Chapter 7

Integral representations of solutions of qDEs

7.1 J_X-function as element of $\mathscr{F}_{\boldsymbol{\kappa}}(X)$

Let X be a variety with nef anticanonical bundle.[1] Introduce the basis $(\beta_1, \ldots, \beta_r)$ of $H_2(X, \mathbb{Z})$ Poincaré dual to $(T^1, \ldots, T^n)$, so that

$$\int_{\beta_i} T_j = \int_X T^i \cup T_j = \delta_{i,j}.$$

Set

$$c_1(X) = \sum_{j=1}^{\mathrm{r}} c^{\alpha_{i_j}} T_{\alpha_{i_j}}, \quad c^{\alpha_{i_j}} \in \mathbb{N}^*.$$

Consider the $\mathbb{C}$-algebra $H^\bullet(X, \mathbb{C})$. For brevity, we set

$$\mathscr{F}_{\boldsymbol{\kappa}}(X) := \mathscr{F}_{\boldsymbol{\kappa}}(H^\bullet(X, \mathbb{C}))$$

for any $\boldsymbol{\kappa} \in (\mathbb{C}^*)^h$.

The J_X-function restricted to the small quantum locus of $QH^\bullet(X)$ admits the following expansion:

$$\begin{aligned} &J_X(\delta + \log z \cdot c_1(X))|_{\substack{\mathbf{Q}=1\\ \hbar=1}} \\ &\quad = e^\delta z^{c_1(X)} + \sum_\alpha \sum_{\beta \neq 0} \sum_{k=0}^\infty e^\delta z^{\int_\beta c_1(X)} z^{c_1(X)} \langle \tau_k T_\alpha, 1 \rangle_{0,2,\beta}^X T^\alpha. \end{aligned}$$

Such a series can be seen as an element of $\mathscr{F}_{\boldsymbol{\kappa}}(X)$ for different choices of $\boldsymbol{\kappa}$. We describe two possible choices. In both cases, we have a series in $\mathscr{F}_{\boldsymbol{\kappa}}(X)$ concentrated at $c_1(X)$.

Choice 1. Set $h = 1$ and $\kappa = c$, where c is a common divisor of the numbers

$$c^{\alpha_{i_1}}, \ldots, c^{\alpha_{i_\mathrm{r}}}.$$

The series can be rearranged as follows:

$$J_X(\delta + \log z \cdot c_1(X))|_{\substack{\mathbf{Q}=1\\ \hbar=1}} = \sum_{d \in \mathbb{N}} J_d(\delta) z^{dc + c_1(X)},$$

[1] We recall that this means $\int_C c_1(X) \geqslant 0$ for all curves C in X. If the strict inequality holds true for any C, then X is Fano by the Nakai–Moishezon theorem. Varieties with nef anticanonical bundle can be thought as an interpolation between Fano and Calabi–Yau varieties.

where

$$J_d(\delta) = e^{\delta} \sum_{\alpha,k} \langle \tau_k T_\alpha, 1 \rangle_{0,2,d\cdot \mathrm{PD}(T)} T^{\alpha}, \quad d \in \mathbb{N},\ T \in H^2(X,\mathbb{Z}),\ c_1(X) = cT.$$

In particular, $J_0(\delta) = e^{\delta}$.

Choice 2. Set $h = \mathrm{r}$ and $\boldsymbol{\kappa} = (c^{\alpha_{i_1}}, \dots, c^{\alpha_{i_\mathrm{r}}})$. By expanding the sum over β over the basis $(\beta_1, \dots, \beta_r)$, the sum above becomes

$$J_X(\delta + \log z \cdot c_1(X))|_{\substack{\mathbf{Q}=1\\ \hbar=1}} = \sum_{d \in \mathbb{N}^\mathrm{r}} J_d(\delta) z^{d_1 c^{\alpha_{i_1}} + \dots + d_\mathrm{r} c^{\alpha_{i_\mathrm{r}}} + c_1(X)},$$

where

$$J_d(\delta) = e^{\delta} \sum_{\alpha,k} \langle \tau_k T_\alpha, 1 \rangle_{0,2,d_1 \beta_{\alpha_{i_1}} + \dots + d_\mathrm{r} \beta_{\alpha_{i_\mathrm{r}}}} T^{\alpha}, \quad d \in \mathbb{N}^\mathrm{r}.$$

In particular, $J_0(\delta) = e^{\delta}$.

7.2 Integral representations of the first kind

Let X be a Fano smooth projective variety. Assume that $\det T_X = L^{\otimes \ell}$ with L ample line bundle. Let $\iota\colon Y \subseteq X$ be a smooth subvariety defined as the zero locus of a regular section of the vector bundle $E = \bigoplus_{j=1}^{s} L^{\otimes d_j}$, where the numbers $d_j \in \mathbb{N}^*$ are such that $\sum_{j=1}^{s} d_j < \ell$.

Theorem 7.2.1. *Let $\delta \in H^2(X,\mathbb{C})$, and let $\mathcal{S}_\delta(X)$ be the corresponding space of master functions of $QH^\bullet(X)$. There exists a complex number $c_\delta \in \mathbb{C}$ such that the space of master functions $\mathcal{S}_{\iota^*\delta}(Y)$ is contained in image of the $\mathbb{C}$-linear map*

$$\mathscr{S}_{(\ell,\boldsymbol{d})}\colon \mathcal{S}_\delta(X) \to \mathcal{O}(\widetilde{\mathbb{C}^*})$$

defined by

$$\mathscr{S}_{(\ell,\boldsymbol{d})}[\Phi](z) := e^{-c_\delta z} \mathscr{L}_{\frac{\ell - \sum_{i=1}^{s} d_i}{d_s}, \frac{d_s}{\ell - \sum_{i=1}^{s-1} d_i}} \circ \dots \circ \mathscr{L}_{\frac{\ell - d_1 - d_2}{d_2}, \frac{d_2}{\ell - d_1}} \circ \mathscr{L}_{\frac{\ell - d_1}{d_1}, \frac{d_1}{\ell}} [\Phi](z).$$

In other words, any element of $\mathcal{S}_{\iota^*\delta}(Y)$ is of the form

$$e^{-c_\delta z} \int_0^\infty \dots \int_0^\infty \Phi\left(z^{\frac{\ell - \sum_{j=1}^{s} d_j}{\ell}} \prod_{i=1}^{s} \zeta_i^{\frac{d_i}{\ell}} \right) e^{-\sum_{i=1}^{s} \zeta_i}\, d\zeta_1 \dots d\zeta_s \tag{7.2.1}$$

for some $\Phi \in \mathcal{S}_\delta(X)$. Moreover, $c_\delta \neq 0$ only if $\sum_j d_j = \ell - 1$.

Proof. Set $\rho := c_1(L)$, and let $\rho^* \in H_2(X,\mathbb{Z})$ be its Poincaré dual homology class. In particular, we have $c_1(X) = \ell\rho$ and $c_1(E) = (\sum_{i=1}^{s} d_i)\rho$. By the adjunction formula, we have $c_1(Y) = \iota^*(c_1(X) - c_1(E))$. From Lemma A.2, we have

$$\begin{aligned} J_X(\delta + \log z \cdot c_1(X))|_{\substack{\mathbf{Q}=1\\ \hbar=1}} &= \sum_{d\in\mathbb{N}} J_{d\rho^*}(\delta) z^{d\ell + c_1(X)} \\ &= \sum_{d\in\mathbb{N}} J_{d\rho^*}(\delta) z^{d\ell+\ell\rho}, \end{aligned} \tag{7.2.2}$$

where $J_{d\rho^*}(\delta) = e^\delta \sum_{\alpha,k} \langle \tau_k T_\alpha, 1\rangle^X_{0,2,d\rho^*} T^\alpha$. Analogously, from (5.3.2) we have

$$\begin{aligned} &I_{X,Y}(\delta + (c_1(X) - c_1(E))\log z)|_{\substack{\mathbf{Q}=1\\ \hbar=1}} \\ &= \sum_{d\in\mathbb{N}} J_{d\rho^*}(\delta + (c_1(X) - c_1(E))\log z) \prod_{i=1}^{s} \prod_{m=1}^{\langle d_i\rho, d\rho^*\rangle} (d_i\rho + m) \\ &= \sum_{d\in\mathbb{N}} J_{d\rho^*}(\delta) z^{d(\ell-\sum d_i) + c_1(X) - c_1(E)} \prod_{i=1}^{s} \prod_{m=1}^{d\cdot d_i} (d_i\rho + m) \\ &= \sum_{d\in\mathbb{N}} J_{d\rho^*}(\delta) z^{d(\ell-\sum d_i) + (\ell - \sum d_i)\rho} \prod_{i=1}^{s} \frac{\Gamma(1 + d_i\rho + d d_i)}{\Gamma(1 + d_i\rho)}. \end{aligned} \tag{7.2.3}$$

On the one hand, from (7.2.2), one can see that $J_X(\delta + \log z \cdot c_1(X)))|_{\mathbf{Q}=1,\hbar=1}$ is the analytification $\hat{\mathrm{J}}_X$ of the series $\mathrm{J}_X \in \mathscr{F}_\ell(X)$, concentrated at $c_1(X) = \ell\rho$, defined by

$$\mathrm{J}_X(Z) = \sum_{d\in\mathbb{N}} J_{d\rho^*}(\delta) Z^{d\ell \oplus c_1(X)}.$$

On the other hand, one recognizes in equation (7.2.3) the analytification of the iteration of Laplace transforms

$$\begin{aligned} \mathrm{I}_{X,Y} := \prod_{i=1}^{s} \frac{1}{\Gamma(1 + d_i\rho)} \cdot \Big(&\mathscr{L}_{\frac{\ell - \sum_{i=1}^s d_i}{d_s}, \frac{d_s}{\ell - \sum_{i=1}^{s-1} d_i}} \\ &\circ \cdots \circ \mathscr{L}_{\frac{\ell - d_1 - d_2}{d_2}, \frac{d_2}{\ell - d_1}} \circ \mathscr{L}_{\frac{\ell - d_1}{d_1}, \frac{d_1}{\ell}} [\mathrm{J}_X] \Big), \end{aligned} \tag{7.2.4}$$

which is an element of $\mathscr{F}_{\frac{\ell - \sum_{i=1}^s d_i}{\ell}}(X)$. By Theorems 5.3.1, 5.3.4, 6.5.1, and Proposition 5.3.5, we have

$$J_Y(\iota^*\delta + c_1(Y)\log z)|_{\substack{\mathbf{Q}=1\\ \hbar=1}} = \iota^* \hat{\mathrm{I}}_{X,Y}(\delta + (c_1(X) - c_1(E))) \exp(-zH(\delta)|_{\mathbf{Q}=1}),$$

where $H(\delta)$ is defined in Proposition 5.3.5. Thus, the components of the right-hand side, with respect to any basis of $H^\bullet(Y, \mathbb{C})$, span the space of master functions $\mathcal{S}_{i^*\delta}(Y)$, by Corollary 5.1.3. The factor $\iota^* \prod_{i=1}^s \Gamma(1 + d_i\rho)^{-1}$ coming from (7.2.4) can be eliminated by a change of basis of $H^\bullet(Y, \mathbb{C})$. By $H^\bullet(X, \mathbb{C})$-linearity of the Laplace (α, β)-transforms, the claim follows by setting $c_\delta := H(\delta)|_{\mathbf{Q}=1}$. ■

Remark 7.2.2. Integral (7.2.1) is convergent for any $z \in \widetilde{\mathbb{C}^*}$. This follows from the exponential asymptotics of Theorem 4.3.2 for $z \to \infty$, the Fano assumption on Y (i.e. $\sum_{j=1}^{s} d_j < \ell$), and the asymptotics $|\Phi(z)| < C|\log z|^{\dim_{\mathbb{C}} X}$ for $z \to 0^+$ (see Theorem 5.1.2 and Corollary 5.1.3).

Remark 7.2.3. Formula (7.2.4) generalizes [37, Lemma 8.1].

7.3 Integral representations of the second kind

Let $X_1, \dots, X_h$ be Fano smooth projective varieties. Assume that $\det T_{X_j} = L_j^{\otimes \ell_j}$ for ample line bundles L_j. Let Y be a smooth subvariety of $X := \prod_{j=1}^{h} X_j$ defined as the zero locus of a regular section of the line bundle

$$E = \boxtimes_{j=1}^{h} L_j^{\otimes d_j},$$

where the numbers $d_j \in \mathbb{N}^*$ are such that $d_j < \ell_j$ for any $j = 1, \dots, h$.

By Künneth isomorphism, any element of $H^2(X, \mathbb{C})$ is of the form

$$\boldsymbol{\delta} = \sum_{i=1}^{h} 1 \otimes \cdots \otimes \delta_i \otimes \cdots \otimes 1 \quad \text{with } \delta_i \in H^2(X_i, \mathbb{C}).$$

Denote by $\iota\colon Y \to X$ the inclusion.

Theorem 7.3.1. *Let $\boldsymbol{\delta} \in H^2(X, \mathbb{C})$, $\delta_i \in H^2(X_i, \mathbb{C})$ be as above, and let $\mathcal{S}_{\delta_i}(X_i)$ be the corresponding space of master functions of $QH^\bullet(X_i)$. There exists a rational number $c_{\boldsymbol{\delta}} \in \mathbb{Q}$ such that the space of master functions $\mathcal{S}_{\iota^*\delta}(Y)$ is contained in image of the $\mathbb{C}$-linear map $\mathscr{P}_{(\boldsymbol{\ell},\boldsymbol{d})}\colon \bigotimes_{j=1}^{h} \mathcal{S}_{\delta_j}(X_j) \to \mathcal{O}(\widetilde{\mathbb{C}^*})$ defined by*

$$\mathscr{P}_{(\ell,\boldsymbol{d})}[\Phi_1, \dots, \Phi_h](z) := e^{-c_\delta z} \mathscr{L}_{\boldsymbol{\alpha},\boldsymbol{\beta}}[\Phi_1, \dots, \Phi_h](z),$$

where

$$(\boldsymbol{\alpha}, \boldsymbol{\beta}) = \left(\frac{\ell_1 - d_1}{d_1}, \dots, \frac{\ell_h - d_h}{d_h}; \frac{d_1}{\ell_1}, \dots, \frac{d_h}{\ell_h}\right).$$

In other words, any element of $\mathcal{S}_{\iota^*\delta}(Y)$ is of the form

$$e^{-c_\delta z} \int_0^{\infty} \prod_{j=1}^{h} \Phi_j\left(z^{\frac{\ell_j - d_j}{\ell_j}} \lambda^{\frac{d_j}{\ell_j}}\right) e^{-\lambda}\, d\lambda \tag{7.3.1}$$

for some $\Phi_j \in \mathcal{S}_{\delta_j}(X)$ with $j = 1, \dots, h$. Moreover, $c_\delta \neq 0$ only if $d_j = \ell_j - 1$ for some j.

Proof. Set $\rho_i := c_1(L_i)$ and let $\rho_i^* \in H_2(X_i, \mathbb{Z})$ be its Poincaré dual homology class, for any $i = 1, \dots, h$. By the Künneth isomorphism, and by the universal property of coproduct of algebras (i.e. tensor product), we have injective[2] maps

$$H^\bullet(X_i,\mathbb{C}) \to H^\bullet(X,\mathbb{C}).$$

In order to ease the computations, in the next formulas we will not distinguish an element of $H^\bullet(X_i,\mathbb{C})$ with its image in $H^\bullet(X,\mathbb{C})$. So, for example we will write

$$c_1(E) = \sum_{p=1}^{h} d_p\rho_p.$$

The same will be applied for elements in $H_2(X,\mathbb{Z})$.

We have

$$\begin{aligned} J_X(\boldsymbol{\delta} + c_1(X)\log z)|_{\substack{\mathbf{Q}=1\\ \hbar=1}} &= \bigotimes_{i=1}^{h} J_{X_i}(\delta_i + c_1(X_i)\log z)|_{\substack{\mathbf{Q}=1\\ \hbar=1}} \\ &= \bigotimes_{i=1}^{h} \sum_{k_i\in\mathbb{N}} J_{i,k_i\rho_i^*}(\delta_i) z^{k_i\ell_i+\ell_i\rho_i}, \end{aligned} \tag{7.3.2}$$

where

$$J_{i,k_i\rho_i^*}(\delta_i) = e^{\delta_i}\sum_{\alpha,j}\langle\tau_j T_{\alpha,i},1\rangle^{X_i}_{0,2,k_i\rho_i^*} T_i^\alpha.$$

Analogously, from (5.3.2), we deduce the formula

$$\begin{aligned} &I_{X,Y}(\boldsymbol{\delta} + (c_1(X)-c_1(E))\log z)|_{\substack{\mathbf{Q}=1\\ \hbar=1}} \\ &= \sum_{k_1,\dots,k_h\in\mathbb{N}} \bigotimes_{i=1}^{h} J_{i,k_i\rho_i^*}(\delta_i) z^{k_i(\ell_i-d_i)+(\ell_i-d_i)\rho_i} \cdot \prod_{m=1}^{\langle\sum_p d_p\rho_p,\sum_p k_p\rho_p^*\rangle}\left(\sum_p d_p\rho_p + m\right) \\ &= \sum_{k_1,\dots,k_h\in\mathbb{N}} \bigotimes_{i=1}^{h} J_{i,k_i\rho_i^*}(\delta_i) z^{k_i(\ell_i-d_i)+(\ell_i-d_i)\rho_i} \cdot \prod_{m=1}^{\sum_p d_pk_p}\left(\sum_p d_p\rho_p + m\right) \\ &= \sum_{k_1,\dots,k_h\in\mathbb{N}} \bigotimes_{i=1}^{h} J_{i,k_i\rho_i^*}(\delta_i) z^{k_i(\ell_i-d_i)+(\ell_i-d_i)\rho_i} \cdot \frac{\Gamma(1+\sum_p d_pk_p + \sum_p d_p\rho_p)}{\Gamma(1+\sum_p d_p\rho_p)}. \end{aligned} \tag{7.3.3}$$

[2]In particular, we have inclusions $\mathscr{F}_{\boldsymbol{k}}(X_j) \to \mathscr{F}_{\boldsymbol{k}}(X)$.

Each element in the tensor product (7.3.2) can easily be recognized as the analytification $\widehat{\mathrm{J}}_{X_i}$ of a series $\mathrm{J}_{X_i} \in \mathscr{F}_{\ell_i}(X)$, for each $i = 1, \dots, h$. The function in equation (7.3.3) can be identified with the analytification of the Laplace $(\boldsymbol{\alpha}, \boldsymbol{\beta})$-multitransform

$$\mathrm{I}_{X,Y} = \left(\bigotimes_{i=1}^{h} \frac{1}{\Gamma(1 + \sum_p d_p \rho_p)} \right) \cup_X \mathscr{L}_{\alpha,\beta} \left[\bigotimes_{i=1}^{h} \mathrm{J}_{X_i} \right], \tag{7.3.4}$$

where

$$(\boldsymbol{\alpha}, \boldsymbol{\beta}) = \left(\frac{\ell_1 - d_1}{d_1}, \dots, \frac{\ell_h - d_h}{d_h}; \frac{d_1}{\ell_1}, \dots, \frac{d_h}{\ell_h} \right).$$

The series $\mathrm{I}_{X,Y}$ can be seen as an element of $\mathscr{F}_{\boldsymbol{\kappa}}(X)$, with $\boldsymbol{\kappa} = (\ell_i - d_i)_{i=1}^h$, via the Künneth isomorphism. By Theorems 5.3.1, 5.3.4, 6.5.1, and Proposition 5.3.5, we have

$$J_Y(\iota^* \boldsymbol{\delta} + c_1(Y) \log z)|_{\substack{\mathbf{Q}=1 \\ \hbar=1}} = \iota^* \widehat{\mathrm{I}}_{X,Y}(\boldsymbol{\delta} + (c_1(X) - c_1(E))) \exp(-z H(\boldsymbol{\delta})|_{\mathbf{Q}=1}).$$

Thus, the components of the right-hand side, with respect to any basis of $H^\bullet(Y, \mathbb{C})$, span the space of master functions $\mathcal{S}_{\iota^*\delta}(Y)$, by Corollary 5.1.3. Notice that the factor $\iota^* \bigotimes_{i=1}^{s} \Gamma(1 + \sum_p d_p \rho_p)^{-1}$ coming from (7.3.4) can be eliminated by a change of basis of $H^\bullet(Y, \mathbb{C})$. By $H^\bullet(X, \mathbb{C})$-linearity of the Laplace $(\boldsymbol{\alpha}, \boldsymbol{\beta})$-multitransform, the claim follows by setting $c_{\boldsymbol{\delta}} := H(\boldsymbol{\delta})|_{\mathbf{Q}=1}$. ∎

Remark 7.3.2. Integral (7.3.1) is convergent for any $z \in \widetilde{\mathbb{C}^*}$. This follows from the exponential asymptotics of Theorem 4.3.2 for $z \to \infty$, the assumption $d_j < \ell_j$ for any $j = 1, \dots, h$, and the asymptotics $|\Phi_j(z)| < C |\log z|^{\dim_{\mathbb{C}} X_j}$ for $z \to 0^+$ (see Theorem 5.1.2 and Corollary 5.1.3).

Remark 7.3.3. Formula (7.3.4) generalizes [37, Lemma 8.1].

7.4 Master functions as Mellin–Barnes integrals

When applied to the case of Fano complete intersections in products of projective spaces, Theorems 7.2.1 and 7.3.1 give explicit Mellin–Barnes integral representations of solutions of the qDE.

Theorem 7.4.1. *Let Y be a Fano complete intersection in $\mathbb{P}^{n-1}$ defined by h homogeneous polynomials of degrees $d_1, \dots, d_h$. There exists a unique $c \in \mathbb{Q}$ such that any master functions in $\mathcal{S}_0(Y)$ is a linear combination of the Mellin–Barnes integrals*

$$G_j(z) = \frac{e^{-cz}}{2\pi\sqrt{-1}} \int_\gamma \Gamma(s)^n \prod_{k=1}^{h} \Gamma(1 - d_k s) z^{-(n - \sum_{k=1}^{h} d_k)s} \varphi_j(s)\, ds$$

for $j = 0, \dots, n-1$. *The path of integration* γ *is a parabola of the form*

$$\operatorname{Re} s = -\rho_1 (\operatorname{Im} s)^2 + \rho_2,$$

for suitable $\rho_1, \rho_2 \in \mathbb{R}_+$, *such that* γ *encircles the poles of* $\Gamma(s)^n$, *and separates them from the poles of the factors* $\Gamma(1 - d_k s)$. *The functions* φ_j *are given by*

- *for* n *even:*

$$\varphi_j(s) := \exp\bigl(2\pi\sqrt{-1}\, js\bigr), \quad j = 0, \dots, n-1,$$

- *for* n *odd:*

$$\varphi_j(s) := \exp\bigl(2\pi\sqrt{-1}\, js + \pi\sqrt{-1}s\bigr), \quad j = 0, \dots, n-1.$$

Moreover, $c \neq 0$ *only if* $\sum_k d_k = n - 1$.

Proof. The functions

$$g_j(z) := \frac{1}{2\pi\sqrt{-1}} \int_\gamma \Gamma(s)^n z^{-ns} \varphi_j(s)\, ds, \quad j = 0, \dots, n-1,$$

are a basis of the space of master functions $\mathcal{S}_0(\mathbb{P}^{n-1})$, see [46, Lemma 5]. The result follows by applying Theorem 7.2.1 to the case $X = \mathbb{P}^{n-1}$, $\ell = n$. ■

Theorem 7.4.2. *Let* Y *be a Fano hypersurface of* $\mathbb{P}^{n_1-1} \times \dots \times \mathbb{P}^{n_h-1}$ *defined by a homogeneous polynomial of multi-degree* $(d_1, \dots, d_h)$. *There exists a unique* $c \in \mathbb{Q}$ *such that any master function in* $\mathcal{S}_0(Y)$ *is a linear combination of the multi-dimensional Mellin–Barnes integrals*

$$H_{\boldsymbol{j}}(z) := \frac{e^{-cz}}{(2\pi\sqrt{-1})^h} \int_{\times\gamma_i} \left[\prod_{i=1}^h \Gamma(s_i)^{n_i} \varphi^i_{j_i}(s_i)\right] \cdot \Gamma\left(1 - \sum_{i=1}^h d_i s_i\right) z^{-\sum_{i=1}^h (n_i - d_i) s_i}\, ds_1 \dots ds_h$$

for $\boldsymbol{j} = (j_1, \dots, j_h) \in \prod_{i=1}^h \{0, \dots, n_i - 1\}$. *The paths* γ_i *are parabolas of the form*

$$\operatorname{Re} s_i = -\rho_{1,i} (\operatorname{Im} s_i)^2 + \rho_{2,i},$$

for suitable $\rho_{1,i}, \rho_{2,i} \in \mathbb{R}_+$, *so that they encircle the poles of the factors* $\Gamma(s_i)^{n_i}$. *The function* $\varphi^i_{j_i}$ *is defined as follows:*

- *for* n_i *even:*

$$\varphi^i_{j_i}(s_i) := \exp\bigl(2\pi\sqrt{-1}\, j_i s_i\bigr), \quad j_i = 0, \dots, n_i - 1,$$

- *for* n_i *odd:*

$$\varphi^i_{j_i}(s_i) := \exp\bigl(2\pi\sqrt{-1}\, j_i s_i + \pi\sqrt{-1} s_i\bigr), \quad j_i = 0, \dots, n_i - 1.$$

Moreover, $c \neq 0$ *only if* $d_i = n_i - 1$ *for some* $i = 1, \dots, h$.

Proof. The result follows by application of Theorem 7.3.1 to the case $X_i = \mathbb{P}^{n_i-1}$, $\ell_i = n_i$. For each factor $\mathbb{P}^{n_i-1}$ a basis of the space $\mathcal{S}_0(\mathbb{P}^{n_i-1})$ is given by the integrals

$$g^i_{j_i}(z) := \frac{1}{2\pi\sqrt{-1}}\int_{\gamma_i}\Gamma(s)^{n_i}z^{-n_i s}\varphi^i_{j_i}(s)\,ds, \quad j_i = 0,\dots,n_i-1. \qquad \blacksquare$$

Example. Consider the complex Grassmannian $\mathbb{G} := \mathbb{G}(2,4)$: it can be realized as a quadric in $\mathbb{P}^5$, by Plücker embedding. It can be shown that the space $\mathcal{S}_0(\mathbb{G})$ is the space of solutions Φ of the qDE given by

$$\vartheta^5\Phi - 1024z^4\vartheta\Phi - 2048z^4\Phi = 0, \quad \vartheta := z\frac{d}{dz}. \tag{7.4.1}$$

By Theorem 7.4.1, any solution of (7.4.1) is a linear combination of the functions

$$G_j(z) = \frac{1}{2\pi\sqrt{-1}}\int_\gamma \Gamma(s)^6\Gamma(1-2s)z^{-4s}\exp\big(2\pi\sqrt{-1}js\big)\,ds, \quad j = 0,\dots,5.$$

Recalling the reflection and duplication formulas for Γ-function (see e.g. [64]),

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin(\pi z)}, \quad \Gamma(2z) = \pi^{-\frac{1}{2}}2^{2z-1}\Gamma(z)\Gamma\left(z+\frac{1}{2}\right),$$

it is easy to see that the function

$$G_0(z) = \frac{2\pi^{\frac{3}{2}}}{2\pi\sqrt{-1}}\int_\gamma \frac{\Gamma(s)^5}{\Gamma\big(s+\frac{1}{2}\big)}\frac{4^{-s}}{\sin(2\pi s)}z^{-4s}\,ds$$

is a solution of (7.4.1). In [23, Section 6] the solutions

$$\Phi_1(z) = \frac{1}{2\pi\sqrt{-1}}\int_\gamma \frac{\Gamma(s)^5}{\Gamma\big(s+\frac{1}{2}\big)}4^{-s}z^{-4s}\,ds$$

and

$$\Phi_2(z) = \frac{1}{2\pi\sqrt{-1}}\int_\gamma \Gamma(s)^5\Gamma\left(\frac{1}{2}-s\right)e^{i\pi s}4^{-s}z^{-4s}\,ds$$

of equation (7.4.1) were found and studied. It is not difficult to see that Φ_1 and Φ_2 are linear combinations of the functions G_j.

Remark 7.4.3. This example can be extended to Grassmannians $\mathbb{G}(k,n)$ and other families of partial flag varieties. In the case of Grassmannians it gives different integral representations of solutions with respect to those obtained from the quantum Satake identification [42, 55]. More in general, it would be interesting to do a comparison with the integral representations of solutions obtained from the Abelian–Nonabelian correspondence [14].