H^1(C, \mathcal{O}(p_1 + \cdots + p_g)) = 0$, one can consider the algebra $E = \mathrm{Ext}^*(G, G)$, for

$$G = \mathcal{O}_C \oplus \mathcal{O}_{p_1} \oplus \cdots \oplus \mathcal{O}_{p_g}.$$

It turns out that up to an isomorphism, the algebra E does not depend on the curve C, but only on the genus g. Furthermore, in Chapter 3 we show that the moduli space of minimal A_∞-structures on E is an affine scheme of finite type, which is isomorphic to the appropriate moduli space of curves (in addition to the marked points, one has to fix nonzero tangent vectors at them to get an identification of $\mathrm{Ext}^*(G, G)$ with a fixed algebra). We also generalize this result to the case of $n \geq g$ marked points.

The fact that every minimal A_∞-structure on some algebra comes from the derived category of coherent sheaves on some variety can help to establish homological mirror symmetry in some cases. In Section 20.3 we will consider one simple example giving a characterization of the A_∞-structures arising from nodal curves of arithmetic genus 1 (which was used in [35] to prove homological mirror symmetry for n-punctured tori).

Philosophically, A_∞-algebras belong to the world of *noncommutative geometry*, so associating with a variety the corresponding A_∞-algebra describing its derived category should be viewed as embedding some commutative moduli problem into a noncommutative one. Thus, the above isomorphism of the moduli spaces in the case of curves is somewhat of an exception. In fact, taking other types of generators in derived categories of coherent sheaves on curves leads to moduli spaces that cannot be identified with some purely commutative moduli problem. One such example is considered in Chapter 4. Namely, if we take the generator $G = \mathcal{O}_C \oplus L$ on an elliptic curve C, where L is a line bundle of degree n, then the corresponding Ext-algebra $\mathrm{Ext}^*(G, G)$ does not depend on the curve. However, in general, not all minimal A_∞-structures on this algebra come from elliptic curves or their degenerations. In Chapter 4 we reformulate this problem as classification of pairs of so-called 1-spherical objects in 1-Calabi–Yau categories and study the corresponding moduli problem. It turns out that the corresponding moduli space is isomorphic to the moduli spaces parametrizing curves equipped with noncommutative orders of certain type. Another interesting connection is that the same moduli space (with the additional constraint of cyclicity) describes solutions of the associative Yang–Baxter equation (AYBE), an equation closely related to the much-studied classical and quantum Yang–Baxter equations.

There are many interesting topics related to A_∞-structures and moduli spaces that are not covered in these notes. For example, they provide a natural framework for studying noncommutative moduli spaces of objects in derived categories, in particular, they help to understand noncommutative thickenings of moduli spaces of vector bundles. Another interesting direction is noncommutative Hodge theory for smooth and proper A_∞-algebras. In a more abstract direction related to Chapter 2, one can study moduli spaces of A_∞-modules over A_∞-algebras. In particular, in the context of Chapter 3 one expects that the curve can be recovered from the corresponding A_∞-algebra as the appropriate moduli space of A_∞-modules.

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