Chapter 6

Borel–Laplace (α, β)-multitransforms

6.1 Algebras of Ribenboim's generalized power series

Let $(M, +, 0)$ be a *monoid*, i.e. a commutative semigroup with neutral element. We say that a partial order relation $\leqslant$ on M defines a *strictly ordered monoid* $(M, +, 0, \leqslant)$ if the following compatibility condition holds true:

$$\text{if } a < b, \text{ then } a + s < b + s \text{ for all } s \in M.$$

Let R be a commutative ring with unit. The set

$$R[\![M]\!] := R^M$$

of all functions $f\colon M \to R$ is equipped with a natural R-module structure, with respect to pointwise addition and multiplication by scalars. An element $f \in R[\![M]\!]$ will usually be denoted by

$$f = \sum_{a \in M} f(a) Z^a,$$

where Z is an indeterminate. Given two functions $f, g \in R[\![M]\!]$, we could be tempted to define their product as

$$f \cdot g := \sum_{s \in M} \Bigg(\sum_{(p,q) \in X_s(f,g)} f(p) \cdot g(q) \Bigg) Z^s, \tag{6.1.1}$$

where we set

$$X_s(f, g) := \{(p, q) \in M \times M \,:\, p + q = s,\ f(p) \neq 0,\ g(q) \neq 0\}.$$

In general the set $X_s(f, g)$ is *not* finite, and consequently the product $f \cdot g$ could be not defined.

Definition 6.1.1. Let $(M, +, 0, \leqslant)$ be a strictly ordered monoid. The R-submodule of $R[\![M]\!]$ which consists of all functions $f\colon M \to R$ whose support

$$\operatorname{supp}(f) := \{s \in M \,:\, f(s) \neq 0\}$$

is

(1) *Artinian*, i.e. every subset of $\operatorname{supp}(f)$ admits a minimal element,

(2) *narrow*, i.e. every subset of $\operatorname{supp}(f)$ of pairwise incomparable elements is finite,

is called the set of *generalized power series* with coefficients in R and exponents in M. It is denoted by $R[\![M, \leqslant]\!]$.

Proposition 6.1.2 ([68, 69]). *Given $f, g \in R[\![M, \leqslant]\!]$, the set $X_s(f, g)$ is finite, and the product (6.1.1) is well defined. The set $R[\![M, \leqslant]\!]$ inherits the structure of an associative R-algebra.*

Remark 6.1.3. If $(M, \leqslant)$ is itself Artinian and narrow, then all its subsets are Artinian and narrow. Consequently, $R[\![M, \leqslant]\!] = R[\![M]\!]$.

6.2 The algebra $\mathscr{F}_{\boldsymbol{\kappa}}(A)$

Let $\boldsymbol{\kappa} := (\kappa_1, \ldots, \kappa_h) \in (\mathbb{C}^*)^h$. Consider an associative, commutative, unitary and finite-dimensional $\mathbb{C}$-algebra $(A, +, \cdot, 1_A)$. Denote by $\mathrm{Nil}(A)$ the nilradical of A, that is,

$$\mathrm{Nil}(A) := \{a \in A : \text{there exists an } n \in \mathbb{N} \text{ such that } a^n = 0\}.$$

Set $\mathbb{N}_A := \{n \cdot 1_A : n \in \mathbb{N}\}$. Define the monoid $M_{A,\boldsymbol{\kappa}}$ as the (external) direct sum of monoids

$$M_{A,\boldsymbol{\kappa}} := \left(\bigoplus_{j=1}^{h} \kappa_j \mathbb{N}_A \right) \oplus \mathrm{Nil}(A).$$

We have two maps $\nu_{\boldsymbol{\kappa}} : M_{A,\boldsymbol{\kappa}} \to \mathbb{N}^h$ and $\iota_{\boldsymbol{\kappa}} : M_{A,\boldsymbol{\kappa}} \to A$ defined by

$$\nu_{\boldsymbol{\kappa}}((\kappa_i n_i 1_A)_{i=1}^h, r) := (n_i)_{i=1}^h$$

and

$$\iota_{\boldsymbol{\kappa}}((\kappa_i n_i 1_A)_{i=1}^h, r) := \sum_{i=1}^{h} \kappa_i n_i 1_A + r.$$

On $M_{A,\boldsymbol{\kappa}}$ we can define the partial order

$$x \leqslant y \iff \nu_{\boldsymbol{\kappa}}(x) \leqslant \nu_{\boldsymbol{\kappa}}(y),$$

the order on $\mathbb{N}^h$ being the lexicographical one. This order makes $(M_{A,\boldsymbol{\kappa}}, \leqslant)$ a strictly ordered monoid.

We denote by $\mathscr{F}_{\boldsymbol{\kappa}}(A)$ the ring $A[\![M_{A,\boldsymbol{\kappa}}, \leqslant]\!]$.

By the universal property of the direct sums of monoids, the natural inclusions $M_{A,\kappa_i} \to M_{A,\boldsymbol{\kappa}}$ induce a unique morphism

$$\rho_{\boldsymbol{\kappa}} : \bigoplus_{i=1}^{h} M_{A,\kappa_i} \to M_{A,\boldsymbol{\kappa}}.$$

Definition 6.2.1. Let $r_o \in \mathrm{Nil}(A)$. We say that an element $f \in \mathscr{F}_{\boldsymbol{\kappa}}(A)$ is *concentrated at r_o* if

$$\mathrm{supp}(f) \subseteq \left(\bigoplus_{i=1}^{h} \kappa_i \mathbb{N}_A \right) \times \{r_o\}.$$

6.3 Formal Borel–Laplace $(\boldsymbol{\alpha}, \boldsymbol{\beta})$-multitransforms

Given two h-tuples $\boldsymbol{\alpha}, \boldsymbol{\beta} \in (\mathbb{C}^*)^h$, we set $\boldsymbol{\alpha} \cdot \boldsymbol{\beta} := (a_i \beta_i)_{i=1}^h$, and $\boldsymbol{\alpha}^{-1} := (\frac{1}{\alpha_i})_{i=1}^h$.

Definition 6.3.1. Let $F \in \mathbb{C}[\![x]\!]$ be a formal power series $F(x) = \sum_{k=0}^{\infty} a_k x^k$. For $\alpha \in \mathrm{Nil}(A)$ define $F(\alpha) \in A$ by the finite sum

$$F(\alpha) = \sum_{k=0}^{\infty} a_k \alpha^k.$$

If F is invertible, i.e. $a_0 \neq 0$, then $F(\alpha)$ is invertible in A.

In what follows we will usually take $F(x) = \Gamma(\lambda + x)$ with $\lambda \in \mathbb{C} \setminus \mathbb{Z}_{\leqslant 0}$, where Γ denotes the Euler Gamma function.

Definition 6.3.2. Let $\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\kappa} \in (\mathbb{C}^*)^h$. We define the *Borel* $(\boldsymbol{\alpha}, \boldsymbol{\beta})$*-multitransform* as the A-linear morphism

$$\mathscr{B}_{\boldsymbol{\alpha},\boldsymbol{\beta}} \colon \bigotimes_{j=1}^{h} \mathscr{F}_{\kappa_j}(A) \to \mathscr{F}_{\boldsymbol{\alpha}^{-1} \cdot \boldsymbol{\beta}^{-1} \cdot \boldsymbol{\kappa}}(A),$$

which is defined, on decomposable elements, by

$$\mathscr{B}_{\boldsymbol{\alpha},\boldsymbol{\beta}}\left(\bigotimes_{j=1}^{h}\left(\sum_{s_j \in M_{A,\kappa_j}} f_{s_j}^j Z^{s_j}\right)\right) := \sum_{\substack{s_j \in M_{A,\kappa_j} \\ j=1,\ldots,h}} \frac{\prod_{i=1}^h f_{s_i}^i}{\Gamma\big(1 + \sum_{\ell=1}^h \iota_{\kappa_\ell}(s_\ell)\beta_\ell\big)} Z^{\rho_{\boldsymbol{\kappa}}\left(\oplus_{\ell=1}^h \frac{s_\ell}{\alpha_\ell \beta_\ell}\right)}.$$

Definition 6.3.3. Let $\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\kappa} \in (\mathbb{C}^*)^h$. We define the *Laplace* $(\boldsymbol{\alpha}, \boldsymbol{\beta})$*-multitransform* as the A-linear morphism

$$\mathscr{L}_{\boldsymbol{\alpha},\boldsymbol{\beta}} \colon \bigotimes_{j=1}^{h} \mathscr{F}_{\kappa_j}(A) \to \mathscr{F}_{\boldsymbol{\alpha} \cdot \boldsymbol{\beta} \cdot \boldsymbol{\kappa}}(A),$$

which is defined, on decomposable elements, by

$$\mathscr{L}_{\boldsymbol{\alpha},\boldsymbol{\beta}}\left(\bigotimes_{j=1}^{h}\left(\sum_{s_j \in M_{A,\kappa_j}} f_{s_j}^j Z^{s_j}\right)\right) := \sum_{\substack{s_j \in M_{A,\kappa_j} \\ j=1,\ldots,h}} \left(\prod_{i=1}^h f_{s_i}^i\right) \Gamma\left(1 + \sum_{\ell=1}^h \iota_{\kappa_\ell}(s_\ell)\beta_\ell\right) Z^{\rho_{\boldsymbol{\kappa}}(\oplus_{\ell=1}^h \alpha_\ell \beta_\ell s_\ell)}.$$

In the case $h = 1$, the Borel–Laplace $(\boldsymbol{\alpha}, \boldsymbol{\beta})$-multitransform simplify as follows.

Definition 6.3.4. Given $\alpha, \beta \in \mathbb{C}^*$, we define two A-linear maps

$$\mathscr{B}_{\alpha,\beta}\colon \mathscr{F}_\kappa(A) \to \mathscr{F}_{\frac{\kappa}{\alpha\beta}}(A), \quad \mathscr{L}_{\alpha,\beta}\colon \mathscr{F}_\kappa(A) \to \mathscr{F}_{\alpha\beta\kappa}(A), \quad \kappa \in \mathbb{C}^*$$

called respectively (α, β)-*Borel and Laplace transforms*, through the formulas

$$\mathscr{B}_{\alpha,\beta}\Bigg[\sum_{s\in M_{A,\kappa}} f_s Z^s\Bigg] := \sum_{s\in M_{A,\kappa}} \frac{f_s}{\Gamma(1+\beta s)} Z^{\frac{s}{\alpha\beta}},$$

$$\mathscr{L}_{\alpha,\beta}\Bigg[\sum_{s\in M_{A,\kappa}} f_s Z^s\Bigg] := \sum_{s\in M_{A,\kappa}} f_s \Gamma(1+\beta s) Z^{\alpha\beta s}.$$

Theorem 6.3.5. *The Borel–Laplace (α, β)-transform are inverses of each other, i.e.*

$$\mathscr{B}_{\alpha,\beta} \circ \mathscr{L}_{\alpha,\beta} = \mathrm{Id}, \quad \mathscr{L}_{\alpha,\beta} \circ \mathscr{B}_{\alpha,\beta} = \mathrm{Id}. \qquad \blacksquare$$

6.4 Analytic Borel–Laplace $(\boldsymbol{\alpha}, \boldsymbol{\beta})$-multitransforms

Definition 6.4.1. Let $\boldsymbol{\alpha}, \boldsymbol{\beta} \in (\mathbb{C}^*)^h$. The Borel $(\boldsymbol{\alpha}, \boldsymbol{\beta})$-multitransform of an h-tuple of A-valued functions $(\Phi_1, \dots, \Phi_h)$ is defined, when the integral exists, by

$$\mathscr{B}_{\boldsymbol{\alpha},\boldsymbol{\beta}}[\Phi_1, \dots, \Phi_h](z) := \frac{1}{2\pi i} \int_\gamma \prod_{j=1}^{h} \Phi_j\left(z^{\frac{1}{\alpha_j \beta_j}} \lambda^{-\beta_j}\right) e^\lambda \frac{d\lambda}{\lambda},$$

where γ is a Hankel-type contour of integration, see Figure 6.1.

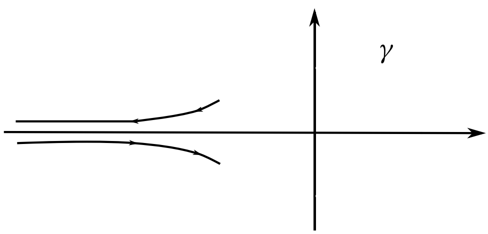

Figure 6.1. Hankel-type contour of integration defining Borel $(\boldsymbol{\alpha}, \boldsymbol{\beta})$-multitransform.

Definition 6.4.2. Let $\boldsymbol{\alpha} := (\alpha_1, \dots, \alpha_h)$ and $\boldsymbol{\beta} := (\beta_1, \dots, \beta_h)$ be h-tuples in $(\mathbb{C}^*)^h$. The $(\boldsymbol{\alpha}, \boldsymbol{\beta})$-Laplace transform of an h-tuple of functions $(\Phi_1, \dots, \Phi_h)$ is defined, when the integral exists, by

$$\mathscr{L}_{\boldsymbol{\alpha},\boldsymbol{\beta}}[\Phi_1, \dots, \Phi_h](z) := \int_0^\infty \prod_{i=1}^{h} \Phi_i(z^{\alpha_i \beta_i} \lambda^{\beta_i}) \exp(-\lambda)\, d\lambda.$$

Proposition 6.4.3. *Let $(e_1, \ldots, e_n)$ be a basis of A and let $\Phi_1, \ldots, \Phi_h$ be A-valued functions. Write $\Phi_i = \sum_j \Phi_i^j e_j$ for $\mathbb{C}$-valued component functions Φ_i^j. The components of $\mathscr{B}_{\boldsymbol{\alpha},\boldsymbol{\beta}}[\Phi_1, \ldots, \Phi_h]$ (resp. $\mathscr{L}_{\boldsymbol{\alpha},\boldsymbol{\beta}}[\Phi_1, \ldots, \Phi_h]$) are $\mathbb{C}$-linear combinations of the $h \cdot n$ $\mathbb{C}$-valued functions $\mathscr{B}_{\boldsymbol{\alpha},\boldsymbol{\beta}}[\Phi_1^{i_1}, \ldots, \Phi_h^{i_h}]$ (resp. $\mathscr{L}_{\boldsymbol{\alpha},\boldsymbol{\beta}}[\Phi_1^{i_1}, \ldots, \Phi_h^{i_h}]$), where $(i_1, \ldots, i_h) \in \{1, \ldots, n\}^{\times h}$.*

Proof. Let $c_{jk}^i \in \mathbb{C}$ be the structure constants of the algebra A, so that

$$e_j e_k = \sum_i c_{jk}^i e_i.$$

We have

$$\mathscr{B}_{\boldsymbol{\alpha},\boldsymbol{\beta}}[\Phi_1, \ldots, \Phi_h] = \sum_{\boldsymbol{a},\boldsymbol{i}} c_{i_1 i_2}^{a_1} c_{a_1 i_3}^{a_2} \cdots c_{a_{h-2} i_h}^{a_{h-1}} e_{a_{h-1}} \mathscr{B}_{\boldsymbol{\alpha},\boldsymbol{\beta}}[\Phi_1^{i_1}, \ldots, \Phi_h^{i_h}].$$

Similarly for the Laplace multitransform. ■

6.5 Analytification of elements of $\mathscr{F}_{\boldsymbol{\kappa}}(A)$

Let $s = ((\kappa_i n_i 1_A)_{i=1}^h, r) \in M_{A,\boldsymbol{\kappa}}$. We define the *analytification* $\widehat{Z^s}$ of the monomial $Z^s \in \mathscr{F}_{\boldsymbol{\kappa}}(A)$ to be the A-valued holomorphic function

$$\widehat{Z^s} \colon \widetilde{\mathbb{C}^*} \to A, \quad \widehat{Z^s}(z) := z^{\sum_{i=1}^h \kappa_i n_i} \sum_{j=1}^{\infty} \frac{r^j}{j!} \log^j z.$$

Notice that the sum is finite, since $r \in \mathrm{Nil}(A)$.

Let $f \in \mathscr{F}_{\boldsymbol{\kappa}}(A)$ be a series

$$f(Z) = \sum_{s \in M_{A,\boldsymbol{\kappa}}} f_a Z^s$$

such that

$$\operatorname{card} \operatorname{supp}(f) \leqslant \aleph_0.$$

The *analytification* $\widehat{f}$ of f is the A-valued holomorphic function defined if the series absolutely converges, by

$$\widehat{f} \colon W \subseteq \widetilde{\mathbb{C}^*} \to A, \quad \widehat{f}(z) := \sum_{s \in M_{A,\boldsymbol{\kappa}}} f_a \widehat{Z^s}(z).$$

Theorem 6.5.1. *Let $f_i \in \mathscr{F}_{\kappa_i}(A)$ such that*

- card supp$(f_i) \leqslant \aleph_0$ *for $i = 1, \ldots, h$,*
- *the functions $\widehat{f_i}$ are well defined on $\mathbb{R}_+$.*

We have

$$\widehat{\mathscr{B}_{\boldsymbol{\alpha},\boldsymbol{\beta}}\left[\bigotimes_{j=1}^{h} f_j\right]} = \mathscr{B}_{\boldsymbol{\alpha},\boldsymbol{\beta}}[\widehat{f}_1, \dots, \widehat{f}_h],$$

$$\widehat{\mathscr{L}_{\boldsymbol{\alpha},\boldsymbol{\beta}}\left[\bigotimes_{j=1}^{h} f_j\right]} = \mathscr{L}_{\boldsymbol{\alpha},\boldsymbol{\beta}}[\widehat{f}_1, \dots, \widehat{f}_h],$$

provided that both sides are well defined.

Proof. It is sufficient to prove the statement on monomials $Z^{s_1}, \dots, Z^{s_h}$. To this end, let $s_j = (\kappa_j n_j 1_A, r_j)$ for $j = 1, \dots, h$. We have

$$\begin{aligned}
&\mathscr{B}_{\boldsymbol{\alpha},\boldsymbol{\beta}}\left[\bigotimes_{j=1}^{h} Z^{s_j}\right] \\
&\quad = \frac{1}{\Gamma\big(1 + \sum_{\ell=1}^{h} \iota_{\kappa_\ell}(s_\ell)\beta_\ell\big)} Z^{\rho_\kappa\left(\bigoplus_{\ell=1}^{h} \frac{s_\ell}{\alpha_\ell \beta_\ell}\right)} \\
&\quad = \frac{1}{\Gamma\big(1 + \sum_{\ell=1}^{h} (\kappa_\ell n_\ell 1_A + r_\ell)\beta_\ell\big)} Z^{\left(\left(\frac{\kappa_j}{\alpha_j \beta_j} n_j 1_A\right)_{j=1}^{h},\ \frac{r_1}{\alpha_1\beta_1} + \cdots + \frac{r_h}{\alpha_h \beta_h}\right)}.
\end{aligned}$$

Hence, we have

$$\begin{aligned}
&\widehat{\mathscr{B}_{\boldsymbol{\alpha},\boldsymbol{\beta}}\left[\bigoplus_{j=1}^{h} Z^{s_j}\right]}(z) \\
&\quad = \frac{z^{\sum_{i=1}^{h} \frac{\kappa_i n_i}{\alpha_i \beta_i}}}{\Gamma\big(1 + \sum_{\ell=1}^{h} (\kappa_\ell n_\ell 1_A + r_\ell)\beta_\ell\big)} \sum_{j=1}^{\infty} \frac{\left(\frac{r_1}{\alpha_1 \beta_1} + \cdots + \frac{r_h}{\alpha_h \beta_h}\right)^j}{j!} \log^j z.
\end{aligned}$$

On the other hand, we have

$$\widehat{Z^{s_j}}(z) = z^{\kappa_j n_j} \sum_{\ell} \frac{r_j^{\ell}}{\ell!} \log^{\ell} z,$$

so that

$$\begin{aligned}
&\mathscr{B}_{\boldsymbol{\alpha},\boldsymbol{\beta}}[\widehat{Z^{s_1}}, \dots, \widehat{Z^{s_h}}](z) \\
&\quad = \frac{1}{2\pi i} \int_{\gamma} \prod_{j=1}^{h} \widehat{Z^{s_j}}\left(z^{\frac{1}{\alpha_j \beta_j}} \lambda^{-\beta_j}\right) e^{\lambda} \frac{d\lambda}{\lambda} \\
&\quad = \frac{1}{2\pi i} \int_{\gamma} e^{\lambda} \frac{d\lambda}{\lambda} \prod_{j=1}^{h} \left(z^{\frac{1}{\alpha_j \beta_j}} \lambda^{-\beta_j}\right)^{\kappa_j n_j} \sum_{\ell} \frac{r_j^{\ell}}{\ell!} \log^{\ell}\left(z^{\frac{1}{\alpha_j \beta_j}} \lambda^{-\beta_j}\right)
\end{aligned}$$

$$= \frac{z^{\sum_{i=1}^{h} \frac{\kappa_i n_i}{\alpha_i \beta_i}}}{2\pi i} \int_\gamma e^\lambda \frac{d\lambda}{\lambda^{1+\sum_{\ell=1}^{h} \kappa_\ell n_\ell \beta_\ell}} \prod_{j=1}^{h} \sum_{\ell} \frac{r_j^\ell}{\ell!} \log^\ell \left(z^{\frac{1}{\alpha_j \beta_j}} \lambda^{-\beta_j} \right)$$

$$= \frac{z^{\sum_{i=1}^{h} \frac{\kappa_i n_i}{\alpha_i \beta_i}}}{2\pi i} \int_\gamma e^\lambda \frac{d\lambda}{\lambda^{1+\sum_{\ell=1}^{h} \kappa_\ell n_\ell \beta_\ell}} \sum_{\ell_1,\ldots,\ell_h} \prod_{j=1}^{h} \frac{r_j^{\ell_j}}{\ell_j!} \log^{\ell_j} \left(z^{\frac{1}{\alpha_j \beta_j}} \lambda^{-\beta_j} \right).$$

We have

$$\prod_{j=1}^{h} \frac{r_j^{\ell_j}}{\ell_j!} \log^{\ell_j} (z^{\frac{1}{\alpha_j \beta_j}} \lambda^{-\beta_j})$$

$$= \prod_{j=1}^{h} \sum_{w,u=0}^{\infty} \frac{r_j^{\ell_j}}{w!u!} \left(\frac{\log z}{\alpha_j \beta_j} \right)^w (-\beta_j \log \lambda)^u \delta_{w+u,\ell_j}$$

$$= \sum_{\substack{w_1,\ldots,w_h \\ u_1,\ldots,u_h}} \prod_{j=1}^{h} \frac{r_j^{\ell_j}}{w_j! u_j!} \left(\frac{\log z}{\alpha_j \beta_j} \right)^{w_j} (-\beta_j \log \lambda)^{u_j} \delta_{w_j+u_j,\ell_j},$$

and

$$\frac{1}{2\pi i} \int_\gamma e^\lambda \frac{d\lambda}{\lambda^{1+\sum_{\ell=1}^{h} \kappa_\ell n_\ell \beta_\ell}} (-\log \lambda)^{u_j} = \left(\frac{1}{\Gamma} \right)^{(u_j)} \left(1 + \sum_{\ell=1}^{h} \kappa_\ell n_\ell \beta_\ell \right),$$

because of the Hankel formula (see e.g. [64])

$$\frac{1}{\Gamma(z)} = \frac{1}{2\pi i} \int_\gamma e^\lambda \frac{d\lambda}{\lambda^z}.$$

Thus, we have

$$\mathscr{B}_{\boldsymbol{\alpha},\boldsymbol{\beta}}[\widehat{Z^{s_1}}, \ldots, \widehat{Z^{s_h}}](z) = z^{\sum_{i=1}^{h} \frac{\kappa_i n_i}{\alpha_i \beta_i}} \sum_{\substack{\ell_1,\ldots,\ell_h \\ w_1,\ldots,w_h \\ u_1,\ldots,u_h}} \prod_{j=1}^{h} \frac{r_j^{\ell_j} \beta_j^{u_j}}{w_j! u_j!} \left(\frac{\log z}{\alpha_j \beta_j} \right)^{w_j}$$

$$\cdot \left(\frac{1}{\Gamma} \right)^{(u_j)} \left(1 + \sum_{\ell=1}^{h} \kappa_\ell n_\ell \beta_\ell \right) \delta_{w_j+u_j,\ell_j}.$$

This coincides with the formula of $widehat\mathscr{B}_{\boldsymbol{\alpha},\boldsymbol{\beta}}[\bigotimes_{j=1}^{h} Z^{s_j}](z)$. The proof for the Laplace multitransform is similar, based on the identity

$$\Gamma(z) = \int_0^\infty \lambda^{z-1} e^{-\lambda} \, d\lambda. \qquad \blacksquare$$