## Chapter 5

# $J$-function and quantum Lefschetz theorem

## 5.1 $J$-function and master functions

**Definition 5.1.1.** The $J$-function of $X$ is the $H^\bullet(X, \Lambda_X)[\![\hbar^{-1}]\!]$-valued function of $\boldsymbol{\tau} \in H^\bullet(X, \mathbb{C})$ defined by

$$J_X(\boldsymbol{\tau}) := 1 + \sum_{\alpha,\lambda=0}^{N} \sum_{n=0}^{\infty} \hbar^{-(n+1)} \langle\!\langle \tau_n T_\alpha, 1 \rangle\!\rangle_0 \eta^{\alpha\lambda} T_\lambda.$$

The following result will be crucial for us. For its proof see Appendix A.

**Theorem 5.1.2.** *Let* $\alpha = 0, \dots, N$ *and* $\delta \in H^2(X, \mathbb{C})$. *The* $(1, \alpha)$*-entry of the matrix* $\eta Z_{\text{top}}(z, \delta)$ *equals*

$$z^{\frac{\dim X}{2}} \int_X T_\alpha \cup J_X(\delta + \log z \cdot c_1(X)) \Big|_{\substack{\mathbf{Q}=1 \\ \hbar=1}}.$$

**Corollary 5.1.3.** *Let* $\delta \in H^2(X, \mathbb{C})$. *The components of the function*

$$J(\delta + \log z \cdot c_1(X)) \Big|_{\substack{\mathbf{Q}=1 \\ \hbar=1}},$$

*with respect to any basis of* $H^\bullet(X, \mathbb{C})$, *span the space of master functions* $\mathcal{S}_\delta(X)$.

*Proof.* The functions $z^{-\frac{\dim X}{2}} [\eta Z_{\text{top}}(z, \delta)]^1_\alpha$ define a generating set of the space of master functions $\mathcal{S}_\delta(X)$. The claim follows by Theorem 5.1.2. ∎

In the notations of Section 3.1, set

$$\delta = \sum_{i=1}^{r} t^i T_i.$$

Any formal differential operator $P \in \mathbb{C}[\![\hbar \frac{\partial}{\partial t^1}, \dots, \hbar \frac{\partial}{\partial t^r}, e^{t^1}, \dots, e^{t^r}, \hbar]\!]$ such that

$$P J_X(\delta) = 0$$

is called a *quantum differential operator*. The equation $PY = 0$ is called a *quantum differential equation*, see e.g. [27, Section 10.3]. By Corollary 5.1.3, the master differential equation, defined as in Section 2.7 at a point $\delta$ of the complement of the $\mathcal{A}_\Lambda$-stratum of $QH^\bullet(X)$, is equivalent to a differential equation for master functions

$$\widetilde{P}_\delta(\vartheta, z)\Phi = 0, \quad \vartheta := z\frac{d}{dz},$$

for a suitable differentiable operator $\widetilde{P}_\delta$.

## 5.2 Twisted Gromov–Witten invariants

Given a holomorphic vector bundle $E \to X$ and an invertible multiplicative[1] characteristic class $\boldsymbol{c}$, one can introduce a $(E, \boldsymbol{c})$-*twisted* version of the Gromov–Witten theory of $X$.

Given $E$, there exists a complex

$$0 \to E^0_{g,n,\beta} \to E^1_{g,n,\beta} \to 0$$

of locally free orbi-sheaves on $\overline{\mathcal{M}}_{g,n}(X,\beta)$ whose cohomology sheaves are

$$R^0\mathrm{ft}_{n+1,*}(\mathrm{ev}^*_{n+1}E) \quad \text{and} \quad R^1\mathrm{ft}_{n+1,*}(\mathrm{ev}^*_{n+1}E),$$

respectively. Here, the forgetful and evaluation morphisms $\mathrm{ft}_{n+1}$, $\mathrm{ev}_{n+1}$ at the last marked point fit in the diagram

$$\begin{array}{ccc} & \overline{\mathcal{M}}_{g,n+1}(X,\beta) & \\ {}^{\mathrm{ft}_{n+1}}\swarrow & & \searrow^{\mathrm{ev}_{n+1}} \\ \overline{\mathcal{M}}_{g,n}(X,\beta) & & X. \end{array}$$

Let us introduce an *obstruction $K$-class*

$$E_{g,n,\beta} \in K^0(\overline{\mathcal{M}}_{g,n}(X,\beta)),$$

defined as the $K$-theoretic difference

$$E_{g,n,\beta} := [E^0_{g,n,\beta}] - [E^1_{g,n,\beta}].$$

It is possible to show that such a difference does not depend on the choice of the complex.

**Definition 5.2.1.** The $(E, \boldsymbol{c})$-*twisted Gromov–Witten invariants (with descendants)* of $X$ are the intersection numbers

$$\langle \tau_1^{d_1}\alpha_1 \otimes \cdots \otimes \tau_n^{d_n}\alpha_n \rangle^{X,E,\boldsymbol{c}}_{g,n,\beta} := \int_{[\overline{\mathcal{M}}_{g,n}(X,\beta)]^{\mathrm{virt}}} \boldsymbol{c}(E_{g,n,\beta}) \cup \prod_{j=1}^{n} \psi_j^{d_j} \cup \mathrm{ev}_j^*(\alpha_j),$$

where $\alpha_1, \ldots, \alpha_n \in H^\bullet(X, \mathbb{C})$.

**Remark 5.2.2.** If $\boldsymbol{c}$ is the trivial characteristic class, then we recover the untwisted Gromov–Witten invariants of $X$.

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[1] A characteristic class $\boldsymbol{c}$ is said to be *multiplicative* if $\boldsymbol{c}(E_1 \oplus E_2) = \boldsymbol{c}(E_1)\boldsymbol{c}(E_2)$. It is *invertible* if $\boldsymbol{c}(E)$ is invertible in $H^\bullet(Y, \mathbb{C})$ for any vector bundle $E$ on a manifold $Y$.

## 5.3 Quantum Lefschetz theorem

Introduce a $\mathbb{C}^*$-action on the total space $E$ defined by fiberwise multiplication. Note that the $\mathbb{C}^*$-equivariant Euler class $\boldsymbol{e}$ is invertible over the field of fractions $\mathbb{Q}(\lambda)$ of $H^\bullet_{\mathbb{C}^*}(\mathrm{pt}) \cong \mathbb{Q}[\lambda]$. Taking $\boldsymbol{c} = \boldsymbol{e}$ we refer to the twisted Gromov–Witten invariants as *Euler-twisted Gromov–Witten invariants*.

Exactly as in the untwisted case, $(E, \boldsymbol{c})$-twisted Gromov–Witten invariants can be collected in generating functions. In particular, we can introduce the *Euler-twisted J-function* as the $H^\bullet(X, \Lambda_X[\lambda])[\![\hbar^{-1}]\!]$-valued function on $H^\bullet(X, \mathbb{C})$ by

$$J_{E,\boldsymbol{e}}(\boldsymbol{\tau}) = 1 + \sum_{\alpha,k,n,\beta} \hbar^{-n-1} \frac{\mathbf{Q}^\beta}{k!} \langle \tau_n T_\alpha, 1, \boldsymbol{\tau}, \dots, \boldsymbol{\tau} \rangle^{X,E,\boldsymbol{e}}_{0,k+2,\beta} T^\alpha.$$

Assume now that the vector bundle $E$ is convex,[2] i.e. $H^1(C, f^*E) = 0$ for all stable maps $f: C \to X$ with $C$ of genus zero. Let $Y$ be a smooth subvariety of $X$ defined by the zero locus of a regular section of $E$.

**Theorem 5.3.1** ([15, 17]). *The non-equivariant limit $J_{E,\boldsymbol{e}}|_{\lambda=0}$ exists. Moreover, it is related to the function $J_Y$ by the equation*

$$\iota^* J_{E,\boldsymbol{e}}|_{\lambda=0}(\boldsymbol{\tau}) \stackrel{\iota_*}{=} J_Y(\iota^* \boldsymbol{\tau}), \quad \tau \in H^\bullet(X, \mathbb{C}), \tag{5.3.1}$$

*where $\iota: Y \hookrightarrow X$ is the inclusion.*

**Remark 5.3.2.** The symbol $\stackrel{\iota_*}{=}$ means that identity (5.3.1) holds true after application of the morphism $\iota_*: \Lambda_X \to \Lambda_Y$ defined by $\mathbf{Q}^\beta \mapsto \mathbf{Q}^{\iota_*\beta}$.

**Remark 5.3.3.** If $\dim_{\mathbb{C}} X > 3$, then $\iota^*$ is an isomorphism, by the hyperplane Lefschetz theorem.

Assume that

$$E = \bigoplus_{i=1}^s L_i,$$

where $L_i$ are nef line bundles on $X$ such that $c_1(E) \leqslant c_1(X)$. In such a case, the quantum Lefschetz theorem prescribes how to compute the non-equivariant limit $J_{E,\boldsymbol{e}}(\delta)|_{\lambda=0}$ at points of the small quantum locus $\delta \in H^2(X, \mathbb{C})$.

Introduce the *hypergeometric modification* $I_{X,Y}$ of the function $J_X$ as follows: write $J_X = \sum_\beta J_\beta \mathbf{Q}^\beta$, and for $\delta \in H^2(X, \mathbb{C})$ define

$$I_{X,Y}(\delta) := \sum_\beta J_\beta(\delta) \mathbf{Q}^\beta \prod_{i=1}^s \prod_{m=1}^{\langle c_1(L_i), \beta \rangle} (c_1(L_i) + m\hbar). \tag{5.3.2}$$

[2] Globally generated vector bundles and direct sums of nef line bundles are automatically convex.

**Theorem 5.3.4** ([17]). *The function $I_{X,Y}$ admits an expansion of the form*

$$I_{X,Y}(\delta) = F(\delta) + \frac{1}{\hbar}G(\delta) + O\left(\frac{1}{\hbar^2}\right), \quad \delta \in H^2(X,\mathbb{C}),$$

*where $F$ is $H^0(X,\Lambda_X)$-valued and $G$ takes values in $H^0(X,\Lambda_X) \oplus H^2(X,\Lambda_X)$. Moreover, we have*

$$J_{E,\boldsymbol{e}}(\varphi(\delta))|_{\lambda=0} = \frac{I_{X,Y}(\delta)}{F(\delta)}, \quad \varphi(\delta) := \frac{G(\delta)}{F(\delta)}.$$

**Proposition 5.3.5** ([16, 17]). *Moreover, if $c_1(X) > c_1(E)$, then we have*

$$\begin{aligned} F(\delta) &\equiv 1, \\ G(\delta) &= \delta + H(\delta)\cdot 1, \\ H(\delta) &= \sum_{\beta}\left(w_\beta \mathbf{Q}^\beta e^{\int_\beta \delta}\right)\cdot \delta_{1,\langle \beta, c_1(X)-c_1(E)\rangle} \end{aligned}$$

*for suitable rational coefficients $w_\beta \in \mathbb{Q}$.*

*Proof.* The function $I_{X,Y}(\delta)$ is homogeneous of degree 0 with respect to the gradings

$$\begin{aligned} \deg \mathbf{Q}^\beta &= \int_\beta c_1(X) - \int_\beta c_1(E), \\ \deg \hbar &= 1, \\ \deg T_\alpha &= k \quad \text{if } T_\alpha \in H^{2k}(X,\mathbb{C}). \end{aligned}$$

This is easily seen from the expansion of $J_X$ given in Lemma A.2. Hence, $F(\delta)$ is given from the only contribution of the term $J_0(\delta) = 1 + \frac{\delta}{\hbar} + \cdots$ and $H(\delta)$ from the terms for which $\deg \mathbf{Q}^\beta = 1$. ∎

## 5.4 Inequality for dimensions of spaces of master functions

Let $Y \subseteq X$ be the zero locus of a regular section of a vector bundle $E \to X$, sum of nef line bundles, with $c_1(E) < c_1(X)$. Denote by $\iota\colon Y \to X$ the inclusion. We always assume that both $X$ and $Y$ have vanishing odd cohomology.

For a point $\boldsymbol{\tau} \in QH^\bullet(X)$, denote by $\mathcal{S}_{\boldsymbol{\tau}}(X) := \mathcal{S}_{\boldsymbol{\tau}}(QH^\bullet(X))$ the space of master functions as $\boldsymbol{\tau}$.

**Theorem 5.4.1.** *Let $\delta \in H^2(X,\mathbb{C})$. We have*

$$\dim_{\mathbb{C}} \mathcal{S}_{\iota^*\delta}(Y) \leqslant \dim_{\mathbb{C}} \mathcal{S}_{\delta+c}(X), \tag{5.4.1}$$

*where $c := c_1(X) - c_1(E)$.*

*Proof.* By the adjunction formula, we have

$$\iota^* c = c_1(Y).$$

The components of the function $J_X(\delta + c \log z)|_{\mathbf{Q}=1,\hbar=1}$, with respect to any basis of $H^\bullet(X,\mathbb{C})$, span the space $\mathcal{S}_{\delta+c}(X)$. Analogously, the components of the function $J_Y(\iota^*\delta + c_1(Y)\log z)|_{\mathbf{Q}=1,\hbar=1}$, with respect to any basis of $H^\bullet(Y,\mathbb{C})$, span the space $\mathcal{S}_{\iota^*\delta}(Y)$.

By Theorems 5.3.1, 5.3.4 and Proposition 5.3.5, we have

$$J_Y(\iota^*\delta + c_1(Y)\log z)|_{\substack{\mathbf{Q}=1\\ \hbar=1}} = e^{-zH(\delta)} \cdot \iota^* I_{X,Y}(\delta + c \log z)|_{\substack{\mathbf{Q}=1\\ \hbar=1}}.$$

The components of the right side are obtained by linear combinations and rescaling of the components of $J_X(\delta + c\log z)|_{\mathbf{Q}=1,\hbar=1}$: such a linear combination is due to the hypergeometric modification (5.3.2), namely the $\cup$-multiplication by an invertible class. The claim follows. ■

**Theorem 5.4.2.** *Let $Y$ be a hyperplane section of $X$. Assume that $d := \dim_{\mathbb{C}} X$ is odd, and that the following inequalities of Betti numbers hold true:*

$$b_{d-1}(X) < \frac{1}{2} b_{d-1}(Y). \tag{5.4.2}$$

*Then $\iota^*(H^2(X,\mathbb{C}))$ is contained in the $\mathcal{A}_\Lambda$-stratum of the Frobenius manifold $QH^\bullet(Y)$. In particular, along $\iota^*(H^2(X,\mathbb{C}))$ the canonical coordinates of $QH^\bullet(Y)$ coalesce.*

*Proof.* From the hyperplane Lefschetz theorem we deduce that (5.4.2) holds true if and only if $\dim_{\mathbb{C}} H^\bullet(X,\mathbb{C}) < \dim_{\mathbb{C}} H^\bullet(Y,\mathbb{C})$. Then for any $\delta \in H^2(X,\mathbb{C})$ we have $\dim_{\mathbb{C}} \mathcal{S}_{\iota^*\delta}(Y) < \dim_{\mathbb{C}} H^\bullet(Y,\mathbb{C})$, by (5.4.1). Hence, the master differential equation of $QH^\bullet(Y)$ at $\iota^*\delta$ is not of order $\dim_{\mathbb{C}} H^\bullet(Y,\mathbb{C})$. This implies that the denominator of $\det\Lambda$ is identically zero at $\iota^*\delta$. The last statement follows from Lemma 2.6.1 and Theorem 2.6.2. ■