Chapter 10

Dubrovin conjecture for Hirzebruch surfaces $\mathbb{F}_{2k}$

10.1 $\mathcal{A}_\Lambda$-stratum and Maxwell stratum of $QH^\bullet(\mathbb{F}_{2k})$

Fix a point $p = t^{1,2k}T_{1,2k} + t^{2,2k}T_{2,2k}$ of the small quantum cohomology of $\mathbb{F}_{2k}$. The matrix form of the tensor $\mathcal{U}$ is given by

$$\mathcal{U}(p) = \begin{pmatrix} 0 & 2q_1 + 2kq_1^k q_2 & 2q_1^k q_2 & 0 \\ 2 & 0 & 0 & 2q_1^k q_2 \\ 2-2k & 0 & 0 & 2q_1 - 2kq_1^k q_2 \\ 0 & 2+2k & 2 & 0 \end{pmatrix}.$$

The canonical coordinates are given by

$$u_1(p) = -2\big(q_1^{\frac12} - q_1^{\frac k2} q_2^{\frac12}\big), \quad u_2(p) = 2\big(q_1^{\frac12} - q_1^{\frac k2} q_2^{\frac12}\big),$$
$$u_3(p) = -2\big(q_1^{\frac12} + q_1^{\frac k2} q_2^{\frac12}\big), \quad u_4(p) = 2\big(q_1^{\frac12} + q_1^{\frac k2} q_2^{\frac12}\big).$$

The Ψ-matrix at the point p is given by

$$\Psi(p) = \begin{pmatrix} -\frac{i q_1^{\frac12(-\frac k2-\frac12)}}{2\sqrt[4]{q_2}} & \frac{i q_1^{\frac12(-\frac k2-\frac12)}(\sqrt{q_1}-kq_1^{\frac k2}\sqrt{q_2})}{2\sqrt[4]{q_2}} & -\frac12 i q_1^{\frac{k-1}{4}}\sqrt[4]{q_2} & \frac12 i q_1^{\frac{k+1}{4}}\sqrt[4]{q_2} \\ -\frac{i q_1^{\frac12(-\frac k2-\frac12)}}{2\sqrt[4]{q_2}} & -\frac{i q_1^{\frac12(-\frac k2-\frac12)}(\sqrt{q_1}-kq_1^{\frac k2}\sqrt{q_2})}{2\sqrt[4]{q_2}} & \frac12 i q_1^{\frac{k-1}{4}}\sqrt[4]{q_2} & \frac12 i q_1^{\frac{k+1}{4}}\sqrt[4]{q_2} \\ \frac{q_1^{\frac12(-\frac k2-\frac12)}}{2\sqrt[4]{q_2}} & -\frac{q_1^{\frac12(-\frac k2-\frac12)}(k\sqrt{q_2}q_1^{\frac k2}+\sqrt{q_1})}{2\sqrt[4]{q_2}} & -\frac12 q_1^{\frac{k-1}{4}}\sqrt[4]{q_2} & \frac12 q_1^{\frac{k+1}{4}}\sqrt[4]{q_2} \\ \frac{q_1^{\frac12(-\frac k2-\frac12)}}{2\sqrt[4]{q_2}} & \frac{q_1^{\frac12(-\frac k2-\frac12)}(k\sqrt{q_2}q_1^{\frac k2}+\sqrt{q_1})}{2\sqrt[4]{q_2}} & \frac12 q_1^{\frac{k-1}{4}}\sqrt[4]{q_2} & \frac12 q_1^{\frac{k+1}{4}}\sqrt[4]{q_2} \end{pmatrix}.$$

Proposition 10.1.1. *The small quantum cohomology of Hirzebruch surfaces $\mathbb{F}_{2k}$ is contained in the $\mathcal{I}^0_\Lambda$-stratum of $QH^\bullet(\mathbb{F}_{2k})$. Moreover, the point p is in the $\mathcal{A}_\Lambda$-stratum of $QH^\bullet(\mathbb{F}_{2k})$ if and only if $q_1 = q_1^k q_2$.*

Proof. By Theorem 2.5.1, the function $\det\Lambda$ takes the form

$$\det\Lambda(z,p) = \frac{z^2}{z^2 A_0(p) + zA_1(p) + A_2(p)},$$

where A_0, A_1, A_2 are holomorphic functions on $QH^\bullet(\mathbb{F}_{2k})$. If p is a point of the small quantum locus, an explicit computation shows that

$$\det\Lambda(z,p) = -\frac{1}{256}(q_1 - q_2 q_1^k)^{-1},$$

so that $A_1(p) = A_2(p) = 0$. The claim follows. ∎

Corollary 10.1.2. *Along the small quantum locus of $QH^\bullet(\mathbb{F}_{2k})$ the $\mathcal{A}_\Lambda$-stratum coincides with the Maxwell stratum $\mathcal{M}_{\mathbb{F}_{2k}}$.*

Proof. If $q_1 = q_1^k q_2$, we have coalescences of canonical coordinates u_1, u_2, u_3, u_4. Any point of the small quantum locus, however, is semisimple. ∎

10.2 Small qDE of $\mathbb{F}_{2k}$

In the coordinates $(t^{\alpha,2k})_{\alpha=0}^3$, the grading tensor μ has matrix $\mu = \mathrm{diag}(-1, 0, 0, 1)$. The isomonodromic system (2.7.3) is

$$\mathcal{H}_k^{\mathrm{ev}}:\quad \begin{cases} \dfrac{\partial \xi_1}{\partial z} = (2-2k)\xi_3 + 2\xi_2 + \dfrac{1}{z}\xi_1, \\ \dfrac{\partial \xi_2}{\partial z} = (2k+2)\xi_4 + \xi_1(2kq_1^k q_2 + 2q_1), \\ \dfrac{\partial \xi_3}{\partial z} = 2\xi_1 q_1^k q_2 + 2\xi_4, \\ \dfrac{\partial \xi_4}{\partial z} = 2\xi_2 q_1^k q_2 + \xi_3(2q_1 - 2kq_1^k q_2) - \dfrac{1}{z}\xi_4. \end{cases}$$

In the complement of the $\mathcal{A}_\Lambda$-stratum, it can be reduced to the single equation in ξ_1, the *master differential equation*

$$\begin{aligned} &z^4 \frac{\partial^4 \xi_1}{\partial z^4} - z^2\left[z^2(8q_1^k q_2 + 8q_1) - 1\right]\frac{\partial^2 \xi_1}{\partial z^2} \\ &\quad - 3z\frac{\partial \xi_1}{\partial z} - \left(-16z^4(q_1 - q_1^k q_2)^2 - 3\right)\xi_1 = 0. \end{aligned} \tag{10.2.1}$$

Given a solution $\xi_1(z,t)$ of equation (10.2.1), we can reconstruct a solution of the system $\mathcal{H}_k^{\mathrm{ev}}$ through the formulas

$$\begin{aligned} \xi_2 = &-\frac{(-4(k+1)q_2 z^2 q_1^k + 4(k+1)q_1 z^2 + k - 1)}{16z^3(q_1 - q_2 q_1^k)}\xi_1 \\ &- \frac{(4(3k-1)q_2 z^2 q_1^k + 4(k-3)q_1 z^2 - k + 1)}{16z^2(q_1 - q_2 q_1^k)}\frac{\partial \xi_1}{\partial z} + \frac{(k-1)}{16(q_1 - q_2 q_1^k)}\frac{\partial^3 \xi_1}{\partial z^3}, \end{aligned}$$

$$\begin{aligned} \xi_3 = &-\frac{(-4q_2 z^2 q_1^k + 4q_1 z^2 + 1)}{16z^3(q_1 - q_2 q_1^k)}\xi_1 - \frac{(12q_2 z^2 q_1^k + 4q_1 z^2 - 1)}{16z^2(q_1 - q_2 q_1^k)}\frac{\partial \xi_1}{\partial z} \\ &+ \frac{1}{16(q_1 - q_2 q_1^k)}\frac{\partial^3 \xi_1}{\partial z^3}, \end{aligned}$$

$$\xi_4 = -\frac{(4q_2 z^2 q_1^k + 4q_1 z^2 - 1)}{8z^2}\xi_1 - \frac{1}{8z}\frac{\partial \xi_1}{\partial z} + \frac{1}{8}\frac{\partial^2 \xi_1}{\partial z^2}.$$

By looking for solution of the form

$$\xi_1(z,t) = z \cdot \Phi(z,t),$$

equation (10.2.1) can be rewritten as the *(small) quantum differential equation*

$$z(\vartheta^4\Phi - 2\vartheta^3\Phi) - 8z^3(q_1 + q_1^k q_2)[\vartheta^2\Phi + \vartheta\Phi] + 16z^5(q_1 - q_1^k q_2)^2\Phi = 0,$$

where $\vartheta := z\frac{\partial}{\partial z}$.

10.3 Proof for $QH^\bullet(\mathbb{F}_{2k})$

Let us specialize the system $\mathcal{H}_k^{\mathrm{ev}}$ at the point $0 \in QH^\bullet(\mathbb{F}_{2k})$, for which $q_1 = q_2 = 1$:

$$\mathcal{H}_k'\colon \quad \begin{cases} \dfrac{\partial \xi_1}{\partial z} = (2-2k)\xi_3 + 2\xi_2 + \dfrac{1}{z}\xi_1, \\ \dfrac{\partial \xi_2}{\partial z} = (2k+2)\xi_4 + \xi_1(2k+2), \\ \dfrac{\partial \xi_3}{\partial z} = 2\xi_1 + 2\xi_4, \\ \dfrac{\partial \xi_4}{\partial z} = 2\xi_2 + \xi_3(2-2k) - \dfrac{1}{z}\xi_4. \end{cases}$$

The point $p = 0$ is in the $\mathcal{A}_\Lambda$-stratum of $QH^\bullet(\mathbb{F}_{2k})$, and so in the Maxwell stratum. Hence, the study of monodromy data of the system of differential equations $\mathcal{H}_k'$ fits in the analysis developed in [22, 23]. In particular, the isomonodromy property is justified by [23, Theorem 4.5]. As explained in Remark 4.5.2, we can reduce the computation of the monodromy data of the system $\mathcal{H}_k'$ to the single case of $\mathcal{H}_0'$. The system $\mathcal{H}_0'$ can in turn be integrated using solutions of the isomonodromic system of $QH^\bullet(\mathbb{P}^1)$ (see [32, Lemma 4.10]).

Proposition 10.3.1. *Let* $(\varphi_1^{(i)}, \varphi_2^{(i)})$ *with* $i = 1, 2$ *be two solutions of system* (2.7.3) *for the quantum cohomology of* $\mathbb{P}^1$*, specialized at* $0 \in H^2(\mathbb{P}^1, \mathbb{C})$*, i.e.*

$$\begin{cases} \dfrac{\partial \varphi_1}{\partial z} = 2\varphi_2 + \dfrac{1}{2z}\varphi_1, \\ \dfrac{\partial \varphi_2}{\partial z} = 2\varphi_1 - \dfrac{1}{2z}\varphi_2. \end{cases}$$

Then the tensor product

$$\begin{pmatrix} \varphi_1^{(1)} \\ \varphi_2^{(1)} \end{pmatrix} \otimes \begin{pmatrix} \varphi_1^{(2)} \\ \varphi_2^{(2)} \end{pmatrix} = \begin{pmatrix} \varphi_1^{(1)} \cdot \varphi_1^{(2)} \\ \varphi_1^{(1)} \cdot \varphi_2^{(2)} \\ \varphi_2^{(1)} \cdot \varphi_1^{(2)} \\ \varphi_2^{(1)} \cdot \varphi_2^{(2)} \end{pmatrix}$$

is a solution of the system $\mathcal{H}_0'$. ∎

Remark 10.3.2. In order to explicitly compute the monodromy data of $\mathcal{H}'_{\mathrm{ev}}$ one could still develop the study of solutions of the small quantum differential equation, and then reconstruct the Stokes solutions of $\mathcal{H}'_k$ doing a similar argument to the one developed in [23, Section 6] for the quantum cohomology of $\mathbb{G}(2,4)$.

Theorem 10.3.3. *The central connection matrix of $QH^\bullet(\mathbb{F}_{2k})$, computed at the point $0 \in QH^\bullet(\mathbb{F}_{2k})$, with respect to an oriented admissible line ℓ of slope $\varphi \in]\frac{\pi}{2}, \frac{3\pi}{2}[$ and for a suitable choice of the determination of the Ψ-matrix, is equal to*

$$C_k = \begin{pmatrix} \frac{1}{2\pi} & \frac{1}{2\pi} & \frac{1}{2\pi} & \frac{1}{2\pi} \\ -i + \frac{\gamma}{\pi} & -i + \frac{\gamma}{\pi} & \frac{\gamma}{\pi} & \frac{\gamma}{\pi} \\ -\frac{(k-1)(\gamma - i\pi)}{\pi} & \frac{i\pi k - \gamma k + \gamma}{\pi} & -i + \frac{\gamma - \gamma k}{\pi} & \frac{\gamma - \gamma k}{\pi} \\ \frac{2(\gamma - i\pi)^2}{\pi} & \frac{2\gamma(\gamma - i\pi)}{\pi} & \frac{2\gamma(\gamma - i\pi)}{\pi} & \frac{2\gamma^2}{\pi} \end{pmatrix},$$

and the corresponding Stokes matrix is equal to

$$S = \begin{pmatrix} 1 & -2 & -2 & 4 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

The matrix C_k is the matrix associated with the morphism

$$\begin{aligned} \mathcal{A}^-_{\mathbb{F}_{2k}} \colon K_0(\mathbb{F}_{2k})_{\mathbb{C}} &\to H^\bullet(\mathbb{F}_{2k}, \mathbb{C}), \\ [\mathscr{F}] &\mapsto \frac{1}{2\pi} \widehat{\Gamma}^-_{\mathbb{F}_{2k}} \cup e^{-\pi i c_1(\mathbb{F}_{2k})} \cup \mathrm{Ch}(\mathscr{F}), \end{aligned}$$

with respect to

- *an exceptional basis $\mathfrak{E} := (E_i)_{i=1}^4$ of $K_0(\mathbb{F}_{2k})_{\mathbb{C}}$,*
- *the basis $(T_{i,2k})_{i=0}^3$ of $H^\bullet(\mathbb{F}_{2k}, \mathbb{C})$.*

The exceptional basis $\mathfrak{E}$ is the one obtained by acting on the exceptional basis

$$\big([\mathcal{O}], [\mathcal{O}(\Sigma_2^{2k})], [\mathcal{O}(\Sigma_4^{2k})], [\mathcal{O}(\Sigma_2^{2k} + \Sigma_4^{2k})]\big)$$

with the element $(J_k^{-1}, b_k) \in (\mathbb{Z}/2\mathbb{Z})^4 \rtimes \mathcal{B}_4$, where

$$J_k := \begin{cases} (1, 1, (-1)^{p+1}, (-1)^p) & \text{if } k = 2p+1, \\ (1, 1, (-1)^p, (-1)^p) & \text{if } k = 2p, \end{cases}$$
$$b_k := \beta_3^k.$$

Proof. We divide the proof into three steps.

Step 1. Let us first show that for suitable choices of the oriented line ℓ and Ψ-matrix, the central connection matrix computed at the point $0 \in QH^\bullet(\mathbb{F}_0)$ is given in the following form:

$$C_0 := \begin{pmatrix} \frac{1}{2\pi} & \frac{1}{2\pi} & \frac{1}{2\pi} & \frac{1}{2\pi} \\ -i + \frac{\gamma}{\pi} & -i + \frac{\gamma}{\pi} & \frac{\gamma}{\pi} & \frac{\gamma}{\pi} \\ -i + \frac{\gamma}{\pi} & \frac{\gamma}{\pi} & -i + \frac{\gamma}{\pi} & \frac{\gamma}{\pi} \\ \frac{2(\gamma - i\pi)^2}{\pi} & \frac{2\gamma(\gamma - i\pi)}{\pi} & \frac{2\gamma(\gamma - i\pi)}{\pi} & \frac{2\gamma^2}{\pi} \end{pmatrix}. \tag{10.3.1}$$

According to [21, Corollary 6.11], the central connection matrix C of $QH^\bullet(\mathbb{P}^1)$ computed at the point 0, with respect to an oriented line ℓ of slope $\varphi \in]\frac{\pi}{2}, \frac{3\pi}{2}[$ and with respect to the following choice of Ψ-matrix

$$\Psi_0 = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{i}{\sqrt{2}} & -\frac{i}{\sqrt{2}} \end{pmatrix},$$

equals

$$C := \frac{i}{\sqrt{2\pi}} \begin{pmatrix} 1 & 1 \\ 2(\gamma - \pi i) & 2\gamma \end{pmatrix}.$$

This is the matrix associated with the morphism

$$\begin{aligned} Д^-_{\mathbb{P}^1} \colon K_0(\mathbb{P}^1)_\mathbb{C} &\to H^\bullet(\mathbb{P}^1, \mathbb{C}), \\ [\mathscr{F}] &\mapsto \frac{i}{(2\pi)^{\frac{1}{2}}} \widehat{\Gamma}^-_{\mathbb{P}^1} \cup e^{-\pi i c_1(\mathbb{P}^1)} \cup \mathrm{Ch}(\mathscr{F}), \end{aligned}$$

with respect to the bases

- $([\mathcal{O}], [\mathcal{O}(1)])$ of $K_0(\mathbb{P}^1)_\mathbb{C}$ (the Beilinson basis),
- $(1, \sigma)$ of $H^\bullet(\mathbb{P}^1, \mathbb{C})$.

By taking the Kronecker tensor square $C^{\otimes 2}$, we obtain the central connection matrix of $QH^\bullet(\mathbb{P}^1 \times \mathbb{P}^1)$ computed at the point 0, with respect to the same line ℓ (which is still admissible) and with respect to the choice of the Ψ-matrix given by the Kronecker tensor square $\Psi_0^{\otimes 2}$:

$$C^{\otimes 2} = \begin{pmatrix} -\frac{1}{2\pi} & -\frac{1}{2\pi} & -\frac{1}{2\pi} & -\frac{1}{2\pi} \\ -\frac{\gamma - i\pi}{\pi} & -\frac{\gamma}{\pi} & -\frac{\gamma - i\pi}{\pi} & -\frac{\gamma}{\pi} \\ -\frac{\gamma - i\pi}{\pi} & -\frac{\gamma - i\pi}{\pi} & -\frac{\gamma}{\pi} & -\frac{\gamma}{\pi} \\ -\frac{2(\gamma - i\pi)^2}{\pi} & -\frac{2\gamma(\gamma - i\pi)}{\pi} & -\frac{2\gamma(\gamma - i\pi)}{\pi} & -\frac{2\gamma^2}{\pi} \end{pmatrix}.$$

By changing all the signs of the rows of the Kronecker tensor square $\Psi_0^{\otimes 2}$, i.e. acting with $(-1, -1, -1, -1) \in (\mathbb{Z}/2\mathbb{Z})^4$ on $C^{\otimes 2}$, we obtain the matrix $-C^{\otimes 2}$ associated with the morphism

$$\begin{aligned} Д^-_{\mathbb{P}^1 \times \mathbb{P}^1} \colon K_0(\mathbb{P}^1 \times \mathbb{P}^1)_\mathbb{C} &\to H^\bullet(\mathbb{P}^1 \times \mathbb{P}^1, \mathbb{C}), \\ [\mathscr{F}] &\mapsto \frac{1}{2\pi} \widehat{\Gamma}^-_{\mathbb{P}^1 \times \mathbb{P}^1} \cup e^{-\pi i c_1(\mathbb{P}^1 \times \mathbb{P}^1)} \cup \mathrm{Ch}(\mathscr{F}), \end{aligned}$$

written with respect to the bases

- $([\mathcal{O}], [\mathcal{O}(1,0)], [\mathcal{O}(0,1)], [\mathcal{O}(1,1)])$ of $K_0(\mathbb{P}^1 \times \mathbb{P}^1)_{\mathbb{C}}$,
- $(1, \sigma \otimes 1, 1 \otimes \sigma, \sigma \otimes \sigma)$ of $H^\bullet(\mathbb{P}^1 \times \mathbb{P}^1, \mathbb{C}) \cong H^\bullet(\mathbb{P}^1, \mathbb{C})^{\otimes 2}$.

See [21, Proposition 5.11]. In the notations introduced before for Hirzebruch surfaces, this exceptional collection is

$$\big(\mathcal{O}, \mathcal{O}(\Sigma_4^0), \mathcal{O}(\Sigma_2^0), \mathcal{O}(\Sigma_2^0 + \Sigma_4^0)\big).$$

It is a 3-block exceptional collection,[1] coherently with the fact that $0 \in QH^\bullet(\mathbb{F}_0)$ is a semisimple coalescing point, see [23, Section 6] and [21, Remark 5.4]. In particular, the braids $\beta_{2,3}$ and $\beta_{2,3}^{-1}$ act as a mere permutation of the central objects, and of the two central columns of the matrix $-C^{\otimes 2}$. Such a permuted matrix is exactly the matrix C_0 in (10.3.1), and it corresponds to the matrix associated with the morphism $\mathit{\Pi}^-_{\mathbb{F}_0}$ with respect to the collection

$$\big(\mathcal{O}, \mathcal{O}(\Sigma_2^0), \mathcal{O}(\Sigma_4^0), \mathcal{O}(\Sigma_2^0 + \Sigma_4^0)\big).$$

In conclusion, we have proved that, for suitable choices of ℓ and Ψ, the central connection matrix computed at $0 \in QH^\bullet(\mathbb{F}_0)$ is

$$C_0 = \begin{pmatrix} \frac{1}{2\pi} & \frac{1}{2\pi} & \frac{1}{2\pi} & \frac{1}{2\pi} \\ -i + \frac{\gamma}{\pi} & -i + \frac{\gamma}{\pi} & \frac{\gamma}{\pi} & \frac{\gamma}{\pi} \\ -i + \frac{\gamma}{\pi} & \frac{\gamma}{\pi} & -i + \frac{\gamma}{\pi} & \frac{\gamma}{\pi} \\ \frac{2(\gamma - i\pi)^2}{\pi} & \frac{2\gamma(\gamma - i\pi)}{\pi} & \frac{2\gamma(\gamma - i\pi)}{\pi} & \frac{2\gamma^2}{\pi} \end{pmatrix},$$

which coincides with the matrix associated with the collection

$$\big(\mathcal{O}, \mathcal{O}(\Sigma_2^0), \mathcal{O}(\Sigma_4^0), \mathcal{O}(\Sigma_2^0 + \Sigma_4^0)\big).$$

Step 2. Equations (9.3.5) and Proposition 4.5.1 imply that the central connection matrix computed at $0 \in QH^\bullet(\mathbb{F}_{2k})$, with respect to the same choices of ℓ and Ψ, is

$$C_k = \begin{pmatrix} \frac{1}{2\pi} & \frac{1}{2\pi} & \frac{1}{2\pi} & \frac{1}{2\pi} \\ -i + \frac{\gamma}{\pi} & -i + \frac{\gamma}{\pi} & \frac{\gamma}{\pi} & \frac{\gamma}{\pi} \\ -\frac{(k-1)(\gamma - i\pi)}{\pi} & \frac{i\pi k - \gamma k + \gamma}{\pi} & -i + \frac{\gamma - \gamma k}{\pi} & \frac{\gamma - \gamma k}{\pi} \\ \frac{2(\gamma - i\pi)^2}{\pi} & \frac{2\gamma(\gamma - i\pi)}{\pi} & \frac{2\gamma(\gamma - i\pi)}{\pi} & \frac{2\gamma^2}{\pi} \end{pmatrix}.$$

[1] An exceptional collection $(E_1, \dots, E_n)$ is a k*-block exceptional collection* if it is possible to decompose it into k exceptional sub-collections $\mathfrak{B}_1, \dots, \mathfrak{B}_k$, called *blocks*, such that

- they are consecutive, i.e. of the form $\mathfrak{B}_1 = (E_1, \dots, E_{j_1})$, $\mathfrak{B}_2 = (E_{j_1+1}, \dots, E_{j_2})$, ..., $\mathfrak{B}_k = (E_{j_{k-1}+1}, \dots, E_{j_k})$, with $1 \leqslant j_1 < j_2 < \dots < j_k \leqslant n$,
- we have $\mathrm{Hom}^\bullet(E_j, E_i) = 0$ if E_i and E_j belong to a same block $\mathfrak{B}_h$.

In particular, inside each block $\mathfrak{B}_h$, mutations act as permutations of exceptional objects. See [21, Section 3.6.4], and references therein.

The corresponding Stokes matrix is independent of k, and it is equal to

$$S = \begin{pmatrix} 1 & -2 & -2 & 4 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{10.3.2}$$

Step 3. Let us define the matrix $J_k \in (\mathbb{Z}/2\mathbb{Z})^4$ as follows:

$$J_k := \begin{cases} (1, 1, (-1)^{p+1}, (-1)^p) & \text{if } k = 2p + 1, \\ (1, 1, (-1)^p, (-1)^p) & \text{if } k = 2p. \end{cases}$$

We claim that by acting on $C_k J_k$ with the braid β_3^{-k} we obtain the matrix associated with $\Pi^-_{\mathbb{F}_{2k}}$ and with respect to the exceptional collection

$$\big(\mathcal{O}, \mathcal{O}(\Sigma_2^{2k}), \mathcal{O}(\Sigma_4^{2k}), \mathcal{O}(\Sigma_2^{2k} + \Sigma_4^{2k})\big),$$

namely the matrix

$$E_k := \begin{pmatrix} \frac{1}{2\pi} & \frac{1}{2\pi} & \frac{1}{2\pi} & \frac{1}{2\pi} \\ -i + \frac{\gamma}{\pi} & -i + \frac{\gamma}{\pi} & \frac{\gamma}{\pi} & \frac{\gamma}{\pi} \\ -\frac{(k-1)(\gamma - i\pi)}{\pi} & \frac{i\pi k - \gamma k + \gamma}{\pi} & -\frac{(k-1)(\gamma - i\pi)}{\pi} & \frac{i\pi k - \gamma k + \gamma}{\pi} \\ \frac{2(\gamma - i\pi)^2}{\pi} & \frac{2\gamma(\gamma - i\pi)}{\pi} & \frac{2\gamma(i\pi(k-1)+\gamma)}{\pi} & \frac{2\gamma(i\pi k + \gamma)}{\pi} \end{pmatrix}.$$

Note that the claim is equivalent to the following statement: the matrix $A^\beta(J_k \cdot S \cdot J_k)$, with $\beta = \beta_3^{-k}$ and S as in (10.3.2), is equal to

$$E_k^{-1} C_k J_k = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & k+1 & k \\ 0 & 0 & -k & 1-k \end{pmatrix} \cdot J_k. \tag{10.3.3}$$

Given a generic 4×4 unipotent upper triangular matrix X, the action of subsequent powers of the braid β_3, or of its inverse β_3^{-1}, simply changes the sign of the entry in position $(3, 4)$: more precisely, we have

$$[X^\beta]_{3,4} = (-1)^n [X]_{3,4} \quad \text{if } \beta = \beta_3^{\pm n}.$$

For example, by acting twice with the braid β_3 we have

$$\begin{pmatrix} 1 & a & b & c \\ 0 & 1 & d & e \\ 0 & 0 & 1 & f \\ 0 & 0 & 0 & 1 \end{pmatrix} \mapsto \begin{pmatrix} 1 & a & c & b - cf \\ 0 & 1 & e & d - ef \\ 0 & 0 & 1 & -f \\ 0 & 0 & 0 & 1 \end{pmatrix} \mapsto \begin{pmatrix} 1 & a & b - cf & c + f(b - cf) \\ 0 & 1 & d - ef & e + f(d - ef) \\ 0 & 0 & 1 & f \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

In particular, the matrix $A^{\beta}(X)$, with $\beta = \beta_3^{-k}$, is equal to

$$\prod_{j=1}^{k} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & (-1)^j x & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad x = X_{3,4}.$$

In the case $X = J_k \cdot S \cdot J_k$, we have

$$x = (-1)^{k+1} 2.$$

So, in conclusion, we have to prove that the following identity holds for all $k \geqslant 0$:

$$\prod_{j=1}^{k} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & (-1)^{j+k+1} 2 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & k+1 & k \\ 0 & 0 & -k & 1-k \end{pmatrix} \cdot J_k.$$

We prove the claim by induction on k. The base case $k = 0$ is evidently true. Let us assume that the statement holds true for $k - 1$, and let us prove it for k. We have

$$\begin{aligned} &\prod_{j=1}^{k} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & (-1)^{j+k+1} 2 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \\ &= \left[\prod_{j=1}^{k-1} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & (-1)^{j+k+1} 2 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \right] \cdot \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -2 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & k & k-1 \\ 0 & 0 & 1-k & 2-k \end{pmatrix} \cdot J_{k-1} \cdot \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -2 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \end{aligned}$$

and in both cases k even/odd, the last term is easily seen to be equal to (10.3.3). ■