Chapter 8

Dubrovin conjecture

8.1 Exceptional collections and exceptional bases

Let X be a smooth complex projective variety, and denote by $\mathcal{D}^b(X)$ the bounded derived category of coherent sheaves on X, see [38, 52]. Given $E, F \in \mathrm{Ob}(\mathcal{D}^b(X))$, define $\mathrm{Hom}^\bullet(E, F)$ as the $\mathbb{C}$-vector space[1]

$$\mathrm{Hom}^\bullet(E, F) := \bigoplus_{k\in\mathbb{Z}} \mathrm{Hom}(E, F[k]).$$

An object $E \in \mathrm{Ob}(\mathcal{D}^b(X))$ is said to be *exceptional* if $\mathrm{Hom}^\bullet(E, E)$ is a one-dimensional $\mathbb{C}$-algebra, generated by the identity morphism.

A collection $\mathfrak{E} = (E_1, \dots, E_n)$ of objects of $\mathcal{D}^b(X)$ is said to be an *exceptional collection* if

(1) each object E_i is exceptional,

(2) we have $\mathrm{Hom}^\bullet(E_j, E_i) = 0$ for $j > i$.

Moreover, an exceptional collection $\mathfrak{E}$ is *full* if it generates $\mathcal{D}^b(X)$, i.e. any triangular subcategory containing all objects of $\mathfrak{E}$ is equivalent to $\mathcal{D}^b(X)$ via the inclusion functor.

Consider the Grothendieck group $K_0(X) \equiv K_0(\mathcal{D}^b(X))$, and let χ to be the Grothendieck–Euler–Poincaré bilinear form

$$\chi([V], [F]) := \sum_k (-1)^k \dim_{\mathbb{C}} \mathrm{Hom}(V, F[k]), \quad V, F \in \mathcal{D}^b(X).$$

Definition 8.1.1. A basis $(e_i)_{i=1}^n$ of $K_0(X)_{\mathbb{C}}$ is called *exceptional* if $\chi(e_i, e_i) = 1$ for $i = 1, \dots, n$, and $\chi(e_j, e_i) = 0$ for $1 \leqslant i < j \leqslant n$.

Lemma 8.1.2. *Let $(E_i)_{i=1}^n$ be a full exceptional collection in $\mathcal{D}^b(X)$. The K-classes $([E_i])_{i=1}^n$ form an exceptional basis of $K_0(X)_{\mathbb{C}}$.* ■

8.2 Mutations and helices

Let $\mathfrak{E} = (E_1, \dots, E_n)$ be an exceptional collection in $\mathcal{D}^b(X)$. For $i = 1, \dots, n-1$ define the collections

$$\mathbb{L}_i\mathfrak{E} := (E_1, \dots, E_{i-1}, E'_{i+1}, E_i, E_{i+2}, \dots, E_n),$$
$$\mathbb{R}_i\mathfrak{E} := (E_1, \dots, E_{i-1}, E_{i+1}, E''_i, E_{i+2}, \dots, E_n),$$

[1] Notice that the category $\mathcal{D}^b(X)$ is a $\mathbb{C}$-linear category.

where the objects E'_{i+1}, E''_i sit in the distinguished triangles

$$E'_{i+1}[-1] \to \mathrm{Hom}^\bullet(E_i, E_{i+1}) \otimes E_i \to E_{i+1} \to E'_{i+1},$$
$$E''_i \to E_i \to \mathrm{Hom}^\bullet(E_i, E_{i+1})^* \otimes E_{i+1} \to E''_i[1].$$

Remark 8.2.1. The object E'_{i+1} (resp. E''_i) is uniquely defined *up to unique isomorphism*, because of the exceptionality of E_i (resp. E_{i+1}), see [21, Section 3.3].

Proposition 8.2.2 ([12, 44]). *For any i, with $0 < i < n$, the collections $\mathbb{L}_i\mathfrak{E}, \mathbb{R}_i\mathfrak{E}$ are exceptional. The mutation operators $\mathbb{L}_i, \mathbb{R}_i$ satisfy the following identities:*

$$\mathbb{L}_i\mathbb{R}_i = \mathbb{R}_i\mathbb{L}_i = \mathrm{Id},$$
$$\mathbb{R}_i\mathbb{R}_j = \mathbb{R}_j\mathbb{R}_i \quad \text{if } |i-j| > 1, \qquad \mathbb{R}_{i+1}\mathbb{R}_i\mathbb{R}_{i+1} = \mathbb{R}_i\mathbb{R}_{i+1}\mathbb{R}_i.$$

Moreover, if $\mathfrak{E}$ is full, then also $\mathbb{L}_i\mathfrak{E}$ and $\mathbb{R}_i\mathfrak{E}$ are full. ∎

Denote by $\beta_1, \dots, \beta_{n-1}$ the generators of the braid group $\mathcal{B}_n$, satisfying the relations

$$\beta_i\beta_{i+1}\beta_i = \beta_{i+1}\beta_i\beta_{i+1}, \qquad \beta_i\beta_j = \beta_j\beta_i \quad \text{if } |i-j| > 1.$$

We define the left action of $\mathcal{B}_n$ on the set of exceptional collections of length n by identifying the action of β_i with $\mathbb{L}_i$.

Definition 8.2.3. Let $\mathfrak{E} = (E_1, \dots, E_n)$ be a full exceptional collection. We define the *helix* generated by $\mathfrak{E}$ to be the infinite family $(E_i)_{i\in\mathbb{Z}}$ of exceptional objects such that

$$(E_{1-kn}, E_{2-kn}, \dots, E_{n-kn}) = \mathfrak{E}^\beta, \quad \beta = (\beta_{n-1}\dots\beta_1)^{kn}, \quad k \in \mathbb{Z}.$$

Any family of n consecutive exceptional objects $(E_{i+k})_{k=1}^n$ is called a *foundation* of the helix.

Lemma 8.2.4 ([44]). *The following statements hold:*

(1) *Any foundation is a full exceptional collection.*

(2) *For $i, j \in \mathbb{Z}$, we have* $\mathrm{Hom}^\bullet(E_i, E_j) \cong \mathrm{Hom}^\bullet(E_{i-n}, E_{j-n})$. ∎

The action of the braid group on the set of exceptional collections in $\mathcal{D}^b(X)$ admits a K-theoretical analogue on the set of exceptional bases of $K_0(X)_\mathbb{C}$, see [21, 44].

8.3 Γ-classes and graded Chern character

Let V be a complex vector bundle on X of rank r, and $\delta_1, \dots, \delta_r$ its Chern roots, so that $c_j(V) = s_j(\delta_1, \dots, \delta_r)$, where s_j is the j-th elementary symmetric polynomial.

Definition 8.3.1. Let Q be an indeterminate, and $F \in \mathbb{C}[\![Q]\!]$ be of the form

$$F(Q) = 1 + \sum_{n \geqslant 1} \alpha_n Q^n.$$

The F-*class* of V is the characteristic class $\widehat{F}_V \in H^\bullet(X)$ defined by

$$\widehat{F}_V := \prod_{j=1}^{r} F(\delta_j).$$

Definition 8.3.2. The $\Gamma^\pm$-*classes* of V are the characteristic classes associated with the Taylor expansions

$$\Gamma(1 \pm Q) = \exp\left(\mp \gamma Q + \sum_{m=2}^{\infty} (\mp 1)^m \frac{\zeta(m)}{m} Q^m \right) \in \mathbb{C}[\![Q]\!],$$

where γ is the Euler–Mascheroni constant and ζ is the Riemann zeta function.

If $V = TX$, then we denote $\widehat{\Gamma}_X^\pm$ its Γ-classes.

Definition 8.3.3. The *graded Chern character* of the complex vector bundle V is the characteristic class $\mathrm{Ch}(V) \in H^\bullet(X)$ defined by $\mathrm{Ch}(V) := \sum_{j=1}^{r} \exp(2\pi\sqrt{-1}\delta_j)$.

8.4 Statement of the conjecture

Let X be a Fano variety. In [31] Dubrovin conjectured that many properties of the qDE of X, in particular its monodromy, Stokes and central connection matrices, are encoded in the geometry of exceptional collections in $\mathcal{D}^b(X)$. The following conjecture is a refinement of the original version in [31].

Conjecture 8.4.1 ([21]). *Let X be a smooth Fano variety of Hodge–Tate type.*

(1) *The quantum cohomology $QH^\bullet(X)$ has semisimple points if and only if there exists a full exceptional collection in $\mathcal{D}^b(X)$.*

(2) *If $QH^\bullet(X)$ is generically semisimple, then for any oriented ray ℓ of slope $\varphi \in [0, 2\pi[$ there is a map from the set of ℓ-chambers to the set of helices with a marked foundation.*

(3) *Let Ω_ℓ be an ℓ-chamber and $\mathfrak{E}_\ell = (E_1, \ldots, E_n)$ the corresponding exceptional collection (the marked foundation). Denote by S and C Stokes and central connection matrices computed in Ω_ℓ with respect to a basis $(T_\alpha)_{\alpha=1}^n$ of $H^\bullet(X, \mathbb{C})$.*

  (a) *The matrix S is the inverse of the Gram matrix of the χ-pairing in $K_0(X)_{\mathbb{C}}$ with respect to the exceptional basis $[\mathfrak{E}_\ell]$,*

$$(S^{-1})_{ij} = \chi(E_i, E_j).$$

(b) *The matrix C coincides with the matrix associated with the $\mathbb{C}$-linear morphism*

$$\mathit{\Pi}_X^-\colon K_0(X)_{\mathbb{C}} \to H^\bullet(X),$$
$$F \mapsto \frac{(\sqrt{-1})^{\overline{d}}}{(2\pi)^{\frac{d}{2}}}\widehat{\Gamma}_X^- \exp(-\pi\sqrt{-1}c_1(X))\mathrm{Ch}(F),$$

where $d := \dim_{\mathbb{C}} X$, and $\overline{d}$ is the residue class $d \pmod 2$. The matrix is computed with respect to the exceptional basis $[\mathfrak{E}_\ell]$ and the pre-fixed basis $(T_\alpha)_{\alpha=1}^n$.

Remark 8.4.2. If point (3.b) holds true, then automatically also point (3.a) holds true. This follows from identity (4.4.2) and the Hirzebruch–Riemann–Roch theorem, see [21, Corollary 5.8].

Remark 8.4.3. In [9], A. Bayer suggested dropping any reference to X being Fano in the formulation of the Dubrovin conjecture. He proved indeed that the semisimplicity of the quantum cohomology preserves under blow-ups at any number of points. It follows that point (1) of Conjecture 8.4.1 (the *qualitative* part) still holds true after blowing up X at an arbitrary number of points, which may yield a non-Fano variety. To the best of our knowledge, however, there is no non-Fano example for which both the Stokes and central connection matrices have been explicitly computed. In Chapters 10 and 11 we will provide the first example, in the case of Hirzebruch surfaces.

Remark 8.4.4. Assume the validity of points (3.a) and (3.b) of Conjecture 8.4.1. The action of the braid group $\mathcal{B}_n$ on the Stokes and central connection matrices (cf. Lemma 4.6.2) is compatible with the action of $\mathcal{B}_n$ on the marked foundations attached at each ℓ-chambers. Different choices of the branch of the Ψ-matrix correspond to shifts of objects of the marked foundation. The matrix M_0^{-1} is identified with the canonical operator $\kappa\colon K_0(X)_{\mathbb{C}} \to K_0(X)_{\mathbb{C}}$, $[F] \mapsto (-1)^d[F \otimes \omega_X]$. Equations (4.4.4) imply that the connection matrices $C^{(m)}$, with $m \in \mathbb{Z}$, correspond to the matrices of the morphism $\mathit{\Pi}_X^-$ with respect to the foundations $(\mathfrak{E}_\ell \otimes \omega_X^{\otimes m})[md]$. The statement $S^{(m)} = S$ coincides with the Hom-periodicity described in point (2) of Lemma 8.2.4, see [21, Theorem 5.9].

Remark 8.4.5. Conjecture 8.4.1 relates two different aspects of the geometry of X, namely its *symplectic structure* (Gromov–Witten theory) and its *complex structure* (the derived category $\mathcal{D}^b(X)$). Heuristically, Conjecture 8.4.1 follows from the homological mirror symmetry conjecture of M. Kontsevich, see [21, Section 5.5].

Remark 8.4.6. In the paper [54] it was underlined the role of Γ-classes for refining the original version of Dubrovin's conjecture [31]. Subsequently, in [34] and [36, Γ-conjecture II] two equivalent versions of point (3.b) above were given. However, in both these versions, different choices of solutions in Levelt form of the qDE at $z = 0$

are chosen with respect to the natural ones in the theory of Frobenius manifolds, see [21, Section 5.6].

Remark 8.4.7. Point (3.b) of Conjecture 8.4.1 allows to identify K-classes with solutions of the joint system of equations (2.7.1)–(2.7.2). Under this identification, Stokes fundamental solutions correspond to exceptional bases of K-theory. In the approach of [26,75], where the equivariant case is addressed, such an identification is more fundamental and *a priori*: it is defined via explicit integral representations of solutions of the joint system of qDE and qKZ equations.

Remark 8.4.8. Note that the existence of a map between ℓ-chambers and helices with a marked foundation, discussed in point (2) of Conjecture 8.4.1, is an important aspect of the Dubrovin conjecture. A careful study of such a correspondence may hide several delicate open problems. Consider, for instance, the study of injectivity and surjectivity of such a map. This study is closely related (possibly equivalent) to the study of the freeness and transitivity of the braid group action on the set of exceptional collections. These are well-known open problems, whose answer is known in a few special cases only, see [44]. In the remaining sections of this paper, we will address the study of point (3) of Conjecture 8.4.1, but not of point (2).