Abstract

In the first part of this paper, we introduce the notion of *cyclic stratum* of a Frobenius manifold M. This is the set of points of the extended manifold $\mathbb{C}^* \times M$ at which the unit vector field is a cyclic vector for the isomonodromic system defined by the flatness condition of the extended deformed connection. The study of the geometry of the complement of the cyclic stratum is addressed. We show that at points of the cyclic stratum, the isomonodromic system attached to M can be reduced to a scalar differential equation, called the *master differential equation* of M. In the case of Frobenius manifolds coming from Gromov–Witten theory, namely quantum cohomologies of smooth projective varieties, such a construction reproduces the notion of quantum differential equation.

In the second part of the paper, we introduce two multilinear transforms, called *Borel–Laplace* $(\boldsymbol{\alpha}, \boldsymbol{\beta})$*-multitransforms*, on spaces of Ribenboim formal power series with exponents and coefficients in an arbitrary finite-dimensional $\mathbb{C}$-algebra A. When A is specialized to the cohomology of smooth projective varieties, the integral forms of the Borel–Laplace $(\boldsymbol{\alpha}, \boldsymbol{\beta})$-multitransforms are used in order to rephrase the Quantum Lefschetz theorem. This leads to explicit Mellin–Barnes integral representations of solutions of the quantum differential equations for a wide class of smooth projective varieties, including Fano complete intersections in projective spaces.

In the third and final part of the paper, as an application, we show how to use the new analytic tools, introduced in the previous parts, in order to study the quantum differential equations of Hirzebruch surfaces. For Hirzebruch surfaces diffeomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$, this analysis reduces to the simpler quantum differential equation of $\mathbb{P}^1$. For Hirzebruch surfaces diffeomorphic to the blow-up of $\mathbb{P}^2$ in one point, the quantum differential equation is integrated via Laplace $(1, 2; \frac{1}{2}, \frac{1}{3})$-multitransforms of solutions of the quantum differential equations of $\mathbb{P}^1$ and $\mathbb{P}^2$, respectively. This leads to explicit integral representations for the Stokes bases of solutions of the quantum differential equations, and finally to the proof of the Dubrovin conjecture for all Hirzebruch surfaces.

In memoria di mio padre

Keywords. Frobenius manifolds, quantum cohomology, isomonodromic deformations, integral transforms, derived categories, Dubrovin conjecture

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