Chapter 4

Monodromy data of quantum cohomology

4.1 Topological-enumerative solution

For $\beta = 0, \dots, N$ and $k \in \mathbb{N}$, introduce the functions

$$\theta_{\beta,k}(\boldsymbol{t}) := \langle\!\langle \tau_k T_\beta, 1 \rangle\!\rangle_0 \big|_{\mathbf{Q}=1}, \quad \theta_\beta(z, \boldsymbol{t}) := \sum_{k=0}^{\infty} \theta_{\beta,k}(\boldsymbol{t}) z^k.$$

Define the matrix $\Theta(z, \boldsymbol{t})$ by

$$\Theta(z, \boldsymbol{t})^\alpha_\beta := \sum_{\lambda=0}^{N} \eta^{\alpha\lambda} \frac{\partial \theta_\beta(z, \boldsymbol{t})}{\partial t^\lambda}, \quad \alpha, \beta = 0, \dots, N.$$

Denote by R the matrix associated with the morphism

$$c_1(X)\cup\colon H^\bullet(X, \mathbb{C}) \to H^\bullet(X, \mathbb{C}), \quad x \mapsto c_1(X) \cup x,$$

with respect to the basis $(T_0, \dots, T_N)$.

Let us consider the joint system (2.7.1)–(2.7.2) attached to the Frobenius manifold $QH^\bullet(X)$.

Theorem 4.1.1 ([23, 32]). *The matrix $Z_{\text{top}}(z, \boldsymbol{t}) := \Theta(z, \boldsymbol{t}) z^\mu z^R$ is a fundamental system of solutions of the joint system* (2.7.1)–(2.7.2). ∎

The fundamental system of solutions $Z_{\text{top}}(z, \boldsymbol{t})$ is called *topological-enumerative solution* of the joint system (2.7.1)–(2.7.2).

Let $M_0(\boldsymbol{t})$ be the monodromy matrix defined by

$$Z_{\text{top}}(e^{2\pi\sqrt{-1}} z, \boldsymbol{t}) = Z_{\text{top}}(z, \boldsymbol{t}) M_0(\boldsymbol{t}), \quad z \in \widetilde{\mathbb{C}^*}.$$

Lemma 4.1.2. *We have*

$$M_0(\boldsymbol{t}) = \exp(2\pi\sqrt{-1}\mu)\exp(2\pi\sqrt{-1}R).$$

In particular, M_0 does not depend on $\boldsymbol{t}$. ∎

4.2 Stokes rays and ℓ-chamber decomposition

Definition 4.2.1. We call *Stokes rays* at a point $p \in \Omega$ the oriented rays $R_{ij}(p)$ in $\mathbb{C}$ defined by

$$R_{ij}(p) := \left\{ -\sqrt{-1}\left(\overline{u_i(p)} - \overline{u_j(p)}\right)\rho : \rho \in \mathbb{R}_+ \right\},$$

where $(u_1(p), \dots, u_n(p))$ is the spectrum of the operator $\mathcal{U}(p)$ (with a fixed arbitrary order).

Fix an oriented ray ℓ in the universal cover $\widetilde{\mathbb{C}^*}$.

Definition 4.2.2. We say that ℓ is *admissible* at $p \in \Omega$ if the projection of the ray ℓ on $\mathbb{C}^*$ does not coincide with any Stokes ray $R_{ij}(p)$.

Definition 4.2.3. Define the open subset O_ℓ of points $p \in \Omega$ by the following conditions:

(1) the eigenvalues $u_i(p)$ are pairwise distinct,

(2) ℓ is admissible at p.

We call *ℓ-chamber* of Ω any connected component of O_ℓ.

4.3 Stokes fundamental solutions at $z = \infty$

Fix an oriented ray $\ell \equiv \{\arg z = \varphi\}$ in $\widetilde{\mathbb{C}^*}$. For $m \in \mathbb{Z}$, define the following sectors in $\widetilde{\mathbb{C}^*}$:

$$\Pi_{L,m}(\varphi) := \{z \in \widetilde{\mathbb{C}^*} : \varphi + 2\pi m < \arg z < \varphi + \pi + 2\pi m\},$$
$$\Pi_{R,m}(\varphi) := \{z \in \widetilde{\mathbb{C}^*} : \varphi - \pi + 2\pi m < \arg z < \varphi + 2\pi m\}.$$

Denote by $\mathcal{B}_X$ the bifurcation diagram of the quantum cohomology of X.

Theorem 4.3.1 ([30, 32]). *There exists a unique formal solution $Z_{\text{form}}(z, \boldsymbol{t})$ of the joint system* (2.7.1)–(2.7.2) *of the form*

$$Z_{\text{form}}(z, \boldsymbol{t}) = \Psi(\boldsymbol{t})^{-1} G(z, \boldsymbol{t}) \exp(zU(\boldsymbol{t})),$$
$$G(z, \boldsymbol{t}) = I + \sum_{k=1}^{\infty} \frac{1}{z^k} G_k(\boldsymbol{t}),$$

where the matrices $G_k(\boldsymbol{t})$ are holomorphic on $\Omega \setminus \mathcal{B}_X$.

Theorem 4.3.2 ([30, 32]). *Let $m \in \mathbb{Z}$. There exist unique fundamental systems of solutions $Z_{L,m}(z, \boldsymbol{t})$ and $Z_{R,m}(z, \boldsymbol{t})$ of the joint system* (2.7.1)–(2.7.2) *with respective asymptotic expansion*

$$Z_{L,m}(z, \boldsymbol{t}) \sim Z_{\text{form}}(z, \boldsymbol{t}), \quad |z| \to \infty, \ z \in \Pi_{L,m}(\varphi),$$
$$Z_{R,m}(z, \boldsymbol{t}) \sim Z_{\text{form}}(z, \boldsymbol{t}), \quad |z| \to \infty, \ z \in \Pi_{R,m}(\varphi).$$

Definition 4.3.3. The solutions $Z_{L,m}(z, \boldsymbol{t})$ and $Z_{R,m}(z, \boldsymbol{t})$ are called *Stokes fundamental solutions* of the joint system (2.7.1), (2.7.2) on the sectors $\Pi_{L,m}(\varphi)$ and $\Pi_{R,m}(\varphi)$, respectively.

4.4 Monodromy data

Let $\ell \equiv \{\arg z = \varphi\}$ be an oriented ray in $\widetilde{\mathbb{C}^*}$ and consider the corresponding Stokes fundamental systems of solutions $Z_{L,m}(z, \boldsymbol{t})$, $Z_{R,m}(z, \boldsymbol{t})$ for $m \in \mathbb{Z}$.

Definition 4.4.1. We define the *Stokes* and *central connection* matrices $S^{(m)}(p)$, $C^{(m)}(p)$, with $m \in \mathbb{Z}$, at the point $p \in O_\ell$ by the identities

$$Z_{L,m}(z, \boldsymbol{t}(p)) = Z_{R,m}(z, \boldsymbol{t}(p))S^{(m)}(p), \quad z \in \widetilde{\mathbb{C}^*},$$

$$Z_{R,m}(z, \boldsymbol{t}(p)) = Z_{\text{top}}(z, \boldsymbol{t}(p))C^{(m)}(p), \quad z \in \widetilde{\mathbb{C}^*}.$$

Set $S(p) := S^{(0)}(p)$ and $C(p) := C^{(0)}(p)$.

Definition 4.4.2. The *monodromy data* at the point $p \in O_\ell$ are defined as the 4-tuple of matrices $(\mu, R, S(p), C(p))$, where

- μ is the matrix associated to the grading operator,
- R is the matrix associated to the operator $c_1(X) \cup : H^\bullet(X, \mathbb{C}) \to H^\bullet(X, \mathbb{C})$,
- $S(p), C(p)$ are the Stokes and central connection matrices at p, respectively.

Definition 4.4.3. Fix a point $p \in O_\ell$ with canonical coordinates $(u_i(p))_{i=1}^n$. Define the oriented rays $L_j(p, \varphi)$, $j = 1, \dots, n$, in the complex plane by the equations

$$L_j(p, \varphi) := \left\{ u_j(p) + \rho e^{\sqrt{-1}(\frac{\pi}{2} - \varphi)} : \rho \in \mathbb{R}_+ \right\}.$$

The ray $L_j(p, \varphi)$ is oriented from $u_j(p)$ to ∞. We say that $(u_i(p))_{i=1}^n$ are in *ℓ-lexicographical order* if $L_j(p, \varphi)$ is on the left of $L_k(p, \varphi)$ for $1 \leqslant j < k \leqslant n$.

In what follows, it is assumed that the ℓ-lexicographical order of canonical coordinates is fixed at all points of ℓ-chambers.

Lemma 4.4.4 ([21, 32]). *If the canonical coordinates $(u_i(p))_{i=1}^n$ are in ℓ-lexicographical order at $p \in O_\ell$, then the Stokes matrices $S^{(m)}(p)$, $m \in \mathbb{Z}$, are upper triangular with ones along the diagonal.*

By Lemma 4.1.2, the matrices μ and R determine the monodromy of solutions of the qDE,

$$M_0 := \exp(2\pi\sqrt{-1}\mu)\exp(2\pi\sqrt{-1}R).$$

Moreover, μ and R do not depend on the point p. The following theorem furnishes a refinement of this property.

Theorem 4.4.5 ([21, 30, 32]). *The monodromy data (μ, R, S, C) are constant in each ℓ-chamber. Moreover, they satisfy the following identities:*

$$CS^T S^{-1} C^{-1} = M_0, \tag{4.4.1}$$

$$S = C^{-1}\exp(-\pi\sqrt{-1}R)\exp(-\pi\sqrt{-1}\mu)\eta^{-1}(C^T)^{-1}, \tag{4.4.2}$$

$$S^T = C^{-1}\exp(\pi\sqrt{-1}R)\exp(\pi\sqrt{-1}\mu)\eta^{-1}(C^T)^{-1}. \tag{4.4.3}$$

Theorem 4.4.6 ([21]). *The Stokes and central connection matrices S_m, C_m, with $m \in \mathbb{Z}$, can be reconstructed from the monodromy data (μ, R, S, C):*

$$S^{(m)} = S, \quad C^{(m)} = M_0^{-m} C, \quad m \in \mathbb{Z}. \tag{4.4.4}$$

Remark 4.4.7. Points of O_ℓ are semisimple. The results of [21,22,24,25] imply that the monodromy data (μ, R, S, C) are well defined also at points $p \in \Omega_{\mathrm{ss}} \cap \mathcal{B}_\Omega$, and that Theorem 4.4.5 still holds true.

Remark 4.4.8. Note that from the knowledge of the monodromy data (μ, R, S, C) the Gromov–Witten potential $F_0^X(\boldsymbol{t})$ can be reconstructed via a Riemann–Hilbert boundary value problem, see [21, 23, 32, 47]. Hence, the monodromy data may be interpreted as a *system of coordinates* in the space of solutions of WDVV equations.

4.5 Natural transformations of monodromy data

The definition of the Stokes and central connection matrices is subordinate to several non-canonical choices:

(1) the choice of an oriented ray ℓ in $\widetilde{\mathbb{C}^*}$,

(2) the choice of an ordering of canonical coordinates $u_1, \dots, u_n$ on each ℓ-chamber,

(3) the choice of signs in (2.2.1), and hence of the branch of the Ψ-matrix on each ℓ-chamber.

Different choices affect the numerical values of the data (S, C).

For different choices of the oriented ray ℓ, the transformation of S and C can be described in terms of an action of the braid group $\mathcal{B}_n$, described in Section 4.6.
For different choices of ordering of canonical coordinates, the Stokes and central connection matrices transform as follows:

$$S \mapsto \Pi S \Pi^{-1}, \quad C \mapsto C \Pi^{-1}, \quad \Pi \text{ permutation matrix.}$$

For different choices of the branch of the Ψ-matrix, we have a transformation of the following type:

$$S \mapsto ISI, \quad C \mapsto CI, \quad I = \mathrm{diag}(\pm 1, \dots, \pm 1).$$

See [21,23] for more details.

Moreover, let us also add that the value of all the monodromy data is affected by different choices of the system of flat coordinates $\boldsymbol{t}$.

Proposition 4.5.1. *Let $(\tilde{t}^0, \dots, \tilde{t}^N)$ be a system of flat coordinates on Ω related to $(t^0, \dots, t^N)$ by the transformations*

$$\tilde{t}^\alpha = A^\alpha_\beta t^\beta + c^\alpha, \quad A^\alpha_\beta, c^\alpha \in \mathbb{C}, \quad \alpha, \beta = 0, \dots, N.$$

The monodromy data $(\widetilde{\mu}, \widetilde{R}, \widetilde{S}, \widetilde{C})$*, computed with respect to the coordinates* $\widetilde{\boldsymbol{t}}$*, are related to the data* (μ, R, S, C)*, computed with respect to* $\boldsymbol{t}$*, as follows:*

$$\widetilde{\mu} = A\mu A^{-1}, \quad \widetilde{R} = ARA^{-1}, \quad \widetilde{S} = S, \quad \widetilde{C} = AC.$$

Proof. The transformation of μ, R is due to their tensorial nature: they are (1,1)-tensors on Ω. Notice that $\widetilde{\Psi} = \Psi A^{-1}$, $\widetilde{Z}_{R,0} = AZ_{R,0}$ and $\widetilde{Z}_{\text{top}} = AZ_{\text{top}}A^{-1}$ so that

$$\widetilde{C} = \widetilde{Z}_{\text{top}}^{-1}\widetilde{Z}_{R,0} = AZ_{\text{top}}^{-1}A^{-1}AZ_{R,0} = AC.$$

Equation (4.4.2), together with $\widetilde{\eta} = (A^{-1})^T\eta A^{-1}$, shows that $\widetilde{S} = S$. ■

Remark 4.5.2. In particular, Proposition 4.5.1 applies in the case of deformations of the complex structures of X. Consider a smooth proper map $f: \mathcal{F} \to B$ with a connected base space B, and set $X_b := f^{-1}(b)$ with $b \in B$. Given $b_1, b_2 \in B$, there exists a diffeomorphism $\varphi: X_{b_1} \to X_{b_2}$, which allows to identify (co)homology groups:

$$\varphi_*: H_\bullet(X_{b_1}, \mathbb{Z}) \to H_\bullet(X_{b_2}, \mathbb{Z})$$

and

$$\varphi^*: H^\bullet(X_{b_2}, \mathbb{Z}) \to H^\bullet(X_{b_1}, \mathbb{Z}).$$

By using the isomorphisms φ_*, φ^*, and by invoking the deformation axiom of Gromov–Witten invariants (see e.g. [27, Section 7.3]), we can identify the quantum cohomologies $QH^\bullet(X_{b_1})$ and $QH^\bullet(X_{b_2})$: the deformation of the complex structure just represents a change of flat coordinates on the same Frobenius manifold.

4.6 Action of the braid group $\mathcal{B}_n$

Consider the braid group $\mathcal{B}_n$ with generators $\beta_1, \dots, \beta_{n-1}$ satisfying the relations

$$\begin{aligned} \beta_i\beta_j &= \beta_j\beta_i, \quad |i-j| > 1, \\ \beta_i\beta_{i+1}\beta_i &= \beta_{i+1}\beta_i\beta_{i+1}. \end{aligned}$$

Let $\mathcal{U}_n$ be the set of upper triangular $(n \times n)$-matrices with ones along the diagonal.

Definition 4.6.1. Given $U \in \mathcal{U}_n$, define the matrices $A^{\beta_i}(U)$, with $i = 1, \dots, n-1$, as follows:

$$\begin{aligned} \left(A^{\beta_i}(U)\right)_{hh} &:= 1, \quad h = 1, \dots, n, \; h \neq i, i+1, \\ \left(A^{\beta_i}(U)\right)_{i+1,i+1} &:= -U_{i,i+1}, \\ \left(A^{\beta_i}(U)\right)_{i,i+1} &:= \left(A^{\beta_i}(U)\right)_{i+1,i} = 1, \end{aligned}$$

and all other entries of $A^{\beta_i}(U)$ are equal to zero.

Lemma 4.6.2 ([30, 32]). *The braid group $\mathcal{B}_n$ acts on $\mathcal{U}_n \times \mathrm{GL}(n, \mathbb{C})$ as follows:*

$$\mathcal{B}_n \times \mathcal{U}_n \times \mathrm{GL}(n, \mathbb{C}) \to \mathcal{U}_n \times \mathrm{GL}(n, \mathbb{C}),$$
$$(\beta_i, U, C) \mapsto (A^{\beta_i}(U) \cdot U \cdot A^{\beta_i}(U),\ C \cdot A^{\beta_i}(U)^{-1}).$$

We denote by $(U, C)^{\beta_i}$ the action of β_i on (U, C). ∎

Fix an oriented ray $\ell_o \equiv \{\arg z = \varphi_o\}$ in $\widetilde{\mathbb{C}^*}$, and denote by $\overline{\ell_o}$ its projection on $\widetilde{\mathbb{C}^*}$. Let $p_o \in O_{\ell_o}$, and let (S_0, C_0) be the Stokes and central connection matrices computed at p_o with respect to ℓ_o, the ℓ_o-lexicographical order of canonical coordinates $u_i(p_o)$, and a suitable determination of the Ψ-matrix at p_o. If we let the oriented ray rotate, so that it crosses some Stokes rays $R_{ij}(p_o)$, the values of (S_0, C_0) will change. We can describe this difference of values in terms of the braid group action of Lemma 4.6.2.

Theorem 4.6.3 ([21, 30, 32]). *Consider a continuous map $\varphi\colon [0, 1] \to \mathbb{R}$, with $\varphi(0) = \varphi_o$, and set $\ell(t) := \{\arg z = \varphi(t)\}$ for any $t \in [0, 1]$. Assume that*

- *the rays $\ell(0)$ and $\ell(1)$ are admissible at p_o,*
- *there exists a unique $t_o \in\,]0, 1[$ such that $\ell(t_o)$ is not admissible at p_o,*
- *there exist $i_1, \dots, i_k \in \{1, \dots, n\}$, with $|i_a - i_b| > 1$ for $a \neq b$, such that the projected ray $\overline{\ell}(t) \subseteq \mathbb{C}$ crosses the rays $(R_{i_j, i_j+1})_{j=1}^k$ in the counterclockwise (resp. clockwise) direction, as $t \to t_o^-$.*

Denote by (S_i, C_i), with $i = 0, 1$, the Stokes and central connection matrices at p_o with respect to the oriented ray $\ell(i)$, with $i = 0, 1$. We have

$$(S_1, C_1) = (S_0, C_0)^\beta, \quad \beta = \prod_{j=1}^k \beta_{i_j} \quad \left(\text{resp. } \beta = \left(\prod_{j=1}^k \beta_{i_j}\right)^{-1}\right). \qquad ∎$$

Remark 4.6.4. An arbitrary rotation of ℓ can be decomposed into the composition of elementary rotations satisfying the assumptions of Theorem 4.6.3.

Furthermore, the braid group action also describes how the values of Stokes and central connection matrices in different ℓ-chambers (for a fixed oriented rays ℓ) are related to each other.

Fix an oriented ray $\ell \equiv \{\arg z = \varphi\}$ in $\widetilde{\mathbb{C}^*}$, and denote by $\overline{\ell}$ its projection on $\mathbb{C}^*$. Let $\Omega_{\ell,1}, \Omega_{\ell,2}$ be two ℓ-chambers and let $p_i \in \Omega_{\ell,i}$ for $i = 1, 2$. The difference of values of the Stokes and central connection matrices (S_1, C_1) and (S_2, C_2), at p_1 and p_2, respectively, can be described by the action of the braid group $\mathcal{B}_n$ of Lemma 4.6.2.

Theorem 4.6.5 ([21, 30, 32]). *Consider a continuous path $\gamma\colon [0, 1] \to \Omega$ such that*

- *$\gamma(0) = p_1$ and $\gamma(1) = p_2$,*
- *there exists a unique $t_o \in [0, 1]$ such that ℓ is not admissible at $\gamma(t_o)$,*

- *there exist* $i_1, \ldots, i_k \in \{1, \ldots, n\}$, *with* $|i_a - i_b| > 1$ *for* $a \neq b$, *such that the rays*[1] $(R_{i_j, i_j+1}(t))_{j=1}^r$ *(resp.* $(R_{i_j, i_j+1}(t))_{j=r+1}^k$*) cross the ray* $\overline{\ell}$ *in the clockwise (resp. counterclockwise) direction, as* $t \to t_o^-$.

Then we have

$$(S_2, C_2) = (S_1, C_1)^\beta, \quad \beta := \left(\prod_{j=1}^r \beta_{i_j}\right) \cdot \left(\prod_{h=r+1}^k \beta_{i_h}\right)^{-1}. \qquad \blacksquare$$

Remark 4.6.6. In the general case, the points p_1 and p_2 can be connected by concatenations of paths γ satisfying the assumptions of Theorem 4.6.5.

Remark 4.6.7. The action of $\mathcal{B}_n$ on (S, C) also describes the analytic continuation of the Frobenius manifold structure on Ω, see [32, Lecture 4].

[1] Here the labeling of Stokes rays is the one prolonged from the initial point $t = 0$.