Chapter 1

Introduction

1.1 Background

In the last decades, we have been witnessing a growing and fruitful interaction between theoretical physics and various branches of geometry, leading to new developments in both disciplines. *Enumerative geometry* – an old subject and an active field in the 19th century – has been revolutionized by new ideas from the physics of string theory. After the categorical axiomatization of physical theories of quantum fields [4, 5, 72], the emergence of new mathematical objects was noticed. In such an inspiring context, rich structures known as Frobenius manifolds naturally arise, together with the construction of several invariants of symplectic and algebraic varieties.

The notion of *Frobenius manifolds* was introduced by B. Dubrovin, who first recognized its emergence in the study of classification of two-dimensional topological field theories [29–31]. A Frobenius manifold consists[1] of a complex manifold M whose tangent spaces admit an associative, commutative, and unital algebra structure $(T_pM, \circ_p)$, holomorphically depending on the point $p \in M$. The structure is further enriched with a non-degenerate symmetric bilinear form η, whose Levi-Civita connection is flat, and which is compatible with the product, that is,

$$\eta(Y \circ W, Z) = \eta(Y, W \circ Z)$$

for any local vector fields Y, W, Z on M. This condition makes $(T_pM, \circ_p, \eta_p)_{p \in M}$ a family of Frobenius algebras. Pretty soon, it was understood that Frobenius manifolds are a unifying notion in mathematics. These structures play a central role in mirror symmetry, theory of unfolding spaces of singularities, and enumerative geometry [48,61,71]. Remarkably enough, results proved for classes of Frobenius manifolds emerging in a certain mathematical theory turn out to be valid in general. This *universality* of Frobenius manifolds usually leads to unexpected connections between the aforementioned mathematical theories [33].

Quantum cohomology, introduced by E. Witten [79] and C. Vafa [77] in their study of topological non-linear sigma model, is one of the most interesting example of Frobenius manifold, associated with any complex smooth projective variety X, or a more general compact symplectic manifold [30, 58, 61]. From the physical point of view, the space X is the target of two-dimensional fields, and the Frobenius algebras that arise are a highly non-linear deformation of the classical cohomological

[1] Precise definitions will be given in the main body of the paper.

ring $H^\bullet(X,\mathbb{C})$. If the classical cohomology ring of a variety encodes information about the intersections of its subvarieties, the non-functorial construction of quantum cohomology is an instrument to understand how they are related by *rational* (or, in the general symplectic case, *pseudo-holomorphic*) curves. This information is codified in the *Gromov–Witten invariants* [45, 79, 80], used to define the quantum perturbation of the product. Gromov–Witten invariants count curves on X: for each $\beta \in H_2(X,\mathbb{Z})/\text{torsion}$, and cycles $Z_1,\dots,Z_n \subseteq X$ in general position, the Gromov–Witten invariant[2]

$$\langle \mathrm{PD}(Z_1),\dots,\mathrm{PD}(Z_n)\rangle^X_{g,n,\beta} \in \mathbb{Q}$$

heuristically equals the number of curves $C \subseteq X$, of genus g, with homology class $[C]=\beta$, and intersecting all the cycles Z_i. Consider the generating function

$$F_0^X(\gamma) = \sum_{n=0}^{\infty}\sum_{\beta}\frac{1}{n!}\langle\underbrace{\gamma,\dots,\gamma}_{n\text{ times}}\rangle^X_{0,n,\beta}, \quad \gamma \in H^\bullet(X,\mathbb{C}),$$

of genus 0 Gromov–Witten invariants of X, and assume that this sum is convergent on a non-empty domain $\Omega \subseteq H^\bullet(X,\mathbb{C})$. The quantum cohomology $QH^\bullet(X)$ is the Frobenius manifold structure on Ω, the flat metric η being given by the Poincaré pairing

$$\eta(Y,W) := \int_X Y \cup W$$

for any local vector fields[3] Y, W on Ω, and the product $Y \circ W$ of vector fields being defined by the identity

$$\eta(Y\circ W, Z) = (YWZ)\,F_0^X$$

for arbitrary flat local vector fields Y, W, Z on Ω.

1.2 The main problem

At the core of the analytic theory of Frobenius manifolds, there is the local identification of semisimple[4] points $p \in M$ with the parameters of isomonodromic deformations of ordinary differential equations with rational coefficients. Such an identification – one of the main points of the theory of Dubrovin – was originally established in [30–32], and subsequently extended in [22–25].

[2]Here $\mathrm{PD}(\alpha)$ denotes the Poincaré dual class of α.

[3]The tangent space $T_p\Omega$ is canonically identified with $H^\bullet(X,\mathbb{C})$ for any $p \in \Omega$. Thus the $\cup$-product $Y \cup W$ of local vector fields is well defined.

[4]A point $p \in M$ is semisimple if the Frobenius algebra $(T_pM, \circ_p, \eta_p)$ is with no nilpotents.

In this paper, we mainly consider the example of analytic Frobenius manifolds given by the quantum cohomology $QH^\bullet(X)$ of a complex smooth projective variety X, see [30,58,61]. In such a case, points $p \in QH^\bullet(X)$ are parameters of isomonodromic deformations of a linear system of differential equations of the form

$$\frac{\partial}{\partial z}\zeta(z,p) = \left(\boldsymbol{\mathcal{U}}(p) + \frac{1}{z}\boldsymbol{\mu}(p)\right)\zeta(z,p). \tag{1.2.1}$$

Here ζ is a z-dependent vector field of $QH^\bullet(X)$, whereas $\boldsymbol{\mathcal{U}}$ and $\boldsymbol{\mu}$ are $(1,1)$-tensors on $QH^\bullet(X)$: the first[5] is the operator of quantum multiplication by the Euler vector field –a distinguished vector field on $QH^\bullet(X)$ which equals the first Chern class $c_1(X)$ along the locus of small quantum cohomology – the second, called *grading operator*, keeps track of the non-vanishing degrees of $H^\bullet(X,\mathbb{C})$.

Equation (1.2.1) is a rich object associated with the variety X: it encapsulates information not only about its *Gromov–Witten theory*, but also (conjecturally) about its *topology*, its *algebraic geometry*, and their mutual relations. The study of the monodromy of solutions of (1.2.1) is the way to disclose such an amount of information, see [21,31,36]. In this paper we address the following:

Main Problem. *Find integral representations of solutions of* (1.2.1) *for Fano complete intersections in Fano varieties.*

We split the main problem into two parts:

(1) reduce the system of differential equations (1.2.1) to a distinguished scalar linear differential equation, the *master differential equation*,

(2) find integral representations of solutions of master differential equations.

The study of these questions leads us to introduce some relevant notions, both in the analytic theory of Frobenius manifolds and in the theory of integral transforms. The first three ingredients are the notions of *cyclic stratum*, *master differential equations* and *master functions* of a Frobenius manifold. The second new analytical tool is a pair of integral multilinear transforms of functions, that we call *Borel–Laplace* $(\boldsymbol{\alpha},\boldsymbol{\beta})$*-multitransforms*. We are going to briefly outline these objects.

1.3 Master functions and master differential equations

The rich geometry of a Frobenius manifold M is (almost) completely encoded in integrability conditions of the *extended deformed connection* or *first structural connection* of M (see [30,32,61]). This is a flat meromorphic connection $\widehat{\nabla}$ defined on the pullback π^*TM of the tangent bundle of M on the extended manifold $\widehat{M} := \mathbb{C}^* \times M$,

[5] Precise definitions will be given in the main body of the paper.

by the natural projection $\pi\colon \widehat{M} \to M$. Equation (1.2.1) is equivalent to the equation

$$\widehat{\nabla}_{\frac{\partial}{\partial z}} \xi = 0, \quad \xi \in \Gamma(\pi^* T^* M), \tag{1.3.1}$$

the one-form ξ and the vector field ζ being identified via a flat metric η on M. We call *master function* at $p \in M$ any function[6] $\Phi_\xi \in \mathcal{O}(\widetilde{\mathbb{C}^*})$ of the form

$$\Phi_\xi(z) = z^{-\frac{d}{2}} \langle \xi(z,p), e(p) \rangle,$$

where ξ is as in (1.3.1), and d is the *charge* of the Frobenius manifold M.

In the first part of the paper, we address the problem of reducing the system of differential equations (1.3.1) to a scalar differential equation, whose coefficients depend on the point $p \in M$. This is a well-known problem in the theory of ordinary differential equations, equivalent to the choice of a *cyclic vector* [28, Lemma II.1.3]. On Frobenius manifold, however, we have a *natural* candidate, namely the unit vector field $e \in \Gamma(TM)$.

In Chapter 2 we introduce the *cyclic stratum* $\widehat{M}^{\text{cyc}} \subseteq \widehat{M}$ defined as the set of points (z,p) at which the iterated covariant derivatives

$$e, \ \widehat{\nabla}_{\frac{\partial}{\partial z}} e, \ \widehat{\nabla}^2_{\frac{\partial}{\partial z}} e, \ \ldots, \ \widehat{\nabla}^{n-1}_{\frac{\partial}{\partial z}} e, \quad n := \dim_{\mathbb{C}} M, \tag{1.3.2}$$

define a basis of the fiber $\pi^* TM|_{(z,p)}$. The complement of $\widehat{M}^{\text{cyc}}$ in $\mathbb{P}^1 \times M$ admits a natural stratification, whose study is addressed in Section 2.6. A particular role is played by the $\mathcal{A}_\Lambda$-stratum of M, defined as the set of points $p \in M$ such that

$$\mathbb{C}^* \times \{p\} \subseteq \widehat{M} \setminus \widehat{M}^{\text{cyc}}.$$

Introducing the cyclic coframe $\omega_0, \ldots, \omega_{n-1} \in \Gamma(\pi^* T^* M)$ as the dual frame of the iterated covariant derivatives (1.3.2), the system of differential equations (1.3.1), specialized at points $p \in M \setminus \mathcal{A}_\Lambda$, reduces to a scalar differential equation – the *master differential equation* – in the function $\langle \xi, e \rangle$. Hence, at points $p \in M \setminus \mathcal{A}_\Lambda$, we obtain a one-to-one correspondence

$$\{\text{Solutions } \xi \text{ of system (1.3.1) specialized at } p\} \Longleftrightarrow \{\text{Master functions } \Phi_\xi \text{ at } p\}.$$

See Theorems 2.7.4 and 2.7.6. Thus, if integral representations for a basis of master functions are found, the main problem is *solved* at points in $M \setminus \mathcal{A}_\Lambda$.

Some motivational comments for introducing these new tools are in order. The notions of master functions and master differential equations define analogs, for an arbitrary Frobenius manifold, of well-known objects in Gromov–Witten and quantum cohomology theories. Namely, in the case of quantum cohomology the components of Givental's J-function (with respect to an *arbitrary* cohomology basis) define a gen-

[6] Here $\widetilde{\mathbb{C}^*}$ denotes the universal cover of $\mathbb{C}^*$.

erating set of master functions. Moreover, the master differential equation is (up to re-scaling of the unknown function) a quantum differential equation as defined, e.g., in [27, Section 10.3], see Chapter 5. In our opinion the concepts of cyclic stratum, master functions, and master differential equations may represent relevant notions in the analytic theory of Frobenius manifolds. For example, any contingent relations with the geometry of distinguished subsets of Frobenius manifolds (e.g., bifurcation diagram, Maxwell stratum, caustic) deserve further investigations. In that regard, it would be interesting to study relations with results of [22,23], concerning the isomonodromic description of Frobenius manifolds at semisimple coalescing points. This point will be addressed in a future publication.

1.4 Borel–Laplace multitransforms

In Chapter 6, we introduce a pair of multilinear transforms in both a formal and an analytical setting.

For $h \in \mathbb{N}^*$, and a given h-tuple $\boldsymbol{\kappa} \in (\mathbb{C}^*)^h$, we introduce a ring $\mathscr{F}_{\boldsymbol{\kappa}}(A)$ of Ribenboim generalized power series [68, 69] with both coefficients and exponents in a finite-dimensional, commutative, associative, and unitary $\mathbb{C}$-algebra A. The numbers κ_i play a role of "weights" for the exponents of the power series. In such a formal setting, given $\boldsymbol{\alpha}, \boldsymbol{\beta} \in (\mathbb{C}^*)^h$, we introduce the *Borel–Laplace* $(\boldsymbol{\alpha}, \boldsymbol{\beta})$*-multitransforms* as two A-multilinear maps rescaling the weights

$$\mathscr{B}_{\boldsymbol{\alpha},\boldsymbol{\beta}}\colon \bigotimes_{j=1}^{h} \mathscr{F}_{\kappa_j}(A) \to \mathscr{F}_{\boldsymbol{\alpha}^{-1}\cdot\boldsymbol{\beta}^{-1}\cdot\boldsymbol{\kappa}}(A), \quad \boldsymbol{\alpha}^{-1}\cdot\boldsymbol{\beta}^{-1}\cdot\boldsymbol{\kappa} := \left(\frac{\kappa_1}{\alpha_1\beta_1}, \dots, \frac{\kappa_h}{\alpha_h\beta_h}\right),$$

$$\mathscr{L}_{\boldsymbol{\alpha},\boldsymbol{\beta}}\colon \bigotimes_{j=1}^{h} \mathscr{F}_{\kappa_j}(A) \to \mathscr{F}_{\boldsymbol{\alpha}\cdot\boldsymbol{\beta}\cdot\boldsymbol{\kappa}}(A), \qquad \boldsymbol{\alpha}\cdot\boldsymbol{\beta}\cdot\boldsymbol{\kappa} := (\alpha_1\beta_1\kappa_1, \dots, \alpha_h\beta_h\kappa_h).$$

See Sections 6.2 and 6.3 for precise definitions.

In the analytical setting, given h functions $\Phi_1, \dots, \Phi_h\colon \widetilde{\mathbb{C}^*} \to A$, we define their Borel–Laplace $(\boldsymbol{\alpha}, \boldsymbol{\beta})$-multitransforms by

$$\mathscr{B}_{\boldsymbol{\alpha},\boldsymbol{\beta}}[\Phi_1, \dots, \Phi_h](z) := \frac{1}{2\pi i}\int_\gamma \prod_{j=1}^{h} \Phi_j\left(z^{\frac{1}{\alpha_j\beta_j}}\lambda^{-\beta_j}\right)e^\lambda \frac{d\lambda}{\lambda},$$

$$\mathscr{L}_{\boldsymbol{\alpha},\boldsymbol{\beta}}[\Phi_1, \dots, \Phi_h](z) := \int_0^\infty \prod_{i=1}^{h} \Phi_i(z^{\alpha_i\beta_i}\lambda^{\beta_i})e^{-\lambda}\, d\lambda,$$

provided that the integrals exist. The contour γ is a Hankel-type contour beginning from $-\infty$, circling the origin once in the positive direction, and returning to $-\infty$ (see Figure 6.1).

1.5 Main results

Consider a Fano smooth projective variety X, and let $\iota\colon Y \to X$ be a Fano subvariety defined as the zero locus of a regular section of a vector bundle $E \to X$. The classical cohomology groups $H^k(Y,\mathbb{C})$ can be (partially) recovered by the cohomology groups $H^k(X,\mathbb{C})$ by the Lefschetz hyperplane theorem. The quantum Lefschetz theorem is a quantum improvement of the classical result: it describes how to reconstruct the Gromov–Witten theory of Y starting from the Gromov–Witten theory of X (see [15, 17, 60]).

In this paper, by using the quantum Lefschetz theorem, we give explicit integral representations of master functions of Y in terms of Laplace $(\boldsymbol{\alpha},\boldsymbol{\beta})$-multitransforms of master functions of the ambient space X under the following assumptions on X and E:

Case 1. We assume that E is a direct sum of fractional powers of the determinant bundle $\det TX$ of X.

Case 2. We assume that $X = X_1 \times \cdots \times X_h$ is a product of Fano varieties X_i, and that E is the external tensor product of fractional powers of the determinant bundles $\det TX_i$.

Our first main result concerns Case 1. Our Theorem 7.2.1 asserts that any master function of Y, at points $\iota^*\delta \in H^2(Y,\mathbb{C})$ of its small quantum cohomology, can be expressed in terms of iterated Laplace (α,β)-transforms (simple transforms of a single function) of master functions of X at the point $\delta \in H^2(X,\mathbb{C})$. More precisely, if $E = \bigoplus_{j=1}^r L^{\otimes d_j}$, and $\det TX = L^\ell$ for an ample line bundle L, then any master function of Y at $\iota^*\delta$ is a $\mathbb{C}$-linear combination of integrals of the form

$$e^{-c_\delta z}\mathscr{L}_{\frac{\ell-\sum_{i=1}^s d_i}{d_s},\frac{d_s}{\ell-\sum_{i=1}^{s-1} d_i}}\circ\cdots\circ\mathscr{L}_{\frac{\ell-d_1-d_2}{d_2},\frac{d_2}{\ell-d_1}}\circ\mathscr{L}_{\frac{\ell-d_1}{d_1},\frac{d_1}{\ell}}[\Phi]$$
$$= e^{-c_\delta z}\int_0^\infty\cdots\int_0^\infty \Phi\left(z^{\frac{\ell-\sum_{j=1}^r d_j}{\ell}}\prod_{i=1}^r \zeta_i^{\frac{d_i}{\ell}}\right)e^{-\sum_{i=1}^r \zeta_i}\,d\zeta_1\dots d\zeta_r,$$

where Φ is a master function of X at δ, and $c_\delta \in \mathbb{C}$ is a complex number depending on δ.

Our second main result concerns Case 2. In particular, Theorem 7.3.1 asserts that any master function of Y, at points $\iota^*\delta \in H^2(Y,\mathbb{C})$ of the small quantum locus, can be expressed in terms of Laplace $(\boldsymbol{\alpha},\boldsymbol{\beta})$-multitransforms of master functions of X_j at the point $\delta_j \in H^2(X,\mathbb{C})$, where

$$\delta = \sum_{j=1}^h 1\otimes\cdots\otimes\delta_j\otimes\cdots\otimes 1.$$

More precisely, if $E = \boxtimes_{j=1}^{h} L_j^{\otimes d_j}$ and $\det TX_j = L_j^{\ell_j}$ for ample line bundles L_j, any master function of Y at $\iota^*\delta$ is a $\mathbb{C}$-linear combination of integrals of the form

$$e^{-c_\delta z}\mathscr{L}_{\boldsymbol{\alpha},\boldsymbol{\beta}}[\Phi_1,\dots,\Phi_h](z) = e^{-c_\delta z}\int_0^\infty \prod_{j=1}^{h} \Phi_j\Big(z^{\frac{\ell_j-d_j}{\ell_j}}\lambda^{\frac{d_j}{\ell_j}}\Big)e^{-\lambda}\,d\lambda,$$

where $(\boldsymbol{\alpha},\boldsymbol{\beta}) = (\frac{\ell_1-d_1}{d_1},\dots,\frac{\ell_h-d_h}{d_h};\frac{d_1}{\ell_1},\dots,\frac{d_h}{\ell_h})$, Φ_j is a master function of X_j at δ_j, and $c_\delta \in \mathbb{C}$ is a complex number depending on δ.

Assumptions of Cases 1 and 2 are clearly satisfied when the varieties X and X_j have Picard rank one. Therefore, Theorems 7.2.1 and 7.3.1 can be applied to all Fano complete intersections in $\mathbb{P}^n$ and Fano hypersurfaces in products of projective spaces, in order to obtain explicit Mellin–Barnes integral representations of master functions. In particular, if $Y \subseteq \mathbb{P}^{n-1}$ is a Fano complete intersection defined by homogeneous polynomials of degrees $d_1,\dots,d_h$, our Theorem 7.4.1 asserts that any master function of Y at $0 \in H^\bullet(Y,\mathbb{C})$ is a linear combination of *one*-dimensional Mellin–Barnes integrals ($j = 0,\dots,n-1$)

$$G_j(z) := \frac{e^{-cz}}{2\pi\sqrt{-1}}\int_\gamma \Gamma(s)^n \prod_{k=1}^{h}\Gamma(1-d_k s) z^{-(n-\sum_{k=1}^h d_k)s}\varphi_j(s)\,ds,$$

where $c \in \mathbb{Q}$, γ is a parabola (of the form $\operatorname{Re} s = -\rho_1(\operatorname{Im} s)^2 + \rho_2$, for suitable $\rho_1, \rho_2 \in \mathbb{R}_+$) encircling the poles of the factor $\Gamma(s)^n$ and separating them from the poles of the factors $\Gamma(1-d_k s)$, and the function $\varphi_j(s)$ are defined by

$$\varphi_j(s) := \begin{cases} \exp(2\pi\sqrt{-1}js), & n \text{ even},\\ \exp(2\pi\sqrt{-1}js + \pi\sqrt{-1}s), & n \text{ odd}.\end{cases}$$

In the case of a Fano hypersurface $Y \subseteq \mathbb{P}^{n_1-1}\times\dots\times\mathbb{P}^{n_h-1}$ defined by a homogeneous polynomial of multi-degree $(d_1,\dots,d_h)$, then our Theorem 7.4.2 asserts that any master function of Y at $0 \in H^\bullet(Y,\mathbb{C})$ is a linear combination of the h-dimensional Mellin–Barnes integrals ($j = 0,\dots,n-1$)

$$H_{\boldsymbol{j}}(z) := \frac{e^{-cz}}{(2\pi\sqrt{-1})^h}\int_{\times\gamma_i}\left[\prod_{i=1}^{h}\Gamma(s_i)^{n_i}\varphi_{j_i}^i(s_i)\right]\Gamma\left(1-\sum_{i=1}^{h} d_i s_i\right) \times z^{-\sum_{i=1}^h (n_i-d_i)s_i}\,ds_1\dots ds_h,$$

where $c \in \mathbb{Q}$, γ_i are parabolas (of the form $\operatorname{Re} s_i = -\rho_{1,i}(\operatorname{Im} s_i)^2 + \rho_{2,i}$, for suitable $\rho_{1,i}, \rho_{2,i} \in \mathbb{R}_+$) encircling the poles of the factors $\Gamma(s_i)^{n_i}$, and the functions $\varphi_{j_i}^i(s_i)$ are defined by

$$\varphi_{j_i}^i(s_i) := \begin{cases} \exp(2\pi\sqrt{-1}j_i s_i), & n_i \text{ even},\\ \exp(2\pi\sqrt{-1}j_i s_i + \pi\sqrt{-1}s_i), & n_i \text{ odd},\end{cases}$$

for any h-tuple $\boldsymbol{j} = (j_1,\dots,j_h)$ with $0 \leqslant j_h \leqslant n_i - 1$.

Some comments are in order. Given a Fano variety X, Mirror Symmetry provides other kinds of integral representations of solutions of equation (1.3.1).[7] These are complex oscillating integrals associated with the Landau–Ginzburg models mirror to X, see [35, 39–41, 50, 57]. In these representations the cycles of integration are multi-dimensional.[8] This fact typically makes more difficult the study of the asymptotic expansions of solutions, and of the determination of the corresponding validity sectors in $\widetilde{\mathbb{C}^*}$. Furthermore, let us recall another technical issue which may be faced: Landau–Ginzburg models may not have enough critical points, and suitable compactification procedures have to be applied in order to recover the right number, see [43, 66, 70]. This could represent a delicate point for the computation of the Stokes bases of solutions of equation (1.2.1), whose exponential growth is ruled by the critical values of the Landau–Ginzburg potential.

We believe that one-dimensional Mellin–Barnes integrals of Theorem 7.4.1 represent a more advantageous representation of the solutions to the purpose of asymptotic analysis. Moreover, even for multi-dimensional Mellin–Barnes integrals of Theorem 7.4.2 the study of their asymptotics is tame: it is equivalent to the study of the asymptotics of one-dimensional generalized Faxén integrals

$$I(\lambda; c_1, \dots, c_r) := \int_0^\infty \exp\left[-\lambda\left(x^\mu + \sum_{k=1}^{r} c_k x^{m_k}\right)\right] dx,$$

with $\mu > m_1 > m_2 > \cdots > m_r > 0$, which have saddle points whose exponential contributions dominate algebraic terms in the asymptotic expansion. See [65, Chapter 7], [53, Section 5] for a detailed asymptotic analysis, and also [7, 13, 81] for some special cases. This will be exemplified in Section 11.6.

1.6 Dubrovin conjecture for Hirzebruch surfaces

Equation (1.2.1) has two singularities: a Fuchsian singularity at $z = 0$ and an irregular singularity at $z = \infty$ of Poincaré rank 1. The monodromy of its solutions is quantified by a finite set of matrices:

- a monodromy matrix M_0, quantifying the monodromy of solutions of (1.2.1) at $z = 0$,

[7] More precisely, for the equations $\hat{\nabla}_{\frac{\partial}{\partial t^\alpha}} \xi = 0$, where $t^1, \dots, t^n$ are coordinates on $QH^\bullet(X)$, and not with respect to the spectral parameter z.

[8] Notice, for example, that already in the case of $\mathbb{P}^n$ these oscillating integrals are over n-dimensional cycles. On the other hand, one-dimensional Mellin–Barnes integral representations of solutions of equation (1.2.1) associated with $\mathbb{P}^n$ were obtained in [46]. Their asymptotics in sectors of $\widetilde{\mathbb{C}^*}$ is easier to study.

- a Stokes matrix S, describing the Stokes phenomenon at $z = \infty$,
- and a central connection matrix C gluing the monodromy data M_0 and S at the two singularities.

Remarkably, the monodromy data define a sort of "system of coordinates" in the space of solutions of WDVV equations: from the knowledge of their numerical values, the whole Frobenius manifold structure can be reconstructed via a Riemann–Hilbert problem [30, 32, 47].

In [31], B. Dubrovin formulated an intriguing conjecture concerning the geometrical meaning of the numerical values of the monodromy data of quantum cohomologies of Fano varieties. In the *qualitative* part of the conjecture, for a given Fano variety X, the semisimplicity condition of $QH^\bullet(X)$ is claimed to be equivalent to the existence of full exceptional collections in the derived category $\mathcal{D}^b(X)$ of coherent sheaves on X. Moreover, in the refined *quantitative* part of the conjecture, formulated in [21, Conjecture 5.2], the Stokes and central connection matrices (S_p, C_p) computed at any point $p \in QH^\bullet(X)$ are claimed to be determined by characteristic classes of X and of objects of a full exceptional collection $\mathfrak{E}_p$ in $\mathcal{D}^b(X)$.

In particular, the central connection matrix C_p is claimed to equal the matrix associated with the morphism

$$\begin{aligned} \mathit{\Pi}_X^-\colon K_0(X)_{\mathbb{C}} &\to H^\bullet(X, \mathbb{C}), \\ F &\mapsto \frac{(\sqrt{-1})^{\overline{d}}}{(2\pi)^{\frac{d}{2}}} \widehat{\Gamma}_X^- \exp(-\pi\sqrt{-1}c_1(X))\mathrm{Ch}(F), \end{aligned} \tag{1.6.1}$$

where $d = \dim_{\mathbb{C}} X$, $\overline{d}$ is its residue class modulo 2, $\widehat{\Gamma}_X^-$ is the characteristic class of X defined by

$$\widehat{\Gamma}_X^- := \prod_{j=1}^{\dim_{\mathbb{C}} X} \Gamma(1-\delta_j), \qquad \delta_j \text{ Chern roots of } TX,$$

where

$$\Gamma(1-t) = \exp\left(\gamma t + \sum_{n=2}^{\infty} \frac{\zeta(n)}{n} t^n\right),$$

and $\mathrm{Ch}(F)$ is the graded Chern character defined on vector bundles by the formula $\mathrm{Ch}(V) := \sum_{j=1}^{\mathrm{rk}V} \exp(2\pi\sqrt{-1}\varepsilon_j)$, ε_j being the Chern roots of V. The matrix of $\mathit{\Pi}_X^-$ is computed with respect to the exceptional basis $[\mathfrak{E}_p]$ of $K_0(X)_{\mathbb{C}}$, defined by the K-theoretical classes of objects of $\mathfrak{E}_p$, and an arbitrary[9] basis of $H^\bullet(X, \mathbb{C})$. Furthermore, if the central connection matrix C_p is related to the morphism $\mathit{\Pi}_X^-$ as explained above, then the Stokes matrix S_p *automatically* equals the inverse of the Gram matrix

[9]The choice of a basis of $H^\bullet(X, \mathbb{C})$ in (1.6.1) corresponds to the choice of a system of flat coordinates on $QH^\bullet(X)$ with respect to which the monodromy data (M_0, S, C) are computed.

of the Grothendieck–Euler–Poincaré χ-pairing on $K_0(X)$ with respect to the exceptional basis $[\mathfrak{E}_p]$, see [21, Corollary 5.8].

It is important to stress that the monodromy data (M_0, S, C) are defined up to several choices: the choice of a system of flat coordinates on the Frobenius manifold $QH^\bullet(X)$, choices of normalizations (at both $z = 0$ and $z = \infty$) of solutions of equation (1.2.1), and the choice of an "admissible ray" in $\mathbb{C}^*$. Remarkably, all these operations have a geometrical counterpart in derived categories, see [21, Theorem 5.9]. Deserving special mention is Γ-conjecture II of [36]: it consists of an equivalent conjectural statement about the central connection matrix, though with respect to a choice of a solution in "Levelt form" at $z = 0$ not *natural* from the point of view of the theory of Frobenius manifolds. See [21, Section 5.6] for details.

The explicit computation of the monodromy data of quantum cohomologies is typically a rather delicate operation. To the best knowledge of the author, the only cases in which the computation of the complete set of monodromy data (S, C) of equation (1.2.1) has been carried out in all the details (including the determination of the corresponding full exceptional collections) are the cases of projective spaces [32, 46] and of complex Grassmannians [21, 36]. We believe that the main results of the current paper, namely the integral representations described in Theorems 7.2.1, 7.3.1, 7.4.1, and 7.4.2, will represent a fundamental tool for the development of this study [20].

As an application, in Chapters 10 and 11, we will show how to use the Laplace $(\boldsymbol{\alpha}, \boldsymbol{\beta})$-multitransform, and the main results described above, in order to prove the quantitative part of the Dubrovin conjecture for Hirzebruch surfaces [49]. These are surfaces $\mathbb{F}_k$, $k \in \mathbb{Z}$, defined as the total space of the projective bundle $\mathbb{P}(\mathcal{O} \oplus \mathcal{O}(-k))$ on $\mathbb{P}^1$. The interest of this example is highlighted by the fact that

- only two Hirzebruch surfaces are Fano varieties (namely $\mathbb{F}_0$ and $\mathbb{F}_1$),
- all others Hirzebruch surfaces are deformation equivalent to either $\mathbb{F}_0$ or $\mathbb{F}_1$.

Results of A. Bayer already suggested the non-necessity of the Fano assumption for the validity of the qualitative part of the Dubrovin conjecture, see [9]. Moreover, X. Hu proved that, in a smooth family of complete varieties, the existence of full exceptional collection on a fiber preserves for the fibers in a neighborhood, see [51]. See also [11, Corollary B] for an analogue result for arbitrary semiorthogonal decompositions. To the best of our knowledge, the study of the monodromy of the isomonodromic systems (1.2.1) associated with Hirzebruch surfaces, developed in Chapters 10 and 11, represents the first example in literature which addresses also the quantitative part of the Dubrovin conjecture, in both the non-Fano case and the case of deformations of the complex structures.

The case of Hirzebruch surfaces $\mathbb{F}_{2k}$ (resp. $\mathbb{F}_{2k+1}$) can be reduced to the single case of $\mathbb{F}_0 = \mathbb{P}^1 \times \mathbb{P}^1$ (resp. $\mathbb{F}_1 = \mathrm{Bl}_{\mathrm{pt}}\mathbb{P}^2$). The monodromy data of $QH^\bullet(\mathbb{F}_0)$ can easily be reconstructed from the monodromy data of $QH^\bullet(\mathbb{P}^1)$, see Theorem 10.3.3.

In the case of $QH^\bullet(\mathbb{F}_1)$, the computation is more delicate, and reduces to the study of the quantum differential equation

$$\begin{aligned}(283z-24)\vartheta^4\Phi &+ (283z^2-590z+24)\vartheta^3\Phi + (-2264z^2+192z+3)\vartheta^2\Phi\\ &- 4z^2(2547z^2+350z-104)\vartheta\Phi\\ &+ z^2(-3113z^3-9924z^2+1476z+192)\Phi = 0,\end{aligned}$$

where $\vartheta := z\frac{d}{dz}$. In Section 11.4, we show that the solutions of this equation can be expressed as linear combinations of integrals of the form

$$e^{-z}\mathscr{L}_{(1,2;\frac{1}{2},\frac{1}{3})}[\Phi_1,\Phi_2;z] = e^{-z}\int_0^\infty \Phi_1\big(z^{\frac{1}{2}}\lambda^{\frac{1}{2}}\big)\Phi_2\big(z^{\frac{2}{3}}\lambda^{\frac{1}{3}}\big)e^{-\lambda}d\lambda,$$

where Φ_1 and Φ_2 are solutions of quantum differential equations of $\mathbb{P}^1$ and $\mathbb{P}^2$, respectively, that is,

$$\vartheta^2\Phi_1 = 4z^2\Phi_1, \quad \vartheta^3\Phi_2 = 27z^3\Phi_2.$$

This allows the study of the asymptotics of solutions in sectors of $\widetilde{\mathbb{C}^*}$, to reconstruct the Stokes bases of solutions of the quantum differential equation of $\mathbb{F}_1$, and finally to the computation of both Stokes and central connection matrices, see Theorem 11.8.2.

From these results, the quantitative part of the Dubrovin conjecture is proved for all Hirzebruch surfaces $\mathbb{F}_k$, by making explicit the exceptional collections in $\mathcal{D}^b(\mathbb{F}_k)$ which arise from the monodromy data, see Theorems 10.3.3 and 11.8.3.

1.7 Plan of the paper

The paper is organized as follows. In Chapter 2, we introduce the notion of *cyclic stratum* in the general context of Frobenius manifolds theory. A first study of the geometry of the cyclic stratum, and its complement in the extended manifold $\mathbb{C}^*\times M$, is addressed.

In Chapter 3, we recall basic definitions in Gromov–Witten theory, including the definition of the Frobenius manifold structure on the quantum cohomology of a smooth projective variety. In Chapter 4, we recall the definitions of topological-enumerative solution of the isomonodromic system (1.2.1), and also of its monodromy data. We also recall the main properties and natural transformations of the complete set of monodromy data.

In Chapter 5, we recall the definition of Givental's J-function, and we explain how it is related to the space of master functions, see Theorem 5.1.2 and Corollary 5.1.3. We recall the formulation of the quantum Lefschetz theorem, and we obtain an upper bound for the dimension of the space of master functions of a Fano hypersurface of a smooth projective variety X, see Theorem 5.4.1.

In Chapter 6, we recall the notion of generalized power series in the sense of P. Ribenboim, and we introduce the ring $\mathscr{F}_{\boldsymbol{\kappa}}(A)$ of generalized power series with coefficients and exponents in a finite-dimensional $\mathbb{C}$-algebra. We introduce the notions of Borel–Laplace $(\boldsymbol{\alpha}, \boldsymbol{\beta})$-multitransforms, in both formal and analytic setting, and we prove the compatibility of the two definitions, see Theorem 6.5.1.

In Chapter 7, we explain how the J-function can be identified (in several ways) with elements of rings of Ribenboim generalized power series. We prove the main results of this paper, Theorems 7.2.1, 7.3.1, 7.4.1 and 7.4.2.

In Chapter 8, we recall the notions of exceptional collections in derived categories of coherent sheaves, exceptional bases in K-theory, their mutations and helices. We then describe the refined statement of the Dubrovin conjecture, as formulated in [21].

In Chapter 9, we describe the classical and quantum cohomology rings of Hirzebruch surfaces.

In Chapter 10, we explicitly compute the monodromy data of the quantum cohomologies $QH^\bullet(\mathbb{F}_{2k})$, and we prove the Dubrovin conjecture for Hirzebruch surfaces $\mathbb{F}_{2k}$.

In Chapter 11, we address the study of the quantum differential equations of Hirzebruch surfaces $\mathbb{F}_{2k+1}$. We show how to use the Laplace $(1, 2; \frac{1}{2}, \frac{1}{3})$-multitransform in order to give integral representations of solutions, how to reconstruct Stokes fundamental solutions, and hence how to compute the monodromy data. This leads to a proof of the Dubrovin conjecture for Hirzebruch surfaces $\mathbb{F}_{2k+1}$.