Chapter 9

Quantum cohomology of Hirzebruch surfaces

9.1 Preliminaries on Hirzebruch surfaces

Hirzebruch surfaces $\mathbb{F}_k$, with $k \in \mathbb{Z}$, are defined as the total space of $\mathbb{P}^1$-projective bundles on $\mathbb{P}^1$, namely

$$\mathbb{F}_k := \mathbb{P}(\mathcal{O} \oplus \mathcal{O}(-k)), \quad k \in \mathbb{Z},$$

where $\mathcal{O}(n)$ are line bundles on $\mathbb{P}^1$. More explicitly, they can be defined as hypersurfaces in $\mathbb{P}^2 \times \mathbb{P}^1$ by

$$\mathbb{F}_k := \{([a_0 : a_1 : a_2], [b_1 : b_2]) \in \mathbb{P}^2 \times \mathbb{P}^1 : a_1 b_1^k = a_2 b_2^k\}, \quad k \in \mathbb{N}. \tag{9.1.1}$$

Hirzebruch surfaces have the following properties:

- the surfaces $(\mathbb{F}_{2k})_{k \in \mathbb{N}}$ are all diffeomorphic,
- the surfaces $(\mathbb{F}_{2k+1})_{k \in \mathbb{N}}$ are all diffeomorphic,
- the surfaces $\mathbb{F}_n$ and $\mathbb{F}_m$ with $n \neq m$ are not biholomorphic,
- the only Fano Hirzebruch surfaces are $\mathbb{F}_0 \cong \mathbb{P}^1 \times \mathbb{P}^1$ and $\mathbb{F}_1 \cong \mathrm{Bl}_{\mathrm{pt}}\mathbb{P}^2$,
- the surfaces $\mathbb{F}_n$ and $\mathbb{F}_m$ are deformation equivalent if and only if n and m have the same parity.

See [10, 49].

Remark 9.1.1. Let $0 \leqslant m \leqslant \frac{1}{2}n$. Consider the family $\mathcal{F}$ defined by the equation

$$\mathcal{F} := \{([a_0 : a_1 : a_2], [b_1 : b_2], t) \in \mathbb{P}^2 \times \mathbb{P}^1 \times \mathbb{C} : a_1 b_1^n - a_2 b_2^n + t a_0 b_1^{n-m} b_2^m = 0\}.$$

The central fiber over $t = 0$ is $\mathbb{F}_n$. Any non-central fiber over $t \neq 0$ is isomorphic to $\mathbb{F}_{n-2m}$. See [56, Example 2.16]. See also [74] and [63, Example 0.1.10].

Remark 9.1.2. The only possible complex structures on $\mathbb{S}^2 \times \mathbb{S}^2$ are the even Hirzebruch surfaces $\mathbb{F}_{2k}$, with $k \in \mathbb{N}$, and the only possible complex structures on the connected sum $\mathbb{P}^2 \# \overline{\mathbb{P}^2}$ are the odd Hirzebruch surfaces $\mathbb{F}_{2k+1}$, with $k \in \mathbb{N}$, see [67].

9.2 Classical cohomology of Hirzebruch surfaces

Using the explicit polynomial description (9.1.1) of the Hirzebruch surfaces, let us define the following subvarieties of $\mathbb{F}_k$:

$$\Sigma_1^k := \{a_1 = a_2 = 0\}, \quad \Sigma_2^k := \{a_2 = b_1 = 0\},$$
$$\Sigma_3^k := \{a_1 = b_2 = 0\}, \quad \Sigma_4^k := \{a_0 = 0\}.$$

Each of these subvarieties naturally define a cycle in $H_2(\mathbb{F}_k, \mathbb{Z})$. Notice that, under the identification

$$\mathbb{F}_k \equiv \mathcal{O}(-k) \cup \infty\text{section},$$

we can

(1) identify Σ_1^k with the 0-section of $\mathcal{O}(-k)$,
(2) identify Σ_4^k with the ∞-section,
(3) identify both Σ_2^k and Σ_3^k with (the compactification of) two fibers of $\mathcal{O}(-k)$.

Using the original notations of Hirzebruch, we denote by

- $\tau_k \in H_2(\mathbb{F}_k, \mathbb{C})$ the homology class defined by Σ_1^k,
- $\varepsilon_k \in H_2(\mathbb{F}_k, \mathbb{C})$ the homology class defined by Σ_4^k,
- $\nu_k \in H_2(\mathbb{F}_k, \mathbb{C})$ the homology class defined by both Σ_2^k and Σ_3^k.

As it is easily seen, the three classes $\tau_k, \varepsilon_k, \nu_k$ are not $\mathbb{Z}$-linearly independent. They are indeed related by the equation

$$\varepsilon_k = \tau_k + k\nu_k. \tag{9.2.1}$$

Finally, let us also introduce a homogeneous basis $(T_{0,k}, T_{1,k}, T_{2,k}, T_{3,k})$ of the classical cohomology $H^\bullet(\mathbb{F}_k, \mathbb{Z})$, where

$$T_{0,k} := 1, \quad T_{1,k} := \mathrm{PD}(\varepsilon_k), \quad T_{2,k} := \mathrm{PD}(\nu_k), \quad T_{3,k} := \mathrm{PD}(\mathrm{pt}),$$

where $\mathrm{PD}(\alpha)$ denotes the Poincaré dual class of $\alpha \in H_\bullet(\mathbb{F}_k, \mathbb{Z})$. We denote the corresponding dual coordinates by $(t^{0,k}, t^{1,k}, t^{2,k}, t^{3,k})$.

By the Leray–Hirsch theorem, the classical cohomology algebra is generated by the classes $(T_{1,k}, T_{2,k})$. More precisely, we have the following result.

Theorem 9.2.1. *In the classical cohomology ring $H^\bullet(\mathbb{F}_k, \mathbb{Z})$, the following identities hold true:*

(1) $T_{1,k}^2 = k \cdot T_{3,k}$,
(2) $T_{2,k}^2 = 0$,
(3) $T_{1,k}T_{2,k} = T_{3,k}$.

Hence, the following presentation of algebras holds:

$$H^\bullet(\mathbb{F}_k, \mathbb{C}) \cong \frac{\mathbb{C}[T_{1,k}, T_{2,k}]}{\langle T_{2,k}^2, T_{1,k}^2 - k \cdot T_{1,k}T_{2,k} \rangle}.$$

The Poincaré metric in the basis $(T_{i,k})_{i=0}^3$ is given by

$$\eta_k = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & k & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}. \tag{9.2.2}$$

Proposition 9.2.2 ([59]). *Let $k \in \mathbb{N}$. The collection*

$$(\mathcal{O}, \mathcal{O}(\Sigma_2^k), \mathcal{O}(\Sigma_4^k), \mathcal{O}(\Sigma_2^k + \Sigma_4^k))$$

is a full exceptional collection in $\mathcal{D}^b(\mathbb{F}_k)$. The corresponding Gram matrix of the χ-pairing is

$$\begin{pmatrix} 1 & 2 & 2+k & 4+k \\ 0 & 1 & k & 2+k \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Proof. The Gram matrix can easily be computed by the Hirzebruch–Riemann–Roch theorem. ∎

9.3 Quantum cohomology of Hirzebruch surfaces

There exist only two classes of deformation equivalence of Hirzebruch surfaces, namely $(\mathbb{F}_{2k})_{k\in\mathbb{N}}$ and $(\mathbb{F}_{2k+1})_{k\in\mathbb{N}}$. Hence, by the deformation axiom of Gromov–Witten invariants [27], the quantum cohomology algebra of $\mathbb{F}_{2k}$ (resp. $\mathbb{F}_{2k+1}$) can be identified with the one of $\mathbb{F}_0$ (resp. $\mathbb{F}_1$), as explained in Remark 4.5.2. Notice that the quantum cohomology algebras of $\mathbb{F}_0$ and $\mathbb{F}_1$ coincide with the corresponding Batyrev rings [8]. This does not hold true for other Hirzebruch surfaces $\mathbb{F}_k$ with $k \neq 0, 1$, being not Fano [73]. See also [6] for a presentation of the quantum cohomology algebra of $\mathbb{F}_1$.

9.3.1 Case of $\mathbb{F}_{2k}$

The diffeomorphism $\varphi_{2k}: \mathbb{F}_{2k} \to \mathbb{F}_0$ induces isomorphisms in homology and cohomology. We have $(\varphi_{2k})_*(\tau_{2k}) = \tau_0$ and $(\varphi_{2k})_*(\nu_{2k}) = \nu_0$, so that from equations (9.2.1) and (9.2.2) we deduce

$$\varphi_{2k}^*(T_{0,0}) = T_{0,2k}, \tag{9.3.1}$$
$$\varphi_{2k}^*(T_{1,0}) = T_{1,2k} - kT_{2,2k}, \tag{9.3.2}$$
$$\varphi_{2k}^*(T_{2,0}) = T_{2,2k}, \tag{9.3.3}$$
$$\varphi_{2k}^*(T_{3,0}) = T_{3,2k}. \tag{9.3.4}$$

Thus, we can identify the quantum cohomologies $QH^\bullet(\mathbb{F}_0)$ and $QH^\bullet(\mathbb{F}_{2k})$ via the change of coordinates

$$\begin{aligned} t^{0,2k} &= t^{0,0}, & t^{1,2k} &= t^{1,0}, \\ t^{2,2k} &= t^{2,0} - kt^{1,0}, & t^{3,2k} &= t^{3,0}. \end{aligned} \tag{9.3.5}$$

Theorem 9.3.1. *For any $k \geqslant 0$, the following isomorphism of algebras holds true:*

$$QH^\bullet(\mathbb{F}_{2k}) \cong \frac{\mathbb{C}[T_{1,2k}, T_{2,2k}, q_1, q_2]}{\langle T_{2,2k}^{\circ 2} - q_1^k q_2, (T_{1,2k} - k \cdot T_{2,2k})^{\circ 2} - q_1 \rangle},$$

where $q_1 = \exp(t^{1,2k})$ and $q_2 = \exp(t^{2,2k})$.

Proof. The assertion follows from the presentation of the quantum cohomology algebra of $QH^\bullet(\mathbb{F}_0) \cong QH^\bullet(\mathbb{P}^1) \otimes QH^\bullet(\mathbb{P}^1)$, and formulas (9.3.1)–(9.3.5). ∎

Lemma 9.3.2. *For all $k \geqslant 0$ we have*

$$T_{1,2k} \circ T_{2,2k} = T_{3,2k} + k q_1^k q_2. \tag{9.3.6}$$

Proof. By homogeneity, let $\lambda_{0,2k}, \lambda_{1,2k}, \lambda_{2,2k}, \lambda_{3,2k}$ be the dual basis of $H_\bullet(\mathbb{F}_{2k}, \mathbb{C})$ of the basis $(T_{i,2k})_{i=0}^3$. By the deformation axiom of Gromov–Witten invariants, for any $r, s \in \mathbb{N}$, we have

$$\begin{aligned}
&\langle T_{1,2k}, T_{2,2k}, T_{3,2k} \rangle_{0,3,r\lambda_{1,2k}+s\lambda_{1,2k}}^{\mathbb{F}_{2k}} \\
&\quad = \langle T_{1,0} + kT_{2,0}, T_{2,0}, T_{3,0} \rangle_{0,3,r\lambda_{0,0}+s(\lambda_{1,0}-k\lambda_{0,0})}^{\mathbb{F}_0} \\
&\quad = \langle T_{1,0}, T_{2,0}, T_{3,0} \rangle_{0,3,(r-sk)\lambda_{0,0}+s\lambda_{1,0}}^{\mathbb{F}_0} + k\langle T_{2,0}, T_{2,0}, T_{3,0} \rangle_{0,3,(r-sk)\lambda_{0,0}+s\lambda_{1,0}}^{\mathbb{F}_0} \\
&\quad = \langle \sigma, 1, \sigma \rangle_{0,3,(r-sk)H}^{\mathbb{P}^1} \langle 1, \sigma, \sigma \rangle_{0,3,(r-sk)H}^{\mathbb{P}^1} \\
&\qquad\qquad + k\langle 1, 1, \sigma \rangle_{0,3,(r-sk)H}^{\mathbb{P}^1} \langle \sigma, \sigma, \sigma \rangle_{0,3,(r-sk)H}^{\mathbb{P}^1} \\
&\quad = k \cdot \delta_{1,2(r-ks)+1}\delta_{3,2s+1}.
\end{aligned}$$

Here we used the class $H \in H_2(\mathbb{P}^1, \mathbb{Z})$ to be the hyperplane class and $\sigma \in H^2(\mathbb{P}^2, \mathbb{Z})$ to be its dual. This gives the quantum correction in (9.3.6). ∎

9.3.2 Case of $\mathbb{F}_{2k+1}$

The diffeomorphism $\varphi_{2k+1}: \mathbb{F}_{2k+1} \to \mathbb{F}_1$ induces an isomorphism φ_{2k+1}^* in cohomology given by

$$\varphi_{2k+1}^*(T_{0,1}) = T_{0,2k+1}, \tag{9.3.7}$$
$$\varphi_{2k+1}^*(T_{1,1}) = T_{1,2k+1} - kT_{2,2k+1}, \tag{9.3.8}$$
$$\varphi_{2k+1}^*(T_{2,1}) = T_{2,2k+1}, \tag{9.3.9}$$
$$\varphi_{2k+1}^*(T_{3,1}) = T_{3,2k+1}. \tag{9.3.10}$$

We can identify the quantum cohomologies $QH^\bullet(\mathbb{F}_1)$ and $QH^\bullet(\mathbb{F}_{2k+1})$ via the change of coordinates

$$\begin{aligned} t^{0,2k+1} &= t^{0,1}, & t^{1,2k+1} &= t^{1,1}, \\ t^{2,2k+1} &= t^{2,1} - kt^{1,1}, & t^{3,2k+1} &= t^{3,1}. \end{aligned} \tag{9.3.11}$$

Theorem 9.3.3. *For any $k \geqslant 0$, the following isomorphism of algebras holds true:*

$$QH^\bullet(\mathbb{F}_{2k+1}) \cong \frac{\mathbb{C}[T_{1,2k+1}, T_{2,2k+1}, q_1, q_2]}{A_k},$$

where

$$\begin{aligned} A_k := \langle & T_{2,2k+1}^{\circ 2} - (T_{1,2k+1} - (k+1)T_{2,2k+1})q_1^k q_2, \\ & (T_{1,2k+1} - kT_{2,2k+1}) \circ (T_{1,2k+1} - (k+1)T_{2,2k+1}) - q_1 \rangle \end{aligned}$$

and $q_1 := \exp(t^{1,2k+1})$ *and* $q_2 := \exp(t^{2,2k+1})$.

Proof. The following presentation for $QH^\bullet(\mathbb{F}_1)$ holds true:

$$QH^\bullet(\mathbb{F}_1) \cong \frac{\mathbb{C}[T_{1,1}, T_{2,1}, q_1, q_2]}{\langle T_{2,1}^{\circ 2} - (T_{1,1} - T_{2,1})q_2,\ T_{1,1}^{\circ 2} - T_{1,1} \circ T_{2,1} - q_1 \rangle}.$$

The result follows by formulas (9.3.7)-(9.3.10) and (9.3.11). ∎