Chapter 2

Cyclic stratum of Frobenius manifolds

2.1 Frobenius manifolds

Given a complex manifold M, we denote by TM (resp. T^*M) its holomorphic tangent (resp. cotangent) bundle. If E is a holomorphic vector bundle on M, we denote by $\bigodot^k E$ its k-th symmetrized tensor power, and by $\Gamma(E)$ the vector space of global holomorphic sections of E.

Definition 2.1.1. A *Frobenius manifold* structure on a complex manifold M of dimension n is defined by giving

(FM1) a symmetric $\mathcal{O}(M)$-bilinear form $\eta \in \Gamma(\bigodot^2 T^*M)$, called *metric*,[1] whose corresponding Levi-Civita connection ∇ is flat,

(FM2) a $(1,2)$-tensor $c \in \Gamma(TM \otimes \bigodot^2 T^*M)$ such that
  - (a) the induced multiplication of vector fields $X \circ Y := c(-, X, Y)$, for $X, Y \in \Gamma(TM)$, is *associative*,
  - (b) $c^\flat \in \Gamma(\bigodot^3 T^*M)$,
  - (c) $\nabla c^\flat \in \Gamma(\bigodot^4 T^*M)$,

(FM3) a vector field $e \in \Gamma(TM)$, called the *unity vector field*, such that
  - (a) the bundle morphism $c(-, e, -)\colon TM \to TM$ is the identity morphism,
  - (b) $\nabla e = 0$,

(FM4) a vector field $E \in \Gamma(TM)$, called the *Euler vector field*, such that
  - (a) $\mathfrak{L}_E c = c$,
  - (b) $\mathfrak{L}_E \eta = (2 - d) \cdot \eta$, where $d \in \mathbb{C}$ is called the *charge* of the Frobenius manifold.

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[1] In what follows, we will denote by $(-)^\flat$ and $(-)^\sharp$ the *musical isomorphisms* induced by the metric η. These are the isomorphisms between vector spaces of mixed tensors. If $v \in \Gamma(TM)$, the one-form $v^\flat \in \Gamma(T^*M)$ is defined by $v^\flat(w) = \eta(w, v)$, where $w \in \Gamma(TM)$. Conversely, if $\xi \in \Gamma(T^*M)$, the vector field $\xi^\sharp \in \Gamma(TM)$ is uniquely defined by the identity

$$\xi(w) = \eta(w, \xi^\sharp),$$

where $w \in \Gamma(TM)$. Thus, $(-)^\flat\colon \Gamma(TM) \to \Gamma(T^*M)$ and $(-)^\sharp\colon \Gamma(T^*M) \to \Gamma(TM)$ are mutually inverse. In components, these operations are also known as "lowering" and "raising" of indices, respectively. These operations naturally extend to mixed tensors. For example, given a $(1,2)$-tensor $c \in \Gamma(TM \otimes T^*M \otimes T^*M)$, the tensor $c^\flat$ is the $(0,3)$-tensor defined by

$$c^\flat(v_1, v_2, v_3) = \eta(v_1, c(v_2, v_3)),$$

where $v_1, v_2, v_3 \in \Gamma(TM)$.

At any point $p \in M$ the triple $(T_pM, \eta_p, \circ_p)$ is a complex *Frobenius algebra*, namely an associative commutative algebra with unity whose product is compatible with the metric, in the sense that

$$\eta_p(a \circ_p b, c) = \eta_p(a, b \circ_p c) \quad \text{for all } a, b, c \in T_pM,$$

by axioms (FM2-a), (FM2-b), (FM3-a). Moreover, there exist an open neighborhood $\Omega \subseteq M$ of p and a function $F\colon \Omega \to \mathbb{C}$ such that

$$c^\flat = \nabla^3 F,$$
$$\eta = \nabla_e \nabla^2 F.$$

This follows from axiom (FM2-b). Any such a function F will be called *potential* of M.

Remark 2.1.2. The Euler vector field E is an affine vector field, i.e.

$$\nabla^2 E = 0.$$

This follows[2] from axioms (FM1) and (FM4-b).

Convention. In this paper, we assume that the flat endomorphism $X \mapsto \nabla_X E$ of TM is *diagonalizable*. By introducing ∇-flat coordinates $\boldsymbol{t} = (t^\alpha)_{\alpha=1}^n$ on M, with respect to which the metric η is constant and the connection ∇ coincides with partial derivatives, we have that

$$E = \sum_{\alpha=1}^n ((1 - q_\alpha)t^\alpha + r_\alpha)\frac{\partial}{\partial t^\alpha}, \quad q_\alpha, r_\alpha \in \mathbb{C}.$$

Following [30–32], we choose flat coordinates $\boldsymbol{t}$ so that $\frac{\partial}{\partial t^1} \equiv e$ and $r_\alpha \neq 0$ only if $q_\alpha = 1$. This can always be done, up to an affine change of coordinates.

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[2]For a generic vector field X on a pseudo-Riemannian manifold (M, g), a simple computation (invoking the first Bianchi identities) shows that

$$\nabla_\beta \nabla_\alpha X_\lambda = \sum_\mu R_{\lambda\alpha\beta\mu} X^\mu + \frac{1}{2}(\nabla_\beta K_{\alpha\lambda} + \nabla_\alpha K_{\beta\lambda} - \nabla_\lambda K_{\alpha\beta}),$$

where

$$K_{\alpha\beta} = (\mathfrak{L}_X g)_{\alpha\beta} = \nabla_\alpha X_\beta + \nabla_\beta X_\alpha.$$

If X is Killing conformal, and $\mathfrak{L}_X g = \omega g$ for a function ω, then

$$\nabla_\beta \nabla_\alpha X_\lambda = \sum_\mu R_{\lambda\alpha\beta\mu} X^\mu + \frac{1}{2}(g_{\alpha\lambda}\partial_\beta \omega + g_{\beta\lambda}\partial_\alpha \omega - g_{\alpha\beta}\partial_\lambda \omega).$$

In our case $R = 0$ and ω is a constant function.

Remark 2.1.3. The associativity of the algebra is equivalent to the following conditions for F, called WDVV-equations:

$$\sum_{\gamma,\delta=1}^{n} \partial_\alpha \partial_\beta \partial_\gamma F \, \eta^{\gamma\delta} \partial_\delta \partial_\epsilon \partial_\nu F = \sum_{\gamma,\delta=1}^{n} \partial_\nu \partial_\beta \partial_\gamma F \, \eta^{\gamma\delta} \partial_\delta \partial_\epsilon \partial_\alpha F,$$

while axiom (FM4) is equivalent to

$$\eta_{\alpha\beta} = \partial_1 \partial_\alpha \partial_\beta F, \quad \mathfrak{L}_E F = (3-d)F + Q(\boldsymbol{t}),$$

with $Q(\boldsymbol{t})$ a quadratic expression in parameters t_α. Conversely, given a solution of the WDVV equations, satisfying the quasi-homogeneity conditions above, a structure of Frobenius manifold is naturally defined on an open subset of the space of parameters t^α.

Definition 2.1.4. Define the *grading operator* of M to be the tensor $\boldsymbol{\mu} \in \Gamma(TM \otimes T^*M)$ defined by

$$\boldsymbol{\mu}(Y) := \frac{2-d}{2} Y - \nabla_Y E, \quad Y \in \Gamma(TM).$$

In what follows we will also denote by $\boldsymbol{\mathcal{U}}$ the $(1,1)$-tensor defined by $\circ$-multiplication by the Euler vector field, i.e.

$$\boldsymbol{\mathcal{U}}(Y) := E \circ Y, \quad Y \in \Gamma(TM).$$

We denote by μ and $\mathcal{U}$ the matrices of components of the tensors $\boldsymbol{\mu}$, and $\boldsymbol{\mathcal{U}}$, respectively, with respect to the system $\boldsymbol{t}$ of ∇-flat coordinates.

2.2 Semisimple points and bifurcation set

Definition 2.2.1. A point $p \in M$ is *semisimple* if and only if the corresponding Frobenius algebra $(T_pM, *_p, \eta_p, \frac{\partial}{\partial t^1}|_p)$ is without nilpotents. Denote by M_{ss} the open dense subset of M of semisimple points.

In this paper, only generically semisimple Frobenius manifolds are considered. In other words, we will always assume $M_{\text{ss}} \neq \emptyset$.

On M_{ss} there are n well-defined idempotent vector fields $\pi_1, \dots, \pi_n \in \Gamma(TM_{\text{ss}})$, satisfying

$$\pi_i * \pi_j = \delta_{ij}\pi_i, \quad \eta(\pi_i, \pi_j) = \delta_{ij}\eta(\pi_i, \pi_i), \quad i, j = 1, \dots, n.$$

Theorem 2.2.2 ([29, 30, 32]). *The idempotent vector fields pairwise commute, that is,* $[\pi_i, \pi_j] = 0$ *for* $i, j = 1, \dots, n$. *Hence, there exist holomorphic local coordinates* $(u_1, \dots, u_n)$ *on* M_{ss} *such that* $\frac{\partial}{\partial u_i} = \pi_i$ *for* $i = 1, \dots, n$.

Definition 2.2.3. The coordinates $(u_1, \dots, u_n)$ of Theorem 2.2.2 are called *canonical coordinates*.

Proposition 2.2.4 ([30,32]). *Canonical coordinates are uniquely defined up to ordering and shifts by constants. The eigenvalues of the tensor* $\boldsymbol{\mathcal{U}}$ *define a system of canonical coordinates in a neighborhood of any semisimple point of* $M_{\rm ss}$.

Definition 2.2.5. Given a Frobenius manifold M, we call *bifurcation set* of M the set $\mathcal{B}_M$ of points $p \in M$ at which the spectrum of the operator $\mathcal{U}(p)$ is not simple, i.e. $u_i(p) = u_j(p)$ for some $i \neq j$.

Following the terminology of [21,23,25], the points of $\mathcal{B}_M$ which are semisimple are called *semisimple coalescing points*. We define the[3] *Maxwell stratum* of M to be the closure of the set of semisimple coalescing points, i.e. $\mathcal{M}_M := \overline{M_{\rm ss} \cap \mathcal{B}_M}$.

The *caustic* of M is the set-theoretic difference $\mathcal{K}_M := \mathcal{B}_M \setminus M_{\rm ss}$.

Lemma 2.2.6. *We have* $\mathcal{B}_M = \mathcal{M}_M \cup \mathcal{K}_M$. ■

Definition 2.2.7. We call *orthonormalized idempotent frame* a frame $(f_i)_{i=1}^n$ of $TM_{\rm ss}$ defined by

$$f_i := \eta(\pi_i, \pi_i)^{-\frac{1}{2}} \pi_i, \quad i = 1, \dots, n, \tag{2.2.1}$$

for arbitrary choices of signs of the square roots. The Ψ-matrix is the matrix of change of tangent frames $(\Psi_{i\alpha})_{i,\alpha=1}^n$, defined by

$$\frac{\partial}{\partial t^\alpha} = \sum_{i=1}^n \Psi_{i\alpha} f_i, \quad \alpha = 1, \dots, n.$$

Remark 2.2.8. In the orthonormalized idempotent frame, the operator $\boldsymbol{\mathcal{U}}$ is represented by a diagonal matrix, and the operator $\boldsymbol{\mu}$ by an antisymmetric matrix:

$$\begin{aligned} U &:= \operatorname{diag}(u_1, \dots, u_n), & \Psi \mathcal{U} \Psi^{-1} &= U, \\ V &:= \Psi \mu \Psi^{-1}, & V^T + V &= 0. \end{aligned}$$

2.3 Extended deformed connection

Given a Frobenius manifold M, we introduce the extended manifold $\widehat{M} := \mathbb{C}^* \times M$, and consider the pullback $\pi^* TM$ of the tangent bundle of M along the obvious projection $\pi \colon \widehat{M} \to M$. We will denote the natural lifts on $\widehat{M}$ of the tensors $\eta, c, e, E, \boldsymbol{\mu}$, $\boldsymbol{\mathcal{U}}$ by the same symbols. Moreover, we also denote by ∇ the pull-backed Levi-Civita connection: it is the connection on the vector bundle $\pi^* TM$, uniquely defined by the

[3] The name is taken from singularity theory: for Frobenius structures defined on the universal space of unfoldings of singularities the two notions coincide, see [1–3].

further requirement that

$$\nabla_{\frac{\partial}{\partial z}} Y = 0 \quad \text{for all } Y \in \pi^{-1}\mathscr{T}_M,$$

where z denotes the natural coordinate on $\mathbb{C}^*$, and $\mathscr{T}_M$ denotes the tangent sheaf of M. We are going now to define a second connection $\widehat{\nabla}$ on π^*TM which is a deformation of ∇.

Definition 2.3.1. We define the *extended deformed connection* $\widehat{\nabla}$ as the connection on π^*TM given by

$$\widehat{\nabla}_X Y = \nabla_X Y + zX \circ Y, \quad \widehat{\nabla}_{\frac{\partial}{\partial z}} Y = \nabla_{\frac{\partial}{\partial z}} Y + \mathcal{U}(Y) - \frac{1}{z}\boldsymbol{\mu}(Y)$$

for all $X, Y \in \Gamma(\pi^*TM)$.

Theorem 2.3.2 ([32]). *The extended deformed connection $\widehat{\nabla}$ is flat. More precisely, its flatness is equivalent to the totality of the following conditions:*

(1) $\nabla c^\flat \in \Gamma(\odot^4 T^*M)$,

(2) *the product on each tangent space of M is associative,*

(3) $\nabla^2 E = 0$,

(4) $\mathfrak{L}_E c = c$. ■

The connection $\widehat{\nabla}$ induces a flat connection on π^*T^*M, denoted by the same symbol.

2.4 Cyclic stratum, and cyclic (co)frame

Definition 2.4.1. Given a Frobenius manifold M, we define infinitely many sections $e_j \in \Gamma(\pi^*TM)$ as

$$e_k := \widehat{\nabla}^k_{\frac{\partial}{\partial z}} e, \quad k \in \mathbb{N}.$$

We will call the *cyclic stratum* $\widehat{M}^{\text{cyc}}$ to be the maximal open subset U of $\widehat{M}$ such that the bundle $\pi^*TM|_U$ is trivial and the collection of sections $(e_k|_U)_{k=0}^{n-1}$ defines a basis of each fiber. On $\widehat{M}^{\text{cyc}}$ we will also introduce the dual coframe $(\omega_j)_{j=0}^{n-1}$, by imposing

$$\langle \omega_j, e_k \rangle = \delta_{jk}. \tag{2.4.1}$$

The frame $(e_k)_{k=0}^{n-1}$ will be called *cyclic frame*, and its dual $(\omega_j)_{j=0}^{n-1}$ *cyclic coframe*.

Definition 2.4.2. Define the matrix-valued function $\Lambda = (\Lambda_{i\alpha}(z, p))$, holomorphic on $\widehat{M}^{\text{cyc}}$, by the equation

$$\frac{\partial}{\partial t^\alpha} = \sum_{i=0}^{n-1} \Lambda_{i\alpha} e_i, \quad \alpha = 1, \dots, n. \tag{2.4.2}$$

Remark 2.4.3. The Λ-matrix should be thought as an analogue of the Ψ-matrix. The former matrix relates the flat coordinate frame $(\frac{\partial}{\partial t^\alpha})_{\alpha=1}^n$ to the cyclic frame $(e_i)_{i=0}^{n-1}$, and the latter matrix relates the flat coordinate frame $(\frac{\partial}{\partial t^\alpha})_{\alpha=1}^n$ to the normalized idempotent frame $(f_i)_{i=1}^n$.

Lemma 2.4.4. *For $j = 1, \dots, n-1$, we have*

$$\widehat{\nabla}_{\frac{\partial}{\partial z}}\omega_j = -\omega_{j-1}.$$

Proof. From (2.4.1), for any $k = 0, \dots, n-2$, we have

$$\begin{aligned}\langle\widehat{\nabla}_{\frac{\partial}{\partial z}}\omega_j, e_k\rangle + \langle\omega_j, e_{k+1}\rangle = 0 &\implies \widehat{\nabla}_{\frac{\partial}{\partial z}}\omega_j, e_k\rangle = -\delta_{j,k+1}\\ &\implies \widehat{\nabla}_{\frac{\partial}{\partial z}}\omega_j = -\omega_{j-1}.\end{aligned}$$

■

Proposition 2.4.5. *The vector fields e_k, with $k \in \mathbb{N}$, have the following form:*

$$e_k = \sum_{j=0}^{k} \frac{1}{z^j} p_j^k(E),$$

where the vector fields $p_j^k(E)$ do not depend on z and satisfy the difference equations

$$\begin{aligned}&p_0^{k+1}(E) = E \circ p_0^k(E),\\ &p_j^{k+1}(E) = E \circ p_j^k(E) - \boldsymbol{\mu}(p_{j-1}^k(E)) + (1-j)p_{j-1}^k(E), \quad j = 1, \dots, k,\\ &p_{k+1}^{k+1}(E) = -\mu(p_k^k(E)) - kp_k^k(E),\end{aligned}$$

with the only initial datum $p_j^0(E) = \delta_{0j} \cdot e$. ■

2.5 Properties of the function det Λ

The holomorphic function $\det\Lambda\colon \widehat{M}^{\text{cyc}} \to \mathbb{C}^*$ extends meromorphically to a function on $\mathbb{P}^1 \times M$.

Theorem 2.5.1. *The function* $\det\Lambda$ *is a meromorphic function on $\mathbb{P}^1 \times M$ of the form*

$$\det\Lambda(z, p) = \frac{z^{\binom{n-1}{2}}}{z^{\binom{n-1}{2}}A_0(p) + \cdots + A_{\binom{n-1}{2}}(p)},$$

where $A_0, \dots, A_{\binom{n-1}{2}}$ are holomorphic functions on M. Moreover, if $n > 2$ and if the eigenvalues of the grading operator $\boldsymbol{\mu}$ are not pairwise distinct, then the function $A_{\binom{n-1}{2}}$ is identically zero.

We need a preliminary result.

Lemma 2.5.2. *For $k \in \{0, \ldots, n-1\}$, the polyvector field*

$$e_0 \wedge \cdots \wedge e_k \in \Gamma\Big(\bigwedge^{k+1} \pi^* TM\Big)$$

admits a pole at $\{0\} \times M$ of order at most $\binom{k}{2}$. More precisely, we have

$$e_0 \wedge \cdots \wedge e_k = w_0 + \frac{1}{z} w_1 + \cdots + \frac{1}{z^{\binom{k}{2}}} w_{\binom{k}{2}}, \quad w_j \in \Gamma\Big(\bigwedge^{k+1} \pi^* TM\Big),$$

with

$$w_{\binom{k}{2}} = (-1)^{\binom{k}{2}} e \wedge E \wedge \boldsymbol{\mu}(E) \wedge \boldsymbol{\mu}^2(E) \wedge \cdots \wedge \boldsymbol{\mu}^{k-1}(E).$$

Proof. By induction on k. For the base cases $k = 0$ and $k = 1$, we have $e_0 = e$ and $e_0 \wedge e_1 = e \wedge E$, respectively. So, for $k = 0, 1$ the claim holds true.

Assume that $e_0 \wedge \cdots \wedge e_{k-1}$ is of the form

$$e_0 \wedge \cdots \wedge e_{k-1} = w_0 + \frac{1}{z} w_1 + \cdots + \frac{1}{z^{\binom{k-1}{2}}} w_{\binom{k-1}{2}}$$

with

$$w_{\binom{k-1}{2}} = (-1)^{\binom{k-1}{2}} e \wedge E \wedge \boldsymbol{\mu}(E) \wedge \boldsymbol{\mu}^2(E) \wedge \cdots \wedge \boldsymbol{\mu}^{k-2}(E).$$

We have

$$e_0 \wedge \cdots \wedge e_k = \left(\sum_{j=0}^{\binom{k-1}{2}} z^{-j} w_j \right) \wedge \left(\sum_{\ell=0}^{k} z^{-\ell} p_\ell^k(E) \right).$$

We claim that the coefficient $w_{\binom{k-1}{2}} \wedge p_k^k(E)$ of $z^{-\binom{k-1}{2}-k}$ vanishes. Indeed, $p_k^k(E)$ is proportional to e: we have

$$p_k^k(E) = \frac{d}{2}\left(\frac{d}{2} - 1\right) \cdots \left(\frac{d}{2} - k + 1\right) e, \quad k \geqslant 0,$$

as it can easily be seen by induction (the key property is $\mu(e) = -\frac{d}{2} e$, together with the last difference equation of Proposition 2.4.5). Consequently, we have

$$w_{\binom{k-1}{2}} \wedge p_k^k(E) = c \cdot (e \wedge \cdots \wedge e) = 0.$$

Hence, the (possibly non-vanishing) most polar term of $e_0 \wedge \cdots \wedge e_k$ equals

$$\begin{aligned} z^{-\binom{k-1}{2}-k+1} \cdot w_{\binom{k-1}{2}} \wedge p_{k-1}^k(E) &= z^{-\binom{k}{2}} \cdot w_{\binom{k-1}{2}} \wedge ((-1)^{k-1} \mu^{k-1}(E)) \\ &= z^{-\binom{k}{2}} (-1)^{\binom{k}{2}} e \wedge E \wedge \boldsymbol{\mu}(E) \wedge \cdots \wedge \boldsymbol{\mu}^{k-1}(E). \end{aligned}$$

For the first equality we have used the difference equation for $p_{k-1}^k(E)$ of Proposition 2.4.5. ∎

Proof of Theorem 2.5.1. The polyvector field $e_0 \wedge \cdots \wedge e_{n-1}$ has the form

$$e_0 \wedge \cdots \wedge e_{n-1} = w_0(p) + \frac{1}{z} w_1(p) + \cdots + \frac{1}{z^{\binom{n-1}{2}}} w_{\binom{n-1}{2}}(p), \tag{2.5.1}$$

where $w_0, w_1, \dots, w_{\binom{n-1}{2}}$ are holomorphic n-vector fields on M, by Lemma 2.5.2. Introduce holomorphic functions $A_0(p), \dots, A_{\binom{n-1}{2}}(p)$ such that

$$w_j(p) = A_j(p) \cdot \frac{\partial}{\partial t^1} \wedge \cdots \wedge \frac{\partial}{\partial t^n}.$$

From the identity

$$\frac{\partial}{\partial t^1} \wedge \cdots \wedge \frac{\partial}{\partial t^n} = \det \Lambda \cdot e_0 \wedge \cdots \wedge e_{n-1},$$

we deduce

$$1 = \det \Lambda(z,p)\left(A_0(p) + \frac{1}{z} A_1(p) + \dots \frac{1}{z^{\binom{n-1}{2}}} A_{\binom{n-1}{2}}(p) \right).$$

The last statement on $A_{\binom{n-1}{2}}$ follows from the explicit formula for $w_{\binom{n-1}{2}}$ given in Lemma 2.5.2. ∎

Theorem 2.5.3. *We have*

$$A_0(p) = \frac{\prod_{i<j}(u_j(p) - u_i(p))}{\mathrm{Jac}(p)}, \qquad \mathrm{Jac}(p) := \det\left(\frac{\partial u_i}{\partial t^\alpha} \right)\bigg|_p .$$

Proof. The polyvector field w_0 in equation (2.5.1) is

$$w_0 = \bigwedge_{j=0}^{n-1} p_0^j(E).$$

By Proposition 2.4.5, we have

$$p_0^j(E) = E^{\circ j}, \quad j \in \mathbb{N},$$

and using the idempotent vielbein $(\frac{\partial}{\partial u_i})_{i=1}^n$, we can write w_0 as follows:

$$\begin{aligned} w_0 &= \begin{vmatrix} 1 & \dots & 1 \\ u_1 & \dots & u_n \\ u_1^2 & \dots & u_n^2 \\ & \vdots & \\ u_1^{n-1} & \dots & u_n^{n-1} \end{vmatrix} \frac{\partial}{\partial u_1} \wedge \cdots \wedge \frac{\partial}{\partial u_n} \\ &= \left(\prod_{i<j} (u_j - u_i) \right) \frac{\partial}{\partial u_1} \wedge \cdots \wedge \frac{\partial}{\partial u_n} \\ &= \left(\prod_{i<j} (u_j - u_i) \right) \cdot \frac{1}{\mathrm{Jac}} \cdot \frac{\partial}{\partial t^1} \wedge \cdots \wedge \frac{\partial}{\partial t^n}. \end{aligned}$$

∎

Remark 2.5.4. We also have

$$\frac{\partial}{\partial t^1} \wedge \cdots \wedge \frac{\partial}{\partial t^n} = \det \Psi f_1 \wedge \cdots \wedge f_n = \frac{\det \Psi}{\prod_{i=1}^n \eta(\frac{\partial}{\partial u_i}, \frac{\partial}{\partial u_i})^{\frac{1}{2}}} \frac{\partial}{\partial u_1} \wedge \cdots \wedge \frac{\partial}{\partial u_n},$$

so that

$$\mathrm{Jac}(p) = \left.\frac{\det \Psi}{\prod_{i=1}^n \eta(\frac{\partial}{\partial u_i}, \frac{\partial}{\partial u_i})^{\frac{1}{2}}}\right|_p = \left.\frac{(\det \eta)^{\frac{1}{2}}}{\prod_{i=1}^n \eta(\frac{\partial}{\partial u_i}, \frac{\partial}{\partial u_i})^{\frac{1}{2}}}\right|_p.$$

The last equality follows from $\Psi^T \Psi = \eta$.

2.6 Geometry of the complement of the cyclic stratum in $\mathbb{P}^1 \times M$

Let us consider the tuple of functions $(A_0, \dots, A_{\binom{n-1}{2}})$, and extend it to the sequence $(A_k)_{k\in\mathbb{N}}$ by setting $A_k = 0$ for $k > \binom{n-1}{2}$. Set

$$\overline{n} := \min\{j \in \mathbb{N} : A_h(p) = 0 \text{ for all } p \in M \text{ and all } h > j\}.$$

We necessarily have $0 \leqslant \overline{n} \leqslant \binom{n-1}{2}$. By Theorem 2.5.1, we have $\overline{n} < \binom{n-1}{2}$ if $\boldsymbol{\mu}$ has not simple spectrum. The function $\det \Lambda$ takes the form

$$\det \Lambda = \frac{z^{\overline{n}}}{z^{\overline{n}} A_0(p) + z^{\overline{n}-1} A_1(p) \cdots + A_{\overline{n}}(p)}.$$

Define the subsets $\mathcal{P}_\Lambda, M_0, M_\infty \subseteq \mathbb{P}^1 \times M$ and $\mathcal{A}_\Lambda, \mathcal{I}_\Lambda^\infty, \mathcal{I}_\Lambda^0 \subseteq M$ by

$$\begin{aligned}
\mathcal{P}_\Lambda &:= \{(z,p) \in \widehat{M} : z^{\overline{n}} A_0(p) + \cdots + A_{\overline{n}}(p) = 0\},\\
M_0 &:= \{0\} \times M,\\
M_\infty &:= \{\infty\} \times M,\\
\mathcal{A}_\Lambda &:= \{p \in M : A_0(p) = \cdots = A_{\overline{n}}(p) = 0\},\\
\mathcal{I}_\Lambda^\infty &:= \{p \in M : A_0(p) = 0\},\\
\mathcal{I}_\Lambda^0 &:= \{p \in M : A_{\overline{n}}(p) = 0\}.
\end{aligned}$$

Lemma 2.6.1. *We have the obvious inclusions*

$$\mathbb{C}^* \times \mathcal{A}_\Lambda \subseteq \mathcal{P}_\Lambda, \quad \mathcal{A}_\Lambda \subseteq \mathcal{I}_\Lambda^0 \cap \mathcal{I}_\Lambda^\infty. \qquad \blacksquare$$

The set $\mathcal{P}_\Lambda$ is an analytic subspace of $\mathbb{P}^1 \times M$ of codimension 1 along which the function $\det \Lambda$ admits a pole. The function $\det \Lambda$ admits poles along a further analytic subspace, namely $\{\infty\} \times \mathcal{I}_\Lambda^\infty$. See Table 2.1 and Figure 2.1.

The set $\mathcal{P}_\Lambda$ is the complement $\widehat{M} \setminus \widehat{M}^{\mathrm{cyc}}$ of the cyclic stratum. The complement of $\widehat{M}^{\mathrm{cyc}}$ in $\mathbb{P}^1 \times M$ is the disjoint union

$$\mathcal{P}_\Lambda \uplus M_0 \uplus M_\infty.$$

The geometry of $\mathcal{P}_\Lambda$ is rather complicated: in general it admits several irreducible components. For example, $\mathcal{A}_\Lambda$ itself does, and consequently also $\mathbb{C}^* \times \mathcal{A}_\Lambda$. The projection $\pi \colon \widehat{M} \to M$, if restricted to $\mathcal{P}_\Lambda \setminus (\mathbb{C}^* \times \mathcal{A}_\Lambda)$, defines a ramified covering of degree $\overline{n}$.

Poles of $\det\Lambda$	$\mathcal{P}_\Lambda \uplus (\{\infty\}\times\mathcal{I}^\infty_\Lambda)$
Zeros of $\det\Lambda$	$M_0 \setminus (\{0\}\times\mathcal{I}^0_\Lambda)$
Indeterminacy locus of $\det\Lambda$	$\{0\}\times\mathcal{I}^0_\Lambda$

Table 2.1. Location of poles, zeros and indeterminacy locus for the meromorphic function $\det\Lambda$ on $\mathbb{P}^1\times M$.

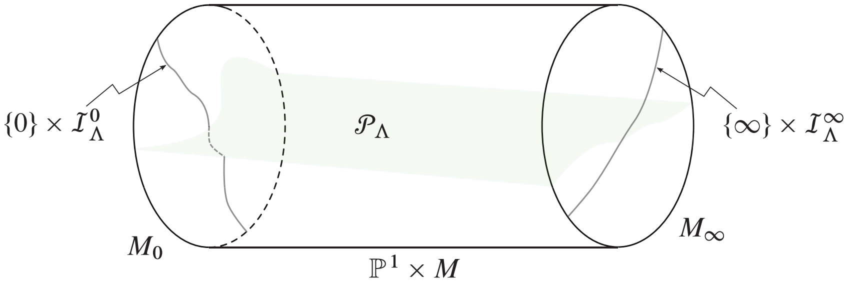

Figure 2.1. Configuration of the sets $\mathcal{P}_\Lambda$, $\{\infty\}\times\mathcal{I}^\infty_\Lambda$, and $\{0\}\times\mathcal{I}^0_\Lambda$ in $\mathbb{P}^1\times M$.

The set $\{0\}\times\mathcal{I}^0_\Lambda$ is an analytic subspace of $\mathbb{P}^1\times M$ of codimension 2 and it is the *indeterminacy locus* of the function $\det\Lambda$.

Each of the sets $\mathcal{I}^\infty_\Lambda, \mathcal{I}^0_\Lambda, \mathcal{A}_\Lambda$ seems to be strictly related to other distinguished subsets of the Frobenius manifold M, namely its bifurcation set $\mathcal{B}_M$, and its two components, the Maxwell stratum $\mathcal{M}_M$ and the caustic $\mathcal{K}_M$. We limit to the following observation.

Theorem 2.6.2. *We have* $\mathcal{I}^\infty_\Lambda \subseteq \mathcal{B}_M$.

Proof. Let $p\notin\mathcal{B}_M$. On the complement of $\mathcal{B}_M$, the eigenvalues $(u_1,\dots,u_n)$ define a holomorphic system of coordinates. Hence, $\mathrm{Jac}(p)\neq 0$. Moreover, by definition we have $\prod_{i<j}(u_j(p)-u_i(p))\neq 0$. Hence, $p\notin\mathcal{I}^\infty_\Lambda$ by Theorem 2.5.3. ■

In order to obtain more precise results on contingent relations between the sets $\mathcal{I}^\infty_\Lambda$, $\mathcal{I}^0_\Lambda$, $\mathcal{A}_\Lambda$ and $\mathcal{B}_M$, $\mathcal{M}_M$, $\mathcal{K}_M$ a more detailed study of the polyvector fields $p^k_j(E)$ of Proposition 2.4.5 is needed. We plan to address this problem in a future project. We conclude this section with three low-dimensional examples.

Example. For two-dimensional Frobenius manifolds, we have $\mathcal{I}^\infty_\Lambda = \mathcal{B}_M$. In this case, indeed, we have

$$e_0 = e,\; e_1 = E + \frac{d}{2z}e \implies e_0\wedge e_1 = e\wedge E.$$

The bivector $e\wedge E$ vanishes if and only if $u_1 = u_2$.

Example. Consider the A_3-Frobenius manifold, that is, the space $M \cong \mathbb{C}^3$ of polynomials $f(x, \boldsymbol{a}) = x^4 + a_2x^2 + a_1x + a_0$, where $\boldsymbol{a} = (a_0, a_1, a_2) \in \mathbb{C}^3$ are natural coordinate. Fix $\boldsymbol{a}_o \in M$, and define the Kodaira–Spencer isomorphism

$$\kappa\colon T_{\boldsymbol{a}_o}M \to \mathbb{C}[x]/\langle \partial_x f(x, \boldsymbol{a}_o)\rangle$$

by identifying ∂_{a_i} with the class of the partial derivative $\partial_{a_i} f(x, \boldsymbol{a}_o)$. This allows to pull back the product of the Jacobi–Milnor algebra $\mathbb{C}[x]/\langle \partial_x f(x, \boldsymbol{a}_o)\rangle$ on $T_{\boldsymbol{a}_o}M$. Consider the Grothendieck residue metric

$$\eta_{\boldsymbol{a}}\left(\frac{\partial}{\partial a_i}, \frac{\partial}{\partial a_j}\right) := \frac{1}{2\pi i}\int_{\Gamma_{\boldsymbol{a}}} \left.\frac{\frac{\partial f}{\partial a_i}\frac{\partial f}{\partial a_j}}{\frac{\partial f}{\partial x}}\right|_{(u,\boldsymbol{a})} du,$$

where $\Gamma_{\boldsymbol{a}}$ is a circle, positively oriented, bounding a disc containing all the roots of $\frac{\partial f}{\partial x}(u, \boldsymbol{a})$. One can show that the coordinates $\boldsymbol{t} = (t_1, t_2, t_3)$ given by

$$t_1 = a_0 - \frac{1}{8}a_2^2, \quad t_2 = a_1, \quad t_3 = a_2,$$

are flat for the metric η. In $\boldsymbol{t}$-coordinates, the Euler vector field is given by

$$E = t_1\frac{\partial}{\partial t_1} + \frac{3t_2}{4}\frac{\partial}{\partial t_2} + \frac{t_3}{2}\frac{\partial}{\partial t_3}.$$

The Maxwell stratum is the set $\{t_2 = 0\}$, and the caustic is the set $\{8t_3^3 + 27t_2^2 = 0\}$. We have the following formulas for the Λ-matrix and for $\det\Lambda$: Setting

$$a := z^2t_3^5 - 21z^2t_2^2t_3^2 - 64z^2t_1^2t_3 - 12t_3 - 18zt_2^2 - 72z^2t_1t_2^2$$

and

$$b := -3z^2t_3^4 - 16zt_3^2 - 64z^2t_1t_3^2 + 63z^2t_2^2t_3 + 192z^2t_1^2 + 48zt_1 + 48,$$

we get

$$\Lambda(z, \boldsymbol{t}) = \begin{pmatrix} 1 & \frac{a}{2zt_2(8zt_3^3-6t_3+27zt_2^2)} & \frac{b}{4z(8zt_3^3-6t_3+27zt_2^2)} \\ 0 & \frac{4(9zt_2^2+16zt_1t_3)}{t_2(8zt_3^3-6t_3+27zt_2^2)} & -\frac{4(-4zt_3^2+24zt_1+3)}{8zt_3^3-6t_3+27zt_2^2} \\ 0 & -\frac{32zt_3}{t_2(8zt_3^3-6t_3+27zt_2^2)} & \frac{48z}{8zt_3^3-6t_3+27zt_2^2} \end{pmatrix}$$

and

$$\det\Lambda(z, \boldsymbol{t}) = \frac{64z}{(8t_2t_3^3 + 27t_2^3)z - 6t_2t_3}.$$

We have

$$\mathcal{I}_\Lambda^\infty = \mathcal{B}_M, \quad \mathcal{I}_\Lambda^0 = \mathcal{M}_M \cup \{t_3 = 0\}, \quad \mathcal{A}_\Lambda = \mathcal{M}_M.$$

Example. The $A_2 \times A_2$-Frobenius manifold is the Frobenius structure M on $\mathbb{C}^4$, with flat coordinates $(\boldsymbol{t}, \boldsymbol{s}) = (t_0, t_1, s_0, s_1)$, defined by the WDVV–potential

$$F(\boldsymbol{t}, \boldsymbol{s}) = \frac{1}{2}(t_0^2t_1 + s_0^2s_1) + \frac{1}{24}(t_1^4 + s_1^4).$$

In these coordinates, the unit vector field is $e = \frac{\partial}{\partial t_0} + \frac{\partial}{\partial s_0}$, and the flat metric η has components

$$\eta = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

The Euler field equals

$$E = t_0 \frac{\partial}{\partial t_0} + \frac{2}{3} t_1 \frac{\partial}{\partial t_1} + s_0 \frac{\partial}{\partial s_0} + \frac{2}{3} s_1 \frac{\partial}{\partial s_1}.$$

The bifurcation diagram $\mathcal{B}_M$ equals $\mathcal{B}_M = \mathcal{M}_M \cup \mathcal{K}_M$, where the Maxwell stratum is

$$\mathcal{M}_M = \left\{-8t_1^3\big(9(s_0 - t_0)^2 + 4s_1^3\big) + \big(4s_1^3 - 9(s_0 - t_0)^2\big)^2 + 16t_1^6 = 0\right\},$$

and the caustic is

$$\mathcal{K}_M = \{t_1 = 0\} \cup \{s_1 = 0\}.$$

After some computations, one finds that

$$\det \Lambda(z, \boldsymbol{t}, \boldsymbol{s}) = 729z^2 \cdot \big(4s_1t_1(z^2(-8t_1^3(9(s_0 - t_0)^2 + 4s_1^3) + (4s_1^3 - 9(s_0 - t_0)^2)^2 + 16t_1^6) + 45(s_0 - t_0)^2)\big)^{-1}.$$

We have

$$\mathcal{I}_\Lambda^\infty = \mathcal{B}_M, \quad \mathcal{I}_\Lambda^0 = \mathcal{K}_M \cup \{s_0 = t_0\}, \quad \mathcal{A}_\Lambda = \mathcal{K}_M \cup \{s_0 = t_0, s_1^3 = t_1^3\}.$$

2.7 Master differential equation and master functions

Let $\xi \in \Gamma(\pi^* T^* M)$ be a $\widehat{\nabla}$-flat section. Consider the corresponding vector field $\zeta \in \Gamma(\pi^* TM)$ via musical isomorphism, i.e. such that

$$\xi(v) = \eta(\zeta, v)$$

for all $v \in \Gamma(\pi^* TM)$.

The vector field ζ satisfies the following system[4] of equations:

$$\frac{\partial}{\partial t^\alpha} \zeta = z \mathcal{C}_\alpha \zeta, \quad \alpha = 1, \dots, n, \tag{2.7.1}$$

$$\frac{\partial}{\partial z} \zeta = \left(\mathcal{U} + \frac{1}{z} \mu\right) \zeta. \tag{2.7.2}$$

Here $\mathcal{C}_\alpha$ is the $(1,1)$-tensor defined by $(\mathcal{C}_\alpha)_\gamma^\beta := c_{\alpha\gamma}^\beta$.

[4]We consider the joint system (2.7.1)–(2.7.2) in matrix notations (ζ is a column vector whose entries are the components $\zeta^\alpha(z, \boldsymbol{t})$ with respect to $\frac{\partial}{\partial t^\alpha}$). Bases of solutions are arranged in invertible $n \times n$-matrices, called *fundamental systems of solutions*.

Multiply by η (on the left) the left-hand and right-hand sides of (2.7.1)–(2.7.2): we obtain the equivalent system of differential equations

$$\begin{cases} \dfrac{\partial}{\partial t^\alpha}\xi = z\mathcal{C}_\alpha^T\xi, \quad \alpha = 1,\dots,n, \\ \dfrac{\partial}{\partial z}\xi = \left(\mathcal{U}^T - \dfrac{1}{z}\mu\right)\xi, \end{cases} \tag{2.7.3}$$

where ξ is a column vector whose entries are the components $\xi_\alpha(z,\boldsymbol{t})$ with respect to dt^α. At points $(z,p) \in \widehat{M}^{\mathrm{cyc}}$, let us introduce the column vector $\overline{\xi}$ by

$$\overline{\xi} = (\Lambda^{-1})^T\xi, \tag{2.7.4}$$

where Λ is defined as in (2.4.2). The entries of $\overline{\xi}$ are the components $\overline{\xi}_j$ with respect to the cyclic coframe ω_j. The vector $\overline{\xi}$ satisfies the system

$$\begin{cases} \dfrac{\partial\overline{\xi}}{\partial t^\alpha} = \left(z(\Lambda^{-1})^T\mathcal{C}_\alpha\Lambda^T + \dfrac{\partial(\Lambda^{-1})^T}{\partial t^\alpha}\Lambda^T\right)\overline{\xi}, \\ \dfrac{\partial\overline{\xi}}{\partial z} = \left((\Lambda^{-1})^T\mathcal{U}^T\Lambda^T - \dfrac{1}{z}(\Lambda^{-1})^T\mu\Lambda^T + \dfrac{\partial(\Lambda^{-1})^T}{\partial z}\Lambda^T\right)\overline{\xi}. \end{cases} \tag{2.7.5}$$

Proposition 2.7.1. *Let $\xi \in \Gamma(\pi^*T^*M)$ be a $\widehat{\nabla}$-flat section, and let $(\overline{\xi}_j(z,p))_{j=0}^{n-1}$ be its components with respect to the cyclic co-frame, i.e. $\xi = \sum_j \overline{\xi}_j\omega_j$. We have*

$$\frac{\partial\overline{\xi}_j}{\partial z} = \overline{\xi}_{j+1}, \quad j = 0,\dots,n-2.$$

Proof. We have

$$\begin{aligned} 0 = \widehat{\nabla}_{\frac{\partial}{\partial z}}\xi &= \sum_j \frac{\partial\overline{\xi}_j}{\partial z}\omega_j + \sum_j \overline{\xi}_j\widehat{\nabla}_{\frac{\partial}{\partial z}}\omega_j \\ &= \sum_j \frac{\partial\overline{\xi}_j}{\partial z}\omega_j - \sum_j \overline{\xi}_j\omega_{j-1}, \end{aligned}$$

by Lemma 2.4.4. The claim follows. ∎

Corollary 2.7.2. *The system of differential equations (2.7.5) is the companion system of a* scalar *differential equation in $\overline{\xi}_0$.* ∎

Remark 2.7.3. Note that $\xi_1 = \overline{\xi}_0$. Indeed, we have $e_0 = e = \frac{\partial}{\partial t^1}$, so that $\Lambda_{i1} = \delta_{i1}$. The claim then follows from (2.7.4).

Theorem 2.7.4. *Consider the system of differential equations (2.7.3), specialized at a point $p \in M \setminus \mathcal{A}_\Lambda$. The system can be reduced to a single scalar ordinary differential equation of order n in the unknown function ξ_1. The scalar differential equation admits at most $\binom{n-1}{2}$ apparent singularities.*

Proof. If $p \in M \setminus \mathcal{A}_\Lambda$, then there exist $\overline{n}$ complex numbers $z_1, \dots, z_{\overline{n}}$, not necessarily distinct, such that $(z_i, p) \notin \widehat{M}^{\rm cyc}$. The numbers z_i are the zeros of the denominator of the function $\det \Lambda(z, p)$. ■

The scalar differential equation to which system (2.7.3) can be reduced will be called the *master differential equation* of M.

Definition 2.7.5. Fix a point $p \in M$. Consider the system of differential equations (2.7.3) specialized at p, and set $\mathcal{X}_p$ be the $\mathbb{C}$-vector space of its solutions. Then let $\nu_p \colon \mathcal{X}_p \to \mathcal{O}(\widetilde{\mathbb{C}^*})$ be the morphism defined by

$$\xi \mapsto \Phi_\xi(z), \quad \Phi_\xi(z) := z^{-\frac{d}{2}} \langle \xi(z, p), e(p) \rangle,$$

where d is the charge of the Frobenius manifold. Set $\mathcal{S}_p(M) := \mathrm{im}(\nu_p)$. Elements of $\mathcal{S}_p(M)$ will be called *master functions* at p.

Theorem 2.7.6. *At points $p \in M \setminus \mathcal{A}_\Lambda$ the morphism ν_p is injective.*

Proof. Given $\Phi_\xi \in \mathcal{S}_p(M)$, the function $\xi_1(z) = z^{\frac{d}{2}} \Phi_\xi(z)$ is a solution of the master differential equation at p. By Theorem 2.7.4, the solution $\xi(z)$ can be reconstructed from the component $\xi_1(z)$ only. ■