Chapter 3

Gromov–Witten theory

3.1 Notations and conventions

Let X be a smooth projective variety over $\mathbb{C}$. In order not to introduce superstructures, in what follows we assume that $H^{\mathrm{odd}}(X,\mathbb{C})=0$. Denote by $b_k(X)$ the k-th Betti number of X.

Attached to X there is an infinite-dimensional $\mathbb{C}$-vector space $\mathscr{P}_X$, called the *big phase space*, defined as the infinite product of countable many copies of the classical cohomology space of X, that is,

$$\mathscr{P}_X := \prod_{n\in\mathbb{N}} H^\bullet(X,\mathbb{C}).$$

Let us fix a homogeneous basis $(T_0,\dots,T_N)$ of $H^\bullet(X,\mathbb{C})$ such that

- $T_0=1$,
- $T_1,\dots,T_r$ is a nef integral basis of $H^2(X,\mathbb{Z})$.

In particular, $b_2(X)=r$. Set $\boldsymbol{t}=(t^0,\dots,t^N)$, the dual coordinates of $H^\bullet(X,\mathbb{C})$.

We denote by $(\tau_pT_0,\dots,\tau_pT_N)$ the corresponding basis of the p-th copy of $H^\bullet(X,\mathbb{C})$ in $\mathscr{P}_X$. The element τ_pT_α will be called a *descendant* of T_α with level p. The coordinate of a point $\boldsymbol{\gamma}\in\mathscr{P}_X$ with respect to the basis $(\tau_pT_\alpha)_{\alpha,p}$ will be denoted by $\boldsymbol{t}^\bullet=(t^{\alpha,p})_{\alpha,p}$. Instead of denoting by $\boldsymbol{\gamma}=(t^{\alpha,p}\tau_pT_\alpha)_{\alpha,p}$ a generic element of $\mathscr{P}_X$ we will usually write this as a formal series

$$\boldsymbol{\gamma}=\sum_{\alpha=1}^{m}\sum_{p=0}^{\infty} t^{\alpha,p}\tau_pT_\alpha.$$

We identify $H^\bullet(X,\mathbb{C})$ with the 0-th factor of $\mathscr{P}_X$, called the *small phase space*. This allow us to identify $t^\alpha\equiv t^{\alpha,0}$ for $\alpha=0,\dots,N$.

Denote by η: $H^\bullet(X,\mathbb{C})\times H^\bullet(X,\mathbb{C})\to H^\bullet(X,\mathbb{C})$ the Poincaré pairing defined by

$$\eta(u,v):=\int_X u\cup v,$$

and we set $\eta_{\alpha\beta}:=\eta(T_\alpha,T_\beta)$ for $\alpha,\beta=0,\dots,N$. The numbers $\eta_{\alpha\beta}$ will be collected in the Gram[1] matrix $\eta=(\eta_{\alpha\beta})_{\alpha,\beta=0}^N$, with inverse matrix $\eta^{-1}=(\eta^{\alpha\beta})_{\alpha,\beta=0}^N$. We also

[1] We denote the metric tensor and its Gram matrix by the same symbol η. This is a standard abuse of notation.

introduce the dual basis $(T^0, \dots, T^N)$ of $H^\bullet(X, \mathbb{C})$, by setting

$$T^\alpha := \sum_{\lambda=0}^{N} T_\lambda \eta^{\lambda\alpha}, \quad \alpha = 0, \dots, N.$$

Define the *Novikov ring* Λ_X as the ring of formal sums

$$\sum_{\beta \in H_2(X,\mathbb{Z})} a_\beta \mathbf{Q}^\beta, \quad a_\beta \in \mathbb{Q},$$

such that

$$\operatorname{card}\left\{\beta : a_\beta \neq 0 \text{ and } \int_\beta \omega < C\right\} < \infty \quad \text{for any } C \in \mathbb{R},$$

where ω is the Kähler form of X.

3.2 Descendant Gromov–Witten invariants

For any given $g, n \in \mathbb{N}$ and $\beta \in H_2(X, \mathbb{Z})$, denote by $\overline{\mathcal{M}}_{g,n}(X, \beta)$ the Kontsevich–Manin moduli stack of genus g, n-pointed stable maps of degree β with target X: it parametrizes isomorphism classes of pairs $((C, \boldsymbol{x}), f)$, where

- C is a genus g nodal connected projective curve,
- $\boldsymbol{x} = (x_1, \dots, x_n)$ is an n-tuple of pairwise distinct points of the smooth locus of C,
- $f : C \to X$ is a morphism with $f_*[C] = \beta$,
- a morphism between two pairs $((C, \boldsymbol{x}), f)$ and $((C', \boldsymbol{x}'), f')$ is a morphism $\sigma : C \to C'$ such that $\sigma(x_i) = x_i'$ for all i, and making commutative the diagram

$$\begin{array}{ccc} C & \xrightarrow{\quad\sigma\quad} & C' \\ & {\scriptstyle f}\searrow \quad \swarrow{\scriptstyle f'} & \\ & X, & \end{array}$$

- the group of automorphisms of $((C, \boldsymbol{x}), f)$ is finite.

The moduli space $\overline{\mathcal{M}}_{g,n}(X, \beta)$ is a proper Deligne–Mumford stack of virtual dimension

$$\operatorname{vir}\dim_{\mathbb{C}} \overline{\mathcal{M}}_{g,n}(X, \beta) := (1 - g)(\dim_{\mathbb{C}} X - 3) + \int_\beta c_1(X) + n.$$

We denote by $\mathcal{L}_i$, with $i = 1, \dots, n$, the i-th *tautological line bundle* on $\overline{\mathcal{M}}_{g,n}(X, \beta)$ whose fiber at the point $[((C, \boldsymbol{x}), f)] \in \overline{\mathcal{M}}_{g,n}(X, \beta)$ is the cotangent space $T^*_{x_i} C$. Set $\psi_j := c_1(\mathcal{L}_j)$ for $j = 1, \dots, n$.

We have naturally defined *evaluation morphisms*

$$\mathrm{ev}_i\colon \overline{\mathcal{M}}_{g,n}(X,\beta) \to X, \quad [((C,\boldsymbol{x}),f)] \mapsto f(x_i)$$

for $i = 1,\dots,n$.

Definition 3.2.1. Let $d_1,\dots,d_n$ be non-negative integers. The *genus g descendant Gromov–Witten invariants* (or *genus g gravitational correlators*) are the rational numbers defined by the integrals

$$\langle \tau_{d_1}\alpha_1,\dots,\tau_{d_n}\alpha_n\rangle^X_{g,n,\beta} := \int_{[\overline{\mathcal{M}}_{g,n}(X,\beta)]^{\mathrm{virt}}} \prod_{j=1}^n \psi_j^{d_j} \cup \mathrm{ev}_j^*(\alpha_j),$$

where $\alpha_1,\dots,\alpha_n \in H^\bullet(X,\mathbb{C})$, and the class

$$[\overline{\mathcal{M}}_{g,n}(X,\beta)]^{\mathrm{virt}} \in \mathrm{CH}_D(\overline{\mathcal{M}}_{g,n}(X,\beta)), \quad D = \mathrm{vir}\dim_{\mathbb{C}} \overline{\mathcal{M}}_{g,n}(X,\beta),$$

denotes the virtual fundamental class of $\overline{\mathcal{M}}_{g,n}(X,\beta)$.

Definition 3.2.2. The *genus g total descendant potential* of X is the generating function $\mathcal{F}_g^X \in \Lambda_X[\![\boldsymbol{t}^\bullet]\!]$ of descendant GW-invariants of X defined by

$$\begin{aligned}\mathcal{F}_g^X(\boldsymbol{t}^\bullet,\mathbf{Q}) &:= \sum_{n=0}^\infty \sum_{\beta\in\mathrm{Eff}(X)} \frac{\mathbf{Q}^\beta}{n!}\langle \boldsymbol{\gamma},\dots,\boldsymbol{\gamma}\rangle^X_{g,n,\beta} \\ &= \sum_{n=0}^\infty \sum_{\beta} \sum_{\alpha_1,\dots,\alpha_n=0}^N \sum_{p_1,\dots,p_n=0}^\infty \frac{t^{\alpha_1,p_1}\dots t^{\alpha_n,p_n}}{n!} \\ &\qquad \cdot \langle \tau_{p_1}T_{\alpha_1},\dots,\tau_{p_n}T_{\alpha_n}\rangle^X_{g,n,\beta}\mathbf{Q}^\beta.\end{aligned}$$

Setting $t^{\alpha,0} = t^\alpha$ and $t^{\alpha,p} = 0$ for $p > 0$, we obtain the *genus g Gromov–Witten potential* of X

$$F_g^X(\boldsymbol{t},\mathbf{Q}) := \sum_{n=0}^\infty \sum_{\beta} \sum_{\alpha_1,\dots,\alpha_n=0}^N \frac{t^{\alpha_1}\dots t^{\alpha_n}}{n!}\langle T_{\alpha_1},\dots,T_{\alpha_n}\rangle^X_{g,n,\beta}\mathbf{Q}^\beta.$$

It will also be convenient to introduce the *genus g correlation functions* defined by the derivatives

$$\langle\!\langle \tau_{d_1}T_{\alpha_1},\dots,\tau_{d_n}T_{\alpha_n}\rangle\!\rangle_g := \frac{\partial}{\partial t^{\alpha_1,d_1}}\cdots\frac{\partial}{\partial t^{\alpha_n,d_n}}\mathcal{F}_g^X(\boldsymbol{t}^\bullet,\mathbf{Q})\Big|_{\substack{t^{\alpha,p}=0 \text{ for } p>0 \\ t^{\alpha,0}=t^\alpha}}.$$

3.3 Quantum cohomology

Let $\beta_1,\dots,\beta_r \in H_2(X,\mathbb{Z})$ be the homology classes dual to $T_1,\dots,T_r$. By the divisor axiom, the genus 0 Gromov–Witten potential $F_0^X(\boldsymbol{t},\mathbf{Q})$ can be seen as an element of the ring $\mathbb{C}[\![t^0, \mathbf{Q}^{\beta_1}e^{t^1}, \dots, \mathbf{Q}^{\beta_r}e^{t^r}, t^{r+1}, \dots, t^N]\!]$. In what follows we will be interested in the cases when F_0^X is a convergent series expansion

$$F_0^X \in \mathbb{C}\{t^0, \mathbf{Q}^{\beta_1}e^{t^1}, \dots, \mathbf{Q}^{\beta_r}e^{t^r}, t^{r+1}, \dots, t^N\}. \tag{3.3.1}$$

Without loss of generality we can put $\mathbf{Q} = 1$. Under assumption (3.3.1), $F_0^X(\boldsymbol{t})$ defines an analytic function in an open neighborhood $\Omega \subseteq H^\bullet(X, \mathbb{C})$ of the point

$$t^i = 0, \quad i = 0, r+1, \dots, N, \qquad \operatorname{Re} t^i \to -\infty, \quad i = 1, 3, \dots, r.$$

The function F_0^X is a solution of WDVV equations [58,61,76,78], and thus it defines an analytic Frobenius manifold structure on Ω. Using the canonical identifications of tangent spaces $T_p\Omega \cong H^\bullet(X; \mathbb{C})$: $\partial_{t^\alpha} \mapsto T_\alpha$, the unit vector field is $e = \partial_{t^0} \equiv 1$, and the Euler vector field is

$$E := c_1(X) + \sum_{\alpha=0}^{N} \left(1 - \frac{1}{2} \deg T_\alpha\right) t^\alpha T_\alpha,$$

which satisfies

$$\mathfrak{L}_E F_0^X = (3 - \dim_{\mathbb{C}} X) F_0^X.$$

The Frobenius manifold structure on Ω can be extended by analytic continuation. The resulting maximal Frobenius structure is called *quantum cohomology of X*, denoted $QH^\bullet(X)$.

In the recent paper [18], a useful convergence criterion for formal power series solutions of WDVV equations is given. In the case of quantum cohomologies of Fano varieties, we have the following result.

Assume that X is Fano, and let us consider the finite-dimensional $\mathbb{C}$-algebra $(H^\bullet(X, \mathbb{C}), \circ_0)$, where the product $\circ_0$ is defined by

$$T_\alpha \circ_0 T_\gamma = \sum_{\lambda=0}^{N} c_{\alpha\gamma}^{\lambda} T_\lambda, \quad \alpha, \gamma = 0, \dots, N,$$

where

$$c_{\alpha\gamma}^{\lambda} := \sum_{\varepsilon=0}^{N} \sum_{\beta \in \mathrm{Eff}(X)} \langle T_\alpha, T_\gamma, T_\varepsilon \rangle_{0,3,\beta}^X \eta^{\varepsilon\lambda}, \quad \alpha, \gamma = 0, \dots, N.$$

Notice that the sums defining the structure constants $c_{\alpha\beta}^{\lambda}$ are finite, due to the Fano assumption.

Theorem 3.3.1 ([18]). *If $(H^\bullet(X, \mathbb{C}), \circ_0)$ is semisimple, then the Gromov–Witten potential $F_0^X(\boldsymbol{t}, \mathbf{Q})$ is convergent. That is, condition (3.3.1) holds.* ■

For a further convergence result, beyond the Fano case, see [18, Sec. 6]. See also [19], where the convergence criteria of [18] have been generalized to solutions of the more general “oriented associativity equations” [62].