Chapter 11

Dubrovin conjecture for Hirzebruch surfaces $\mathbb{F}_{2k+1}$

11.1 $\mathcal{A}_\Lambda$-stratum and Maxwell stratum of $QH^\bullet(\mathbb{F}_{2k+1})$

Fix a point

$$p = t^{1,2k+1}T_{1,2k+1} + t^{2,2k+1}T_{2,2k+1}$$

of the small quantum cohomology of $\mathbb{F}_{2k+1}$. The matrix associated to the $\mathcal{U}$-tensor at p is

$$\mathcal{U}(p) = \begin{pmatrix} 0 & 2q_1 & 0 & 3q_1^{k+1}q_2 \\ 2 & kq_1^k q_2 & q_1^k q_2 & 0 \\ 1-2k & k(-kq_2q_1^k - q_1^k q_2) & -kq_2q_1^k - q_1^k q_2 & 2q_1 \\ 0 & 2k+3 & 2 & 0 \end{pmatrix}.$$

The canonical coordinates are the roots $u_1(p), u_2(p), u_3(p), u_4(p)$ of the polynomial

$$j(u) := u^4 + u^3 q_1^k q_2 - 8q_1u^2 - 36uq_1^{k+1}q_2 - 27q_2^2q_1^{2k+1} + 16q_1^2.$$

Hence the bifurcation set $\mathcal{B}_{\mathbb{F}_{2k+1}}$, along the small quantum cohomology, is defined by the zero locus of the discriminant of $j(u)$, i.e.

$$\mathcal{B}_{\mathbb{F}_{2k+1}} = \{p : q_1^{2k+2}q_2^2(27q_2^2q_1^{2k} + 256q_1)^3 = 0\}.$$

Since any point of the small quantum cohomology of $\mathbb{F}_{2k+1}$ is semisimple, the set above actually coincides with the Maxwell stratum $\mathcal{M}_{\mathbb{F}_{2k+1}}$. The determinant of the Λ-matrix is given by

$$\det \Lambda(z, p) = -\frac{z}{(27q_1^{2k}q_2^2 + 256q_1)z - 24q_2q_1^k}.$$

Hence, the $\mathcal{A}_\Lambda$-stratum is given by

$$\mathcal{A}_\Lambda := \{p : 27q_1^{2k}q_2^2 + 256q_1 = 0\}. \tag{11.1.1}$$

Also in this case, the Maxwell stratum and the $\mathcal{A}_\Lambda$-stratum coincide along the small quantum cohomology of $\mathbb{F}_{2k+1}$.

11.2 Small qDE of $\mathbb{F}_1$

At the point p, the grading operator μ has matrix

$$\mu = \text{diag}(-1, 0, 0, 1).$$

Hence the isomonodromic system of differential equations (2.7.3) for $QH^\bullet(\mathbb{F}_{2k+1})$ is given by

$$\mathcal{H}_k^{\text{od}}\colon \begin{cases} \dfrac{\partial \xi_1}{\partial z} = (1-2k)\xi_3 + 2\xi_2 + \dfrac{\xi_1}{z}, \\ \dfrac{\partial \xi_2}{\partial z} = (2k+3)\xi_4 + k\xi_2 q_2 q_1^k + k\xi_3(-kq_2q_1^k - q_2q_1^k) + 2\xi_1 q_1, \\ \dfrac{\partial \xi_3}{\partial z} = \xi_2 q_2 q_1^k + \xi_3(-kq_2q_1^k - q_2q_1^k) + 2\xi_4, \\ \dfrac{\partial \xi_4}{\partial z} = 3\xi_1 q_2 q_1^{k+1} + 2\xi_3 q_1 - \dfrac{\xi_4}{z}. \end{cases}$$

As explained in Remark 4.5.2, the computation of the monodromy data of $\mathcal{H}_k^{\text{od}}$ can be reduced to the single case $\mathcal{H}_0^{\text{od}}$.

The point $0 \in QH^\bullet(\mathbb{F}_1)$ is not in the $\mathcal{A}_\Lambda$-stratum, as it follows from (11.1.1). At the point $0 \in QH^\bullet(\mathbb{F}_1)$, indeed, the system $\mathcal{H}_0^{\text{od}}$ can be reduced to the *small quantum differential equation*

$$\begin{aligned}(283z-24)\vartheta^4\Phi &+ (283z^2-590z+24)\vartheta^3\Phi \\ &+ (-2264z^2+192z+3)\vartheta^2\Phi \\ &- 4z^2(2547z^2+350z-104)\vartheta\Phi \\ &+ z^2(-3113z^3-9924z^2+1476z+192)\Phi = 0.\end{aligned} \tag{11.2.1}$$

Given a solution $\Phi(z)$ of (11.2.1), the corresponding solution of the system $\mathcal{H}_0^{\text{od}}$ can be reconstructed by the formulas

$$\xi_1(z) = z\cdot\Phi(z), \tag{11.2.2}$$

$$\begin{aligned}\xi_2(z) = \frac{1}{z^2(283z-24)}\big(&169z^3\xi_1'(z) + z^3\xi_1''(z) + 204z^3\xi_1(z) \\ &- 8z^3{\xi_1}^{(3)}(z) - 9z^2\xi_1'(z) - 105z^2\xi_1(z) - 8z\xi_1'(z) \\ &+ 9z\xi_1(z) + 8\xi_1(z)\big),\end{aligned} \tag{11.2.3}$$

$$\begin{aligned}\xi_3(z) = \frac{1}{z^2(283z-24)}\big(&-55z^3\xi_1'(z) - 2z^3\xi_1''(z) - 408z^3\xi_1(z) \\ &+ 16z^3{\xi_1}^{(3)}(z) - 6z^2\xi_1'(z) - 73z^2\xi_1(z) + 16z\xi_1'(z) \\ &+ 6z\xi_1(z) - 16\xi_1(z)\big),\end{aligned} \tag{11.2.4}$$

$$\begin{aligned}\xi_4(z) = \frac{1}{z^2(283z-24)}\big(&-28z^3\xi_1'(z) + 35z^3\xi_1''(z) - 218z^3\xi_1(z) \\ &+ 3z^3{\xi_1}^{(3)}(z) - 35z^2\xi_1'(z) - 3z^2\xi_1''(z) + 16z^2\xi_1(z) \\ &+ 6z\xi_1'(z) + 35z\xi_1(z) - 6\xi_1(z)\big).\end{aligned} \tag{11.2.5}$$

These formulas are obtained by the identity

$$\xi = \Lambda^T \begin{pmatrix} \xi_1 \\ \xi_1' \\ \xi_1'' \\ \xi_1^{(3)} \end{pmatrix},$$

where the Λ-matrix at $0 \in QH^\bullet(\mathbb{F}_1)$ is

$$\Lambda(z,0) = \begin{pmatrix} 1 & \frac{204z^3-105z^2+9z+8}{z^2(283z-24)} & \frac{-408z^3-73z^2+6z-16}{z^2(283z-24)} & \frac{-218z^3+16z^2+35z-6}{z^2(283z-24)} \\ 0 & \frac{169z^2-9z-8}{z(283z-24)} & \frac{-55z^2-6z+16}{z(283z-24)} & \frac{-28z^2-35z+6}{z(283z-24)} \\ 0 & \frac{z}{283z-24} & -\frac{2z}{283z-24} & \frac{35z-3}{283z-24} \\ 0 & -\frac{8z}{283z-24} & \frac{16z}{283z-24} & \frac{3z}{283z-24} \end{pmatrix}.$$

Remark 11.2.1. The quantum differential equation (11.2.1) has one apparent singularity at $z = \frac{24}{283}$. This coincides with the zero of the denominator of the determinant of the Λ-matrix:

$$\det \Lambda(z,0) = \frac{z}{24 - 283z}.$$

The Ψ-matrix at the point $0 \in QH^\bullet(\mathbb{F}_1)$ is given by

$$\Psi = \begin{pmatrix} \alpha_1^{\frac{1}{2}}\varepsilon_1 & \alpha_1^{\frac{1}{2}}\delta_1 & \alpha_1^{\frac{1}{2}}\sigma_1 & \alpha_1^{\frac{1}{2}}\upsilon_1 \\ \alpha_2^{\frac{1}{2}}\varepsilon_2 & \alpha_2^{\frac{1}{2}}\delta_2 & \alpha_2^{\frac{1}{2}}\sigma_2 & \alpha_2^{\frac{1}{2}}\upsilon_2 \\ \alpha_3^{\frac{1}{2}}\varepsilon_3 & \alpha_3^{\frac{1}{2}}\delta_3 & \alpha_3^{\frac{1}{2}}\sigma_3 & \alpha_3^{\frac{1}{2}}\upsilon_3 \\ \alpha_4^{\frac{1}{2}}\varepsilon_4 & \alpha_4^{\frac{1}{2}}\delta_4 & \alpha_4^{\frac{1}{2}}\sigma_4 & \alpha_4^{\frac{1}{2}}\upsilon_4 \end{pmatrix},$$

where the numbers $\alpha_i, \varepsilon_i, \delta_i, \sigma_i, \upsilon_i$ satisfy the algebraic equations

$$\begin{aligned} \alpha_i^4 + \alpha_i^3 - 6\alpha_i^2 - 283 &= 0, \\ 283\varepsilon_i^4 + 6\varepsilon_i^2 - \varepsilon_i - 1 &= 0, \\ 283\delta_i^4 - 2\delta_i^2 - 9\delta_i - 1 &= 0, \\ 283\sigma_i^4 - 32\sigma_i^2 - \sigma_i + 1 &= 0, \\ 283\upsilon_i^4 - 283\upsilon_i^3 + 105\upsilon_i^2 - 17\upsilon_i + 1 &= 0. \end{aligned}$$

Their numerical approximations are

$$\begin{aligned} \alpha_1 &\approx 4.21193, & \varepsilon_1 &\approx 0.237421, \\ \alpha_2 &\approx -0.204399 - 3.73457i, & \varepsilon_2 &\approx -0.0146116 + 0.266969i, \\ \alpha_3 &\approx -0.204399 + 3.73457i, & \varepsilon_3 &\approx -0.0146116 - 0.266969i, \\ \alpha_4 &\approx -4.80313, & \varepsilon_4 &\approx -0.208197, \end{aligned}$$

$$\begin{aligned}
\delta_1 &\approx 0.353808, & \sigma_1 &\approx 0.194489,\\
\delta_2 &\approx -0.122264 - 0.276482i, & \sigma_2 &\approx -0.240929 - 0.0719476i,\\
\delta_3 &\approx -0.122264 + 0.276482i, & \sigma_3 &\approx -0.240929 + 0.0719476i,\\
\delta_4 &\approx -0.10928, & \sigma_4 &\approx 0.28737,\\
\upsilon_1 &\approx 0.28983, & &\\
\upsilon_2 &\approx 0.279666 - 0.0511337i, & &\\
\upsilon_3 &\approx 0.279666 + 0.0511337i, & &\\
\upsilon_4 &\approx 0.150837. & &
\end{aligned}$$

The reader can check that $\Psi^T\Psi = \eta$, and that

$$\Psi\mathcal{U}\Psi^{-1} = \operatorname{diag}(x_1, x_2, x_3, x_4),$$

where the canonical coordinates x_i are the roots of the polynomial

$$x^4 + x^3 - 8x^2 - 36x - 11 = 0.$$

Their numerical approximations are

$$\begin{aligned}
x_1 &\approx 3.7996,\\
x_2 &\approx -2.23455 + 1.94071i,\\
x_3 &\approx -2.23455 - 1.94071i,\\
x_4 &\approx -0.3305.
\end{aligned}$$

11.3 Coordinates on $\mathcal{S}(\mathbb{P}^1) \otimes \mathcal{S}(\mathbb{P}^2)$

Consider the spaces $\mathcal{S}(\mathbb{P}^1)$ and $\mathcal{S}(\mathbb{P}^2)$ of solutions of the qDEs of $\mathbb{P}^1$ and $\mathbb{P}^2$ specialized at the origins of $H^2(\mathbb{P}^1, \mathbb{C})$ and $H^2(\mathbb{P}^2, \mathbb{C})$, respectively: these equations are

$$\vartheta^2\Phi_1 = 4z^2\Phi_1, \tag{11.3.1}$$
$$\vartheta^3\Phi_2 = 27z^3\Phi_2. \tag{11.3.2}$$

Solutions $\Phi_1(z)$ of equation (11.3.1) have the following expansion at $z = 0$:

$$\Phi_1(z) = \sum_{m=0}^{\infty}(A_{m,1} + A_{m,0}\log z)\frac{z^{2m}}{(m!)^2}, \tag{11.3.3}$$

where $A_{0,0}$ and $A_{0,1}$ are arbitrary complex numbers, and the other coefficients are uniquely determined by the difference equations

$$A_{m-1,0} = A_{m,0}, \tag{11.3.4}$$
$$A_{m-1,1} = \frac{A_{m,0}}{m} + A_{m,1}. \tag{11.3.5}$$

In particular, notice that from equation (11.3.5) we deduce that

$$A_{m,1} = A_{0,1} - A_{0,0} H_m, \quad m \geqslant 0,$$

where $H_m := \sum_{i=1}^{m} \frac{1}{i}$ denotes the m-th harmonic number.

Analogously, solutions $\Phi_2(z)$ of equation (11.3.2) have the following expansion at $z = 0$:

$$\Phi_2(z) = \sum_{n=0}^{\infty} (B_{n,2} + B_{n,1} \log z + B_{n,0} \log^2 z) \frac{z^{3n}}{(n!)^3}, \tag{11.3.6}$$

where $B_{0,0}, B_{0,1}, B_{0,2}$ are arbitrary complex numbers, and the other coefficients are uniquely determined by the difference equations

$$B_{n-1,0} = B_{n,0}, \tag{11.3.7}$$

$$B_{n-1,1} = \frac{2}{n} B_{n,0} + B_{n,1}, \tag{11.3.8}$$

$$B_{n-1,2} = \frac{2}{3n^2} B_{n,0} + \frac{1}{n} B_{n,1} + B_{n,2}. \tag{11.3.9}$$

From the difference equation (11.3.8) we deduce that

$$B_{n,1} = B_{0,1} - 2B_{0,0} H_n.$$

The products $A_{0,i} B_{0,j}$, with $i = 0, 1$ and $j = 0, 1, 2$, define a natural system of coordinates on the tensor product $\mathcal{S}(\mathbb{P}^1) \otimes_{\mathbb{C}} \mathcal{S}(\mathbb{P}^2)$.

11.4 Solutions of qDE of $\mathbb{F}_1$ as Laplace $(1, 2; \frac{1}{2}, \frac{1}{3})$-multitransforms

According to Theorem 7.3.1, the space of solutions of the quantum differential equation (11.2.1) can be reconstructed from the spaces of solutions of the qDEs (11.3.1) and (11.3.2). From the polynomial equation (9.1.1), indeed, it follows that Theorem 7.3.1 applies with the specialization of the parameters $h = 2$, $\boldsymbol{\ell} = (2, 3)$ and $\boldsymbol{d} = (1, 1)$.

Hence, we expect to reconstruct the solutions of the differential equation (11.2.1) via a $\mathbb{C}$-bilinear operator

$$\mathscr{P} \colon \mathcal{S}(\mathbb{P}^1) \otimes \mathcal{S}(\mathbb{P}^2) \to \mathcal{O}(\widetilde{\mathbb{C}^*})$$

involving the Laplace $(1, 2; \frac{1}{2}, \frac{1}{3})$-multitransform:

$$\mathscr{P}[\Phi_1, \Phi_2](z) := e^{-cz} \mathscr{L}_{(1,2;\frac{1}{2},\frac{1}{3})}[\Phi_1, \Phi_2] = e^{-cz} \int_0^{\infty} \Phi_1\big(z^{\frac{1}{2}} \lambda^{\frac{1}{2}}\big) \Phi_2\big(z^{\frac{2}{3}} \lambda^{\frac{1}{3}}\big) e^{-\lambda}\, d\lambda,$$

for a suitable number $c \in \mathbb{Q}$ to be determined.

Lemma 11.4.1. *We have $c = 1$.*

Proof. Along the locus of small quantum cohomology, the J-function of $\mathbb{P}^{n-1}$ is

$$J_{\mathbb{P}^{n-1}}(\delta) = e^{\frac{\delta}{\hbar}} \sum_{d=0}^{\infty} \mathbf{Q}^d e^{dt} \frac{1}{\left(\prod_{k=1}^{d}(H + k\hbar)\right)^n}, \quad \delta = tH,$$

where $H \in H^2(\mathbb{P}^{n-1}, \mathbb{C})$ denotes the hyperplane class. It follows that the I-function $I_{\mathbb{P}^1\times\mathbb{P}^2,\mathbb{F}_1}$ equals

$$\begin{aligned} &I_{\mathbb{P}^1\times\mathbb{P}^2,\mathbb{F}_1}(\delta_1 \otimes 1 + 1 \otimes \delta_2) \\ &= e^{\frac{\delta_1}{\hbar}} \otimes e^{\frac{\delta_2}{\hbar}} \cdot \sum_{d_1,d_2\geqslant 0} \mathbf{Q}_1^{d_1}\mathbf{Q}_2^{d_2} \frac{e^{t^1 d_1}}{\left(\prod_{k=1}^{d_1}(H_1 + k\hbar)\right)^2} \otimes \frac{e^{t^2 d_2}}{\left(\prod_{k=1}^{d_2}(H_2 + k\hbar)\right)^3} \\ &\qquad \cdot \prod_{j=1}^{d_1+d_2} (H_1 \otimes 1 + 1 \otimes H_2 + j\hbar) \\ &= 1 + \frac{1}{\hbar}\left(\mathbf{Q}_1^{d_1} e^{t^1} + \delta_1 \otimes 1 + 1 \otimes \delta_2\right) + O\left(\frac{1}{\hbar^2}\right), \end{aligned}$$

where we set:

- $H_1 \in H^2(\mathbb{P}^1, \mathbb{C})$ and $H_2 \in H^2(\mathbb{P}^2, \mathbb{C})$ are the hyperplane classes,
- $\delta_1 = t^1 H_1$ and $\delta_2 = t^2 H_2$ with $t^1, t^2 \in \mathbb{C}$,
- $\mathbf{Q}_i = \mathbf{Q}^{\beta_i}$, β_i being the dual homology class of H_i, for $i = 1, 2$.

In the notations of Proposition 5.3.5, we have $H(\delta_1 \otimes 1 + 1 \otimes \delta_2) = \mathbf{Q}_1^{d_1} e^{t^1}$. The number c equals

$$c = H(0)|_{\mathbf{Q}=1} = 1. \qquad \blacksquare$$

For brevity, in all the remaining part of this section, we will simply write $\mathscr{L}$ to denote the Laplace $(1, 2; \frac{1}{2}, \frac{1}{3})$-multitransform.

11.4.1 The subspace $\mathscr{H}$

The space $\mathcal{S}(\mathbb{P}^1) \otimes \mathcal{S}(\mathbb{P}^2)$ has dimension 6. We are going to identify a subspace $\mathscr{H}$ of dimension 4 which is isomorphically mapped to the space $\mathcal{S}(\mathbb{F}_1)$ via the operator $\mathscr{P}$.

Theorem 11.4.2. *Let $\Phi_1(z)$ and $\Phi_2(z)$ be two solutions of the quantum differential equations of $\mathbb{P}^1$ and $\mathbb{P}^2$, respectively, namely*

$$\vartheta^2\Phi_1(z) = 4z^2\Phi_1(z), \quad \vartheta^3\Phi_2(z) = 27z^3\Phi_2(z).$$

The function

$$\Phi(z) := e^{-z}\mathscr{L}[\Phi_1, \Phi_2; z]$$

is a solution of the quantum differential equation of $\mathbb{F}_1$ *if the following vanishing conditions are satisfied:*

$$\mathscr{D}_1[\Phi_1,\Phi_2;z]=0,\quad \mathscr{D}_2[\Phi_1,\Phi_2;z]=0,$$

where

$$\begin{aligned}\mathscr{D}_1[\Phi_1,\Phi_2;z]&:=2z^2\mathscr{L}[\vartheta\Phi_1,\Phi_2;z]-\frac{2}{9}\mathscr{L}[\vartheta\Phi_1,\vartheta^2\Phi_2;z]\\&\quad+\frac{4}{9}z\mathscr{L}[\Phi_1,\vartheta^2\Phi_2;z],\\ \mathscr{D}_2[\Phi_1,\Phi_2;z]&:=z^3\mathscr{L}[\Phi_1,\Phi_2;z]-\frac{z^2}{3}\mathscr{L}[\Phi_1,\vartheta\Phi_2;z]\\&\quad-\frac{z}{9}\mathscr{L}[\Phi_1,\vartheta^2\Phi_2;z]+\frac{z}{6}\mathscr{L}[\vartheta\Phi_1,\vartheta\Phi_2;z].\end{aligned}$$

Proof. Let us look for solutions of equation (11.2.1) in the form

$$\Phi(z)=e^{-z}\mathscr{L}_{(1,2;\frac{1}{2},\frac{1}{3})}[\Phi_1,\Phi_2;z],$$

where Φ_1 and Φ_2 are solutions of the quantum differential equation for $\mathbb{P}^1$ and $\mathbb{P}^2$, respectively, that is,

$$\vartheta^2\Phi_1=4z^2\Phi_1, \tag{11.4.1}$$

$$\vartheta^3\Phi_2=27z^3\Phi_2. \tag{11.4.2}$$

Given arbitrary functions f and g, we have

$$\begin{aligned}\mathscr{L}[s^2f(s),g(s);z]=z\Big\{&\mathscr{L}[f(s),g(s);z]+\frac{1}{2}\mathscr{L}[\vartheta_sf(s),g(s);z]\\&+\frac{1}{3}\mathscr{L}[f(s),\vartheta_sg(s);z]-\mathcal{I}(f,g)\Big\},\end{aligned}$$

with

$$\mathcal{I}(f,g):=\lambda\cdot f(z^{\frac{1}{2}}\lambda^{\frac{1}{2}})g(z^{\frac{2}{3}}\lambda^{\frac{1}{3}})e^{-\lambda}\Big|_{\lambda=0}^{\lambda=\infty}. \tag{11.4.3}$$

Applying the previous identity to Φ_1 and Φ_2, and using equations (11.4.1)–(11.4.2), we deduce the following identities:

$$\begin{aligned}\mathscr{L}[\vartheta^2\Phi_1,\Phi_2;z]&=4z\Big\{\mathscr{L}[\Phi_1,\Phi_2;z]+\frac{1}{2}\mathscr{L}[\vartheta\Phi_1,\Phi_2;z]\\&\quad+\frac{1}{3}\mathscr{L}[\Phi_1,\vartheta\Phi_2;z]\Big\}+\mathscr{R}_1,\\ \mathscr{L}[\vartheta^3\Phi_1,\Phi_2;z]&=8(z+z^2)\mathscr{L}[\Phi_1,\Phi_2;z]+(8z+4z^2)\mathscr{L}[\vartheta\Phi_1,\Phi_2;z]\\&\quad+\frac{8}{3}(z+z^2)\mathscr{L}[\Phi_1,\vartheta\Phi_2;z]\\&\quad+\frac{4}{3}z\mathscr{L}[\vartheta\Phi_1,\vartheta\Phi_2;z]+\mathscr{R}_2,\end{aligned}$$

$$
\begin{aligned}
\mathscr{L}[\vartheta^4\Phi_1,\Phi_2;z] &= 16(z+4z^2+z^3)\mathscr{L}[\Phi_1,\Phi_2;z] \\
&\quad + 8(3z+5z^2+z^3)\mathscr{L}[\vartheta\Phi_1,\Phi_2;z] \\
&\quad + \frac{16}{3}(z+5z^2+z^3)\mathscr{L}[\Phi_1,\vartheta\Phi_2;z] \\
&\quad + \frac{16}{3}(z+z^2)\mathscr{L}[\vartheta\Phi_1,\vartheta\Phi_2;z] \\
&\quad + \frac{16}{9}z^2\mathscr{L}[\Phi_1,\vartheta^2\Phi_2;z] + \mathscr{R}_3, \\
\mathscr{L}[\Phi_1,\vartheta^3\Phi_2;z] &= 27z^2\Big\{\mathscr{L}[\Phi_1,\Phi_2;z] + \frac{1}{2}\mathscr{L}[\vartheta\Phi_1,\Phi_2;z] \\
&\quad + \frac{1}{3}\mathscr{L}[\Phi_1,\vartheta\Phi_2;z]\Big\} + \mathscr{R}_4, \\
\mathscr{L}[\Phi_1,\vartheta^4\Phi_2;z] &= \frac{9}{2}z^2\Big\{18\mathscr{L}[\Phi_1,\Phi_2;z] + 12\mathscr{L}[\Phi_1,\vartheta\Phi_2;z] \\
&\quad + 2\mathscr{L}[\Phi_1,\vartheta^2\Phi_2;z] + 9\mathscr{L}[\vartheta\Phi_1,\Phi_2;z] \\
&\quad + 3\mathscr{L}[\vartheta\Phi_1,\vartheta\Phi_2;z]\Big\} + \mathscr{R}_5, \\
\mathscr{L}[\vartheta\Phi_1,\vartheta^3\Phi_2;z] &= 54z^3\mathscr{L}[\Phi_1,\Phi_2;z] + 27(z^2+z^3)\mathscr{L}[\vartheta\Phi_1,\Phi_2;z] \\
&\quad + 18z^3\mathscr{L}[\Phi_1,\vartheta\Phi_2;z] \\
&\quad + 9z^2\mathscr{L}[\vartheta\Phi_1,\vartheta\Phi_2;z] + \mathscr{R}_6, \\
\mathscr{L}[\vartheta^2\Phi_1,\vartheta^2\Phi_2;z] &= 36z^3\mathscr{L}[\Phi_1,\Phi_2;z] + 18z^3\mathscr{L}[\vartheta\Phi_1,\Phi_2;z] \\
&\quad + 12z^3\mathscr{L}[\Phi_1,\vartheta\Phi_2;z] + 4z\mathscr{L}[\Phi_1,\vartheta^2\Phi_2;z] \\
&\quad + 2z\mathscr{L}[\vartheta\Phi_1,\vartheta^2\Phi_2;z] + \mathscr{R}_7, \\
\mathscr{L}[\vartheta^3\Phi_1,\vartheta\Phi_2;z] &= 8(z+z^2)\mathscr{L}[\Phi_1,\vartheta\Phi_2;z] \\
&\quad + (8z+4z^2)\mathscr{L}[\vartheta\Phi_1,\vartheta\Phi_2;z] \\
&\quad + \frac{8}{3}(z+z^2)\mathscr{L}[\Phi_1,\vartheta^2\Phi_2;z] \\
&\quad + \frac{4}{3}z\mathscr{L}[\vartheta\Phi_1,\vartheta^2\Phi_2;z] + \mathscr{R}_8, \\
\mathscr{L}[\vartheta^2\Phi_1,\vartheta\Phi_2;z] &= 4z\Big\{\mathscr{L}[\Phi_1,\vartheta\Phi_2;z] + \frac{1}{2}\mathscr{L}[\vartheta\Phi_1,\vartheta\Phi_2;z] \\
&\quad + \frac{1}{3}\mathscr{L}[\Phi_1,\vartheta^2\Phi_2;z]\Big\} + \mathscr{R}_9,
\end{aligned}
$$

where $\mathscr{R}_j$ with $j = 1,\dots,9$ denote some negligible boundary terms due to the cumulations of terms like (11.4.3). Using these identities, after some computations, we can rewrite the quantum differential equation (11.2.1) as follows:

$$(-72+1674z+283z^2)\mathscr{D}_1[\Phi_1,\Phi_2] + (36+724z+4811z^2)\mathscr{D}_2[\Phi_1,\Phi_2] = 0. \quad \blacksquare$$

An explicit computation shows that $\mathscr{D}_1[\Phi_1,\Phi_2;z]$ and $\mathscr{D}_2[\Phi_1,\Phi_2;z]$ have the following expansions:

$$\mathscr{D}_1[\Phi_1,\Phi_2;z] = \Theta_1(z)\log^3 z + \Theta_2(z)\log^2 z + \Theta_3(z)\log z + \Theta_4(z),$$
$$\mathscr{D}_2[\Phi_1,\Phi_2;z] = \Lambda_1(z)\log^3 z + \Lambda_2(z)\log^2 z + \Lambda_3(z)\log z + \Lambda_4(z),$$

where the functions $\Theta_i(z)$ and $\Lambda_i(z)$ are of the form

$$\Theta_i(z) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(m+n)!}{(m!)^2(n!)^3}\big(\mathcal{A}_1^{(i)}(m,n) + \mathcal{A}_2^{(i)}(m,n)z + \mathcal{A}_3^{(i)}(m,n)z^2\big)z^{m+2n}, \tag{11.4.4}$$

$$\Lambda_i(z) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(m+n)!}{(m!)^2(n!)^3}\big(\mathcal{B}_1^{(i)}(m,n) + \mathcal{B}_2^{(i)}(m,n)z + \mathcal{B}_3^{(i)}(m,n)z^2\big)z^{m+2n+1} \tag{11.4.5}$$

for $i = 1,2,3,4$. See Appendix B for the explicit expressions of the coefficients $\mathcal{A}_j^{(i)}$ and $\mathcal{B}_j^{(i)}$.

Lemma 11.4.3. *For all $m,n \geqslant 1$ and $i = 1,2,3,4$, the following identities hold true:*

$$(m+n)\mathcal{A}_1^{(i)}(m,n) + m^2\mathcal{A}_1^{(i)}(m-1,n) + n^3\mathcal{A}_1^{(i)}(m,n-1) = 0, \tag{11.4.6}$$
$$(m+n)\mathcal{B}_1^{(i)}(m,n) + m^2\mathcal{B}_1^{(i)}(m-1,n) + n^3\mathcal{B}_1^{(i)}(m,n-1) = 0, \tag{11.4.7}$$
$$\mathcal{A}_1^{(i)}(m,0) + m\mathcal{A}_2^{(i)}(m-1,0) = 0, \tag{11.4.8}$$
$$\mathcal{B}_1^{(i)}(m,0) + m\mathcal{B}_2^{(i)}(m-1,0) = 0, \tag{11.4.9}$$
$$\mathcal{A}_1^{(i)}(0,n) + n^2\mathcal{A}_3^{(i)}(0,n-1) = 0, \tag{11.4.10}$$
$$\mathcal{B}_1^{(i)}(0,n) + n^2\mathcal{B}_3^{(i)}(0,n-1) = 0. \tag{11.4.11}$$

Proof. The reader can check the validity of these identities using the explicit expressions in Appendix B, equations (11.3.4), (11.3.5), (11.3.7), (11.3.8), (11.3.9), and the following identities (see e.g. [64]):

$$\psi^{(k)}(z+1) = \psi^{(k)}(z) + \frac{(-1)^k k!}{z^{k+1}}, \quad k \geqslant 0,$$
$$\psi^{(0)}(n) = H_{n-1} - \gamma, \quad n \geqslant 1, \quad \psi(z) := \frac{\Gamma'(z)}{\Gamma(z)}.$$
■

Theorem 11.4.4. *Let $\Phi_1(z) \in \mathcal{S}(\mathbb{P}^1)$, $\Phi_2(z) \in \mathcal{S}(\mathbb{P}^2)$ be as in equations (11.3.3) and (11.3.6), respectively. Then the function $\Phi(z) := e^{-z}\mathscr{L}[\Phi_1,\Phi_2;z]$ is a solution of the qDE of $\mathbb{F}_1$ if*

$$A_{0,0}B_{0,0} = 0, \quad 4A_{0,1}B_{0,0} = 3A_{0,0}B_{0,1}. \tag{11.4.12}$$

Proof. Let us rearrange the double series (11.4.4) as follows:

$$\begin{aligned}\Theta_i(z) = \Bigg\{ & \mathcal{A}_1^{(i)}(0,0) + \underbrace{\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{(m+n)!}{(m!)^2(n!)^3}\mathcal{A}_1^{(i)}(m,n) z^{m+2n}}_{(\star)} \\ & + \underbrace{\sum_{m=1}^{\infty} \frac{1}{m!}\mathcal{A}_1^{(i)}(m,0) z^{m}}_{(\star\star)} + \underbrace{\sum_{n=1}^{\infty} \frac{1}{(n!)^2}\mathcal{A}_1^{(i)}(0,n) z^{2n}}_{(\star\star\star)} \\ & + \underbrace{\sum_{m=0}^{\infty}\sum_{n=1}^{\infty} \frac{(m+n)!}{(m!)^2(n!)^3}\mathcal{A}_2^{(i)}(m,n) z^{1+m+2n}}_{(\star)} \\ & + \underbrace{\sum_{m=0}^{\infty} \frac{1}{m!}\mathcal{A}_2^{(i)}(m,0) z^{1+m}}_{(\star\star)} \\ & + \underbrace{\sum_{m=1}^{\infty}\sum_{n=0}^{\infty} \frac{(m+n)!}{(m!)^2(n!)^3}\mathcal{A}_3^{(i)}(m,n) z^{2+m+2n}}_{(\star)} \\ & + \underbrace{\sum_{n=0}^{\infty} \frac{1}{(n!)^2}\mathcal{A}_3^{(i)}(0,n) z^{2+2n}}_{(\star\star\star)} \Bigg\},\end{aligned}$$

where

(1) the $(\star)$-labelled summands cancel by equation (11.4.6),

(2) the $(\star\star)$-labelled summands cancel by equation (11.4.8),

(3) the $(\star\star\star)$-labelled summands cancel by equation (11.4.10).

The proof for $\Lambda_i(z)$ is identical. ∎

Definition 11.4.5. Let $\mathcal{H}$ denote the four-dimensional subspace of $\mathcal{S}(\mathbb{P}^1) \otimes \mathcal{S}(\mathbb{P}^2)$ defined by the linear equations (11.4.12).

Corollary 11.4.6. *The space $\mathcal{H}$ is isomorphic to the space of solutions $\mathcal{S}(\mathbb{F}_1)$ via the operator $\mathscr{P}$.* ∎

11.4.2 Bases of $\mathcal{S}(\mathbb{P}^1)$

Define

$$g(z) := \frac{1}{2\pi i}\int_{\mathcal{L}_1} \Gamma\left(\frac{s}{2}\right)^2 z^{-s}\, ds, \tag{11.4.13}$$

where $\mathcal{L}_1$ is a (positively oriented) parabola $\operatorname{Re} s = -c \cdot (\operatorname{Im} s)^2 + c'$, for suitable $c, c' \in \mathbb{R}_+$ so that it encircles all the poles of the integrand at $s \in 2\mathbb{Z}_{\leqslant 0}$. It is easy to see that the integral in (11.4.13) converges for all $z \in \widetilde{\mathbb{C}^*}$ and that its value does not depend on the particular choice of c, c'.

Proposition 11.4.7. *The functions $g(e^{-i\pi}z)$ and $g(z)$ define a basis of solutions of the qDE of $\mathbb{P}^1$.* ■

Define the bases $(g_1(z), g_2(z))$ and $(s_1(z), s_2(z))$ of $\mathcal{S}(\mathbb{P}^1)$ by

$$\begin{pmatrix} g_1(z) \\ g_2(z) \end{pmatrix} = M_1 \begin{pmatrix} g(e^{-\pi i}z) \\ g(z) \end{pmatrix}, \quad \begin{pmatrix} s_1(z) \\ s_2(z) \end{pmatrix} = M_2 \begin{pmatrix} g(e^{-\pi i}z) \\ g(z) \end{pmatrix},$$

where

$$M_1 := \begin{pmatrix} -\frac{i\gamma}{4\pi} & \frac{i(\gamma+i\pi)}{4\pi} \\ \frac{i}{4\pi} & -\frac{i}{4\pi} \end{pmatrix}, \quad M_2 := \begin{pmatrix} -1 & 2 \\ 0 & 1 \end{pmatrix}.$$

Lemma 11.4.8. *For $z \to 0$, the following asymptotic expansions hold true:*

$$g_1(z) = \log z + O(z^2 \log z),$$
$$g_2(z) = 1 + O(z^2 \log z).$$

Proof. The proof is a simple computation of residues: by modifying the paths of integration $\mathcal{L}_1$, one obtains the asymptotic expansions of g as a sum of residues of the integrand. ■

Lemma 11.4.9. *We have*

$$g(z) \sim \frac{2\pi^{\frac{1}{2}}}{z^{\frac{1}{2}}} e^{-2z}, \quad z \to \infty,$$

in the sector $|\arg z| < \frac{3}{2}\pi$.

Proof. The estimate follows from application of steepest descent method. ■

11.4.3 Bases of $\mathcal{S}(\mathbb{P}^2)$

Define

$$h(z) := \frac{1}{2\pi i} \int_{\mathcal{L}_2} \Gamma\left(\frac{s}{3}\right)^3 e^{\frac{\pi i s}{3}} z^{-s}\, ds, \tag{11.4.14}$$

where $\mathcal{L}_2$ is a (positively oriented) parabola $\operatorname{Re} s = -c \cdot (\operatorname{Im} s)^2 + c'$, for suitable $c, c' \in \mathbb{R}_+$ so that it encircles all the poles of the integrand at $s \in 3\mathbb{Z}_{\leqslant 0}$. It is easy to see that the integral in (11.4.14) converges for all $z \in \widetilde{\mathbb{C}^*}$ and that its value does not depend on the particular choice of c, c'.

Proposition 11.4.10. *The functions $h(e^{-\frac{2i\pi}{3}}z), h(z), h(e^{\frac{2i\pi}{3}}z)$ define a basis of solutions of the qDE of $\mathbb{P}^2$.* ■

Define the bases $(h_1(z), h_2(z), h_3(z))$ and $(p_1(z), p_2(z), p_3(z))$ of $\mathcal{S}(\mathbb{P}^2)$ by

$$\begin{pmatrix} h_1(z) \\ h_2(z) \\ h_3(z) \end{pmatrix} = N_1 \begin{pmatrix} h(e^{-\frac{2i\pi}{3}} z) \\ h(z) \\ h(e^{\frac{2i\pi}{3}} z) \end{pmatrix}, \quad \begin{pmatrix} p_1(z) \\ p_2(z) \\ p_3(z) \end{pmatrix} = N_2 \begin{pmatrix} h(e^{-\frac{2i\pi}{3}} z) \\ h(z) \\ h(e^{\frac{2i\pi}{3}} z) \end{pmatrix}, \tag{11.4.15}$$

where

$$N_1 := \begin{pmatrix} \frac{-18\gamma^2-\pi^2}{216\pi^2} & \frac{-18\gamma^2-24i\gamma\pi+7\pi^2}{216\pi^2} & \frac{18\gamma^2+12i\gamma\pi+5\pi^2}{108\pi^2} \\ \frac{\gamma}{12\pi^2} & \frac{3\gamma+2i\pi}{36\pi^2} & \frac{-3\gamma-i\pi}{18\pi^2} \\ -\frac{1}{12\pi^2} & -\frac{1}{12\pi^2} & \frac{1}{6\pi^2} \end{pmatrix},$$

$$N_2 := \begin{pmatrix} -1 & 3 & -3 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

The basis (p_1, p_2, p_3) will be studied later, in Section 11.7, where it will be used to construct Stokes bases of solutions. We now focus on the properties of the basis (h_1, h_2, h_3).

Lemma 11.4.11. *For $z \to 0$, the following asymptotic expansions hold true:*

$$\begin{aligned} h_1(z) &= \log^2 z + O(z^3 \log^2 z), \\ h_2(z) &= \log z + O(z^3 \log^2 z), \\ h_3(z) &= 1 + O(z^3 \log^2 z). \end{aligned}$$

Proof. The proof is a simple computations of residues: by modifying the paths of integration $\mathcal{L}_2$, one obtains the asymptotic expansions of h as a sum of residues of the integrand. ■

Lemma 11.4.12. *We have*

$$h(z) \sim e^{-\frac{5}{3}\pi i} \frac{\sqrt{3}}{z} \exp\left(3e^{\frac{2\pi i}{3}} z\right), \quad z \to \infty,$$

in the sector $-\pi < \arg z < \frac{5}{3}\pi$.

Proof. The estimate follows from the steepest descent method. ■

11.5 Basis of solutions Υ of $\mathcal{S}(\mathbb{F}_1)$

Theorem 11.5.1. *The tensors*

$$\frac{1}{3} g_1 \otimes h_2 + \frac{1}{4} g_2 \otimes h_1, \quad g_1 \otimes h_3, \quad g_2 \otimes h_2, \quad g_2 \otimes h_3 \tag{11.5.1}$$

define a basis of the subspace $\mathcal{H}$.

Proof. Each of the vectors given in (11.5.1) satisfy the constraints (11.4.12), by Lemmata 11.4.8 and 11.4.11. ■

Corollary 11.5.2. *The functions*

$$\begin{aligned}
\Upsilon_1 &:= \mathscr{P}\left(\frac{1}{3}g_1 \otimes h_2 + \frac{1}{4}g_2 \otimes h_1\right),\\
\Upsilon_2 &:= \mathscr{P}(g_1 \otimes h_3),\\
\Upsilon_3 &:= \mathscr{P}(g_2 \otimes h_2),\\
\Upsilon_4 &:= \mathscr{P}(g_2 \otimes h_3)
\end{aligned}$$

define a basis of solutions of the qDE of $\mathbb{F}_1$. ■

Remark 11.5.3. Explicit double Mellin–Barnes integral representations of solutions $\Upsilon_1, \dots, \Upsilon_4$ can be obtained: for any j, k we have

$$\mathscr{P}\left(g(e^{\pi k i}z) \otimes h\left(e^{\frac{2\pi j i}{3}}z\right)\right) = \frac{e^{-z}}{(2\pi i)^2}\int_{\mathscr{L}_1\times\mathscr{L}_2} \Gamma\left(\frac{s}{2}\right)^2 \Gamma\left(\frac{t}{3}\right)^3 \Gamma\left(1-\frac{s}{2}-\frac{t}{3}\right) \cdot e^{-\pi i k s + \frac{\pi i}{3}t(1-2j)} z^{-\frac{s}{2}-\frac{2t}{3}}\, dt\, ds.$$

The functions Υ_i are linear combinations of the integrals above, in accordance with Theorem 7.4.2.

11.6 Asymptotics of Laplace $(1, 2; \frac{1}{2}, \frac{1}{3})$-multitransforms

Consider the integral

$$\mathcal{I}(z) := \int_0^\infty \Phi_1(z^{\frac{1}{2}}\lambda^{\frac{1}{2}})\Phi_2(z^{\frac{2}{3}}\lambda^{\frac{1}{3}})e^{-\lambda}\, d\lambda,$$

where

$$\Phi_1(z) = z^{D_1}\exp(zu_1), \quad \Phi_2(z) = z^{D_2}\exp(zu_2),$$

with $D_1, D_2, u_1, u_2 \in \mathbb{C}$. The integral $\mathcal{I}(z)$ is convergent for all $z \in \widetilde{\mathbb{C}^*}$.

Set $z = re^{i\kappa}$ with $r > 0$, and change variable of integration $\lambda = \alpha z$:

$$\mathcal{I}(z) = z^{1+D_1+D_2}\int_0^{e^{-i\kappa}\infty} \alpha^{\frac{D_1}{2}+\frac{D_2}{3}} \exp\left\{z(-\alpha + u_1\alpha^{\frac{1}{2}} + u_2\alpha^{\frac{1}{3}})\right\} d\alpha.$$

Change variable $\alpha = \beta^6$, by taking the principal determination of the sixth root:

$$\mathcal{I}(z) = 6z^{1+D_1+D_2}\int_0^{e^{-\frac{i\kappa}{6}}\infty} \beta^{5+3D_1+2D_2}\exp\left\{z(-\beta^6+u_1\beta^3+u_2\beta^2)\right\} d\beta. \quad (11.6.1)$$

Define

$$f(\beta; u_1, u_2) := -\beta^6 + u_1\beta^3 + u_2\beta^2 \quad \text{for } \beta \in \mathbb{C},$$

and consider the z-dependent downward flow in the β-plane defined by

$$\frac{d\beta}{dt} = -\bar{z}\frac{\overline{\partial f}}{\partial \overline{\beta}}, \quad \frac{d\overline{\beta}}{dt} = -z\frac{\partial f}{\partial \beta}. \tag{11.6.2}$$

The equilibria points β_c are the critical points of f, that is,

$$\left.\frac{\partial f}{\partial \beta}\right|_{\beta=\beta_c} = 0.$$

For a fixed z, we associate to each critical point β_c a curve $\mathcal{L}_c$, a *Lefschetz thimble*, defined as the set-theoretic union of the trajectories of the flow (11.6.2) starting at β_c for $t \to -\infty$. Morse and Picard–Lefschetz theory guarantee that the cycles $\mathcal{L}_c$ are smooth one-dimensional submanifolds of $\mathbb{C}$, piecewise smoothly dependent on the parameter z, and they represent a basis for the inverse limit of relative homology groups

$$\varprojlim_T H_1(\mathbb{C}, \mathbb{C}_{T,z}), \quad \mathbb{C}_{T,z} := \{\beta \in \mathbb{C} : \operatorname{Re}(zf(\beta; u_1, u_2)) < -T\}, \ T \in \mathbb{R}_+.$$

Lemma 11.6.1. *The Lefschetz thimble $\mathcal{L}_c$ is the steepest descent path at β_c: the function $t \mapsto \operatorname{Im}(zf(\beta; u_1, u_2))$ is constant on $\mathcal{L}_c$ and the function $t \mapsto \operatorname{Re}(zf(\beta; u_1, u_2))$ is strictly decreasing along the flow.*

Proof. We have

$$\frac{d}{dt}[\operatorname{Im}(zf)] = \left(\frac{d\beta}{dt}\frac{\partial}{\partial\beta} + \frac{d\overline{\beta}}{dt}\frac{\partial}{\partial\overline{\beta}}\right)\left[\frac{zf - \overline{zf}}{2i}\right] = 0,$$

$$\frac{d}{dt}[\operatorname{Re}(zf)] = \left(\frac{d\beta}{dt}\frac{\partial}{\partial\beta} + \frac{d\overline{\beta}}{dt}\frac{\partial}{\partial\overline{\beta}}\right)\left[\frac{zf + \overline{zf}}{2}\right] = -\left|z\frac{\partial f}{\partial\beta}\right|^2. \qquad \blacksquare$$

We are interested in the following cases, by Lemmata 11.4.9 and 11.4.12:

$$u_1 = \pm 2, \quad u_2 = 3\zeta_3^k, \quad \zeta_3 := \exp\frac{2\pi i}{3}, \quad k = 0, 1, 2. \tag{11.6.3}$$

For any possible pair (u_1, u_2), define β_+ as the critical point of $f(\beta; u_1, u_2)$ with maximal real part (the bold one in Table 11.1).

Lemma 11.6.2. *We have*

$$\mathcal{I}(z) \sim 6z^{\frac{1}{2}+D_1+D_2}\beta_+^{5+2D_1+3D_2}\left(\frac{2\pi}{9u_1\beta_+ + 8u_2}\right)^{\frac{1}{2}} \exp z(-\beta_+^6 + u_1\beta_+^3 + u_2\beta_+^2)$$

for $|z| \to \infty$ in the sector $|\arg z - \arg \overline{f(\beta_+)}| < \pi$.

u_1	u_2	β_c	$f(\beta_c)$	$f(\beta_c)-1$
2	3	-0.724492	0.6695	-0.3305
2	3	0.	0.	$-1.$
2	3	**1.22074**	4.7996	3.7996
2	3	$-0.248126-1.03398i$	$-1.23455+1.94071i$	$-2.23455+1.94071i$
2	3	$-0.248126+1.03398i$	$-1.23455-1.94071i$	$-2.23455-1.94071i$
2	$3e^{\frac{2i\pi}{3}}$	0.	0.	$-1.$
2	$3e^{\frac{2i\pi}{3}}$	$-0.771392-0.731875i$	$-1.23455-1.94071i$	$-2.23455-1.94071i$
2	$3e^{\frac{2i\pi}{3}}$	$-0.610372+1.0572i$	4.7996	3.7996
2	$3e^{\frac{2i\pi}{3}}$	$0.362246-0.627428i$	0.6695	-0.3305
2	$3e^{\frac{2i\pi}{3}}$	$\mathbf{1.01952+0.302108}\boldsymbol{i}$	$-1.23455+1.94071i$	$-2.23455+1.94071i$
2	$3e^{-\frac{1}{3}(2i\pi)}$	0.	0.	$-1.$
2	$3e^{-\frac{1}{3}(2i\pi)}$	$-0.771392+0.731875i$	$-1.23455+1.94071i$	$-2.23455+1.94071i$
2	$3e^{-\frac{1}{3}(2i\pi)}$	$-0.610372-1.0572i$	4.7996	3.7996
2	$3e^{-\frac{1}{3}(2i\pi)}$	$0.362246+0.627428i$	0.6695	-0.3305
2	$3e^{-\frac{1}{3}(2i\pi)}$	$\mathbf{1.01952-0.302108}\boldsymbol{i}$	$-1.23455-1.94071i$	$-2.23455-1.94071i$
-2	3	-1.22074	4.7996	3.7996
-2	3	0.	0.	$-1.$
-2	3	**0.724492**	0.6695	-0.3305
-2	3	$0.248126-1.03398i$	$-1.23455-1.94071i$	$-2.23455-1.94071i$
-2	3	$0.248126+1.03398i$	$-1.23455+1.94071i$	$-2.23455+1.94071i$
-2	$3e^{\frac{2i\pi}{3}}$	0.	0.	$-1.$
-2	$3e^{\frac{2i\pi}{3}}$	$-1.01952-0.302108i$	$-1.23455+1.94071i$	$-2.23455+1.94071i$
-2	$3e^{\frac{2i\pi}{3}}$	$-0.362246+0.627428i$	0.6695	-0.3305
-2	$3e^{\frac{2i\pi}{3}}$	$0.610372-1.0572i$	4.7996	3.7996
-2	$3e^{\frac{2i\pi}{3}}$	$\mathbf{0.771392+0.731875}\boldsymbol{i}$	$-1.23455-1.94071i$	$-2.23455-1.94071i$
-2	$3e^{-\frac{1}{3}(2i\pi)}$	0.	0.	$-1.$
-2	$3e^{-\frac{1}{3}(2i\pi)}$	$-1.01952+0.302108i$	$-1.23455-1.94071i$	$-2.23455-1.94071i$
-2	$3e^{-\frac{1}{3}(2i\pi)}$	$-0.362246-0.627428i$	0.6695	-0.3305
-2	$3e^{-\frac{1}{3}(2i\pi)}$	$0.610372+1.0572i$	4.7996	3.7996
-2	$3e^{-\frac{1}{3}(2i\pi)}$	$\mathbf{0.771392-0.731875}\boldsymbol{i}$	$-1.23455+1.94071i$	$-2.23455+1.94071i$

Table 11.1. For any possible value of the pair (u_1, u_2), we list the corresponding critical points β_c of the function $f(\beta; u_1, u_2)$, and the corresponding critical values $f(\beta_c)$. Notice that the numbers $f(\beta_c) - 1$, with $\beta_c \neq 0$, equal all possible values of the canonical coordinates x_1, x_2, x_3, x_4 at the origin of $QH^\bullet(\mathbb{F}_1)$. In bold, we represent the critical point β_+ with maximal real part.

Proof. After choosing an orientation for each Lefschetz thimble, the path of integration $\gamma_z \equiv e^{-i\frac{\kappa}{6}} \cdot \mathbb{R}_+$, defining the function $\mathcal{I}$ in equation (11.6.1), can be expressed as integer combination, $\gamma_z = \sum_{j=1}^{5} n_j(z)\mathcal{L}_j$ with $n_j \in \mathbb{Z}$, of the thimbles $\mathcal{L}_c$ for any value of z not on a Stokes ray $\mathcal{R}_{ij}$, defined by

$$\mathcal{R}_{ij} := \left\{z \in \widetilde{\mathbb{C}^*} : z = r\left(\overline{f(\beta_{c,i})} - \overline{f(\beta_{c,j})}\right),\ r \in \mathbb{R}_+\right\}, \quad i, j = 1, \dots, 5,$$

where $\beta_{c,i}$ are the critical points of (11.6.2). If we let z vary, the Lefschetz thimbles change. When z crosses a Stokes ray $\mathcal{R}_{ij}$, Lefschetz thimbles jump discontinuously: in particular, for z on a Stokes ray there exists a flow line of (11.6.2) connecting two critical points β_c. A detailed analysis of the phase portrait of the flow (11.6.2), for each pair (u_1, u_2) as in (11.6.3), shows that in the sector $|\arg z - \arg \overline{f(\beta_+)}| < \pi$ we have $\gamma_z = \pm\mathcal{L}_{\beta_+} \pm \mathcal{L}_0^1 \pm \mathcal{L}'$, where $\mathcal{L}_0^1$ is only one half of the Lefschetz thimble $\mathcal{L}_0$, and $\mathcal{L}'$ denotes the sum of Lefschetz thimbles attached to other critical points β_c. Hence, we have three contributions in the asymptotics of $\mathcal{I}(z)$: one from the integration along $\mathcal{L}_{\beta_+}$, one from other critical points, the last one from the integration along $\mathcal{L}_0^1$. The last two contributions are easily seen to be negligible with respect to the first one. So, by the steepest descent method, we obtain the estimate

$$\mathcal{I}(z) \sim \pm 6 z^{\frac{1}{2}+D_1+D_2} \beta_+^{5+2D_1+3D_2} \left(-\frac{2\pi}{f''(\beta_+)}\right)^{\frac{1}{2}} \exp z f(\beta_+).$$

See Figure 11.1. ∎

Remark 11.6.3. Note that the arbitrariness of the orientations of the Lefschetz thimbles can be incorporated in the choice of the entries of the Ψ-matrix. Consequently, it will affect the monodromy data by the action of the group $(\mathbb{Z}/2\mathbb{Z})^4$.

Proposition 11.6.4. *Let now Φ_1, Φ_2 be two functions with asymptotic expansions*

$$\Phi_1(z) \sim z^{D_1} \exp(z u_1), \quad \Phi_2(z) \sim z^{D_2} \exp(z u_2) \tag{11.6.4}$$

for $|z| \to \infty$ in the sectors

$$A_1 < \arg z < B_1, \quad A_2 < \arg z < B_2, \tag{11.6.5}$$

respectively. We have

$$\mathcal{L}_{(1,2;\frac{1}{2},\frac{1}{3})}[\Phi_1, \Phi_2; z] \sim C z^{\frac{1}{2}+D_1+D_2} \exp z(-\beta_+^6 + u_1\beta_+^3 + u_2\beta_+^2),$$

where

$$C := 6\beta_+^{5+2D_1+3D_2}\left(\frac{2\pi}{9u_1\beta_+ + 8u_2}\right)^{\frac{1}{2}},$$

for $|z| \to \infty$ in the sector $A' < \arg z < B'$, where

$$A' := \max\left\{A_1 - 3\arg\beta_+,\ A_2 - 2\arg\beta_+,\ \arg\overline{f(\beta_+)} - \pi\right\},$$
$$B' := \min\left\{B_1 - 3\arg\beta_+,\ B_2 - 2\arg\beta_+,\ \arg\overline{f(\beta_+)} + \pi\right\}.$$

(See Table 11.2.)

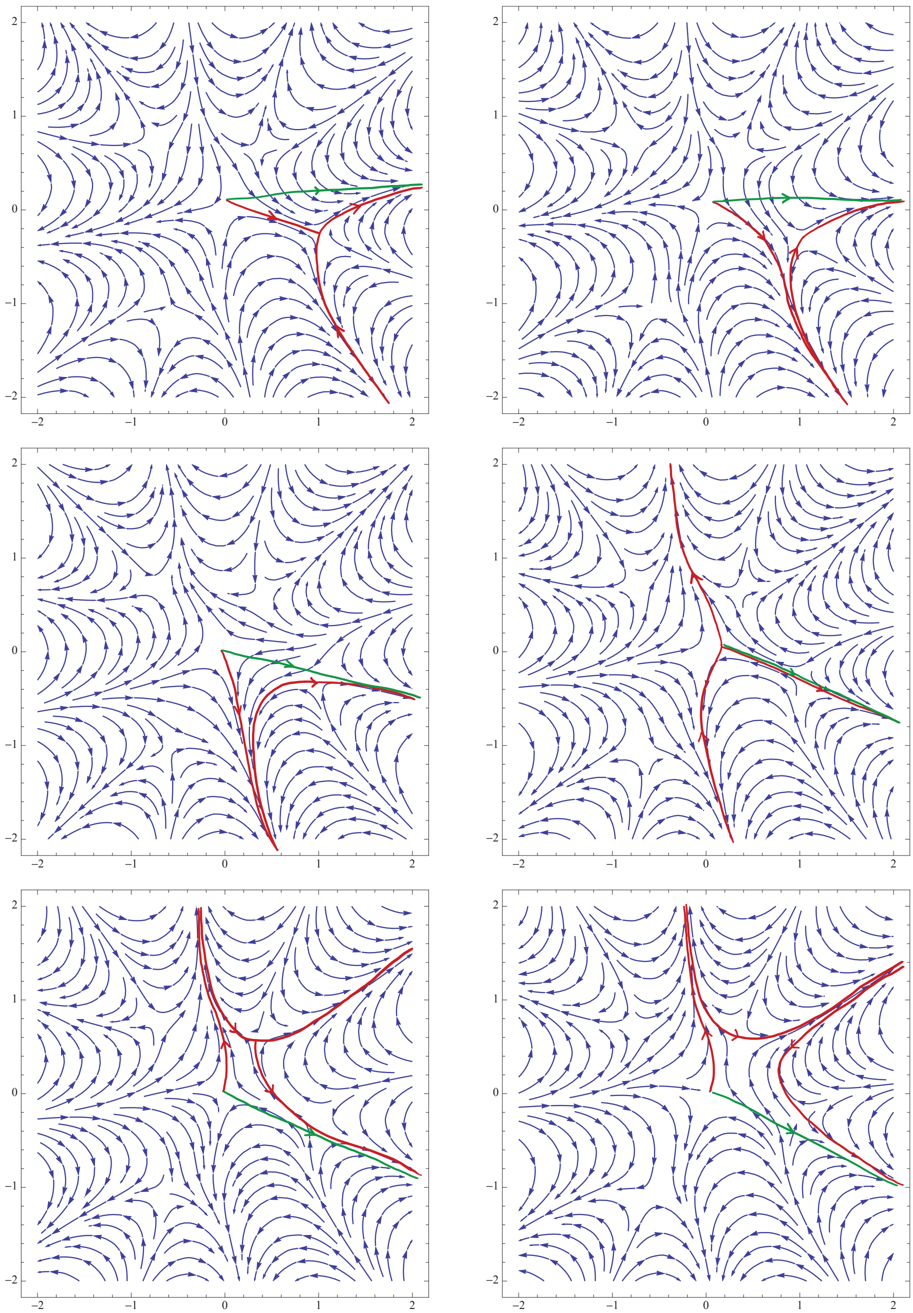

Figure 11.1. In this figure we represent the downward flow (11.6.2) and the mutations of Lefschetz thimbles for $|z| = 10^5$, and $|\arg z - \arg \overline{f(\beta_+)}| < \pi$ for the pair $(u_1, u_2) = (2, 3e^{\frac{4\pi i}{3}})$. Lefschetz thimbles are in red. The path of integration in equation (11.6.1) is drawn in green. *To be continued on the next page.*

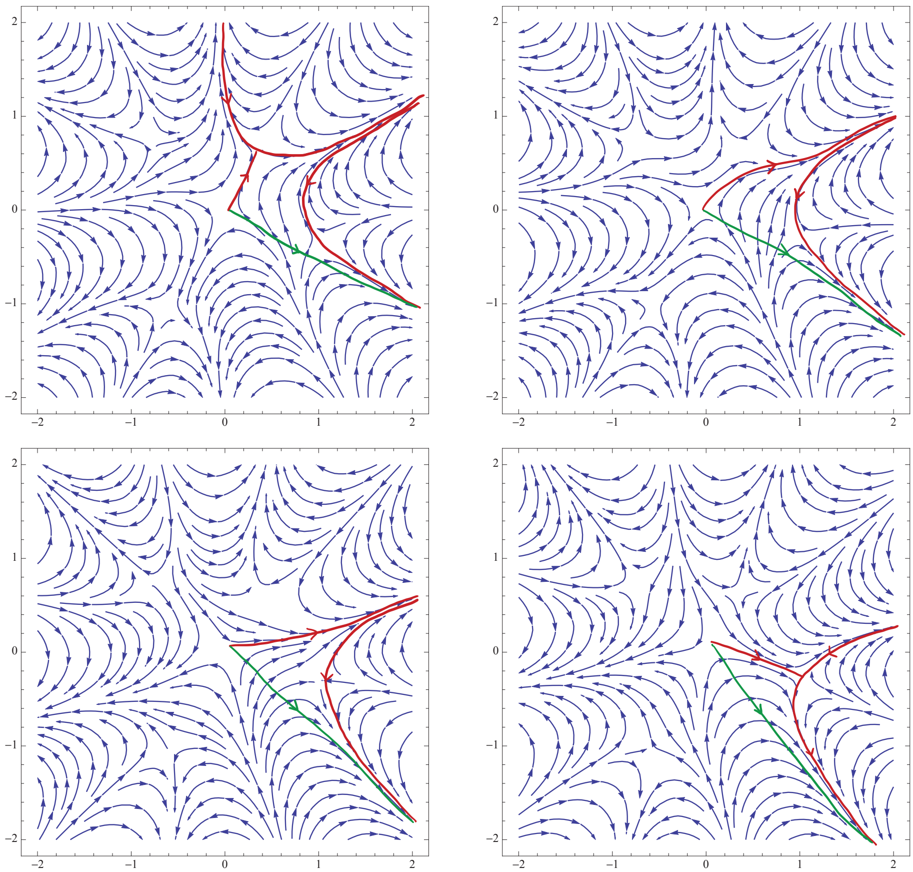

Figure 11.1 (continued). Notice that, for a certain range of values of arg z, there is also a contribution in the asymptotic expansion coming from a third critical point. Such a term is negligible, since it is dominated by the exponential term from the critical point β_+.

u_1	u_2	A'	B'
2	3	$-\pi$	$\frac{\pi}{3}$
2	$3e^{\frac{2\pi i}{3}}$	−3.71775	0.471036
2	$3e^{\frac{4\pi i}{3}}$	−1.00423	1.62336
−2	3	$-\pi$	$\frac{\pi}{3}$
−2	$3e^{\frac{2\pi i}{3}}$	−1.00423	−0.706554
−2	$3e^{\frac{4\pi i}{3}}$	−1.62336	1.00423

Table 11.2. In this table we represent the values A' and B' predicted in Proposition 11.6.4 for all possible values of u_1 and u_2.

Proof. The statement follows by application of the steepest descent path method and Lemma 11.6.2. Notice that the sector $A' < \arg z < B'$ is chosen so that the critical point of the logarithm of the integrand lies in the region (11.6.5) of validity of the asymptotic expansions (11.6.4). ■

11.7 Stokes basis of the qDE of $\mathbb{F}_1$

Set

$$s_{ij} := s_i \otimes p_j \in \mathcal{S}(\mathbb{P}^1) \otimes \mathcal{S}(\mathbb{P}^2)$$

for $i = 1, 2$ and $j = 1, 2, 3$. See equation (11.4.15).

Theorem 11.7.1. *The following linear combinations of the tensors s_{ij} define a basis of $\mathcal{H}$:*

$$s_{11} - 5s_{22} - 6s_{23}, \quad s_{12} + s_{23}, \quad s_{13} - s_{22} - 2s_{23}, \quad s_{21} - 4s_{22} - 5s_{23}.$$

Proof. Define the column vectors

- $\boldsymbol{g} = (g_1, g_2)^T$ and $\boldsymbol{s} = (s_1, s_2)^T$, bases of $\mathcal{S}(\mathbb{P}^1)$,
- $\boldsymbol{h} = (h_1, h_2, h_3)^T$ and $\boldsymbol{p} = (p_1, p_2, p_3)^T$, bases of $\mathcal{S}(\mathbb{P}^2)$, respectively.

In what follow we denote by $A \otimes B$ the Kronecker tensor product of two matrices A and B. Hence we denote

- by $\boldsymbol{g} \otimes \boldsymbol{h}$ the basis $(g_i \otimes h_j)_{i,j}$ of $\mathcal{S}(\mathbb{P}^1) \otimes \mathcal{S}(\mathbb{P}^2)$,
- by $\boldsymbol{s} \otimes \boldsymbol{p}$ the basis $(s_i \otimes p_j)_{i,j}$ of $\mathcal{S}(\mathbb{P}^1) \otimes \mathcal{S}(\mathbb{P}^2)$.

We have

$$\boldsymbol{g} \otimes \boldsymbol{h} = [(M_1 M_2^{-1}) \otimes (N_1 N_2^{-1})] \boldsymbol{s} \otimes \boldsymbol{p}, \tag{11.7.1}$$

where we represent the basis $\boldsymbol{g} \otimes \boldsymbol{h}$ and $\boldsymbol{s} \otimes \boldsymbol{p}$ as column vectors. Multiply on the left both sides of (11.7.1) by the matrix

$$E_1 := \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & \frac{1}{3} & 0 & \frac{1}{4} & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

We thus obtain the relation

$$\boldsymbol{s} \otimes \boldsymbol{p} = X \begin{pmatrix} g_1 \otimes h_1 \\ g_1 \otimes h_2 \\ g_1 \otimes h_3 \\ \frac{1}{3} g_1 \otimes h_2 + \frac{1}{4} g_2 \otimes h_1 \\ g_2 \otimes h_2 \\ g_2 \otimes h_3 \end{pmatrix}, \tag{11.7.2}$$

where X is the matrix

$$X = [(M_1 M_2^{-1}) \otimes (N_1 N_2^{-1})]^{-1} E_1^{-1}$$
$$= \begin{pmatrix} 54 & 36(\gamma + 11i\pi) & * & * & * & * \\ -54 & -36(\gamma + i\pi) & * & * & * & * \\ 54 & 36(\gamma + 3i\pi) & * & * & * & * \\ 54 & 36(\gamma + 9i\pi) & * & * & * & * \\ -54 & -36(\gamma - i\pi) & * & * & * & * \\ 54 & 36(\gamma + i\pi) & * & * & * & * \end{pmatrix}.$$

Multiply on the left each sides of (11.7.2) by the matrix

$$E_2 = \begin{pmatrix} 1 & 0 & 0 & 0 & -5 & -6 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & -1 & -2 \\ 0 & 0 & 0 & 1 & -4 & -5 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

We obtain

$$\begin{pmatrix} s_{11} - 5s_{22} - 6s_{23} \\ s_{12} + s_{23} \\ s_{13} - s_{22} - 2s_{23} \\ s_{21} - 4s_{22} - 5s_{23} \\ s_{22} + s_{23} \\ s_{23} \end{pmatrix} = E_2 X \begin{pmatrix} g_1 \otimes h_1 \\ g_1 \otimes h_2 \\ g_1 \otimes h_3 \\ \frac{1}{3} g_1 \otimes h_2 + \frac{1}{4} g_2 \otimes h_1 \\ g_2 \otimes h_2 \\ g_2 \otimes h_3 \end{pmatrix},$$

and we have

$$E_2 X = \left(\begin{array}{cc|cccc} 0 & 0 & & & & \\ 0 & 0 & & C_1 & & \\ 0 & 0 & & & & \\ 0 & 0 & & & & \\ \hline 0 & 72i\pi & * & * & * & * \\ 54 & 36(\gamma + i\pi) & * & * & * & * \end{array}\right). \qquad (11.7.3)$$

This proves the claim. ∎

Remark 11.7.2. The matrix C_1 in equation (11.7.3) is

$$\begin{pmatrix} 24(-3i\gamma - 2\pi)\pi & -216i\pi & 36\pi(-5i\gamma + 9\pi) & 3\pi(-42i\gamma^2 + 92\gamma\pi + 17i\pi^2) \\ 72i\gamma\pi & 216i\pi & 36\pi(5i\gamma + \pi) & 3\pi(42i\gamma^2 + 12\gamma\pi - i\pi^2) \\ -72i\gamma\pi & -216i\pi & 36\pi(-5i\gamma + \pi) & 3\pi(-42i\gamma^2 + 12\gamma\pi + i\pi^2) \\ -48\pi^2 & 0 & 0 & -48\gamma\pi^2 \end{pmatrix}.$$

Corollary 11.7.3. *The functions*

$$\Sigma_1 := \mathscr{P}(s_{11} - 5s_{22} - 6s_{23}), \quad \Sigma_2 := \mathscr{P}(s_{12} + s_{23}),$$
$$\Sigma_3 := \mathscr{P}(s_{13} - s_{22} - 2s_{23}), \quad \Sigma_4 := \mathscr{P}(s_{21} - 4s_{22} - 5s_{23})$$

define a basis of solutions of the qDE of $\mathbb{F}_1$. ∎

Proposition 11.7.4. *The Stokes basis* Ξ_R *of* $\mathscr{H}_0^{\mathrm{od}}$ *on the sector* $\Pi_R(\varepsilon)$ *can be reconstructed, using formulas* (11.2.2)–(11.2.5), *from a basis* $\boldsymbol{\Sigma_\lambda}$ *of solutions of the qDE of* $\mathbb{F}_1$ *of the form*

$$\lambda_1\Sigma_2, \quad \lambda_2\Sigma_3+\lambda_3\Sigma_2, \quad \lambda_4\Sigma_4+\lambda_5\Sigma_3+\lambda_6\Sigma_2, \quad \lambda_7\Sigma_1+\lambda_8\Sigma_4+\lambda_9\Sigma_3+\lambda_{10}\Sigma_2,$$

for a suitable choice of the coefficients $\lambda_j \in \mathbb{C}$, *with* $j = 1, \ldots, 10$.

Proof. The canonical coordinates x_1, x_2, x_3, x_4 are in lexicographical order with respect to a line of slope $\varepsilon > 0$ sufficiently small. The functions above have the expected exponential growth $\exp(x_i z)$ in the sector $\Pi_R(\varepsilon)$ defined by an oriented line of slope ε. This follows from the data in Tables 11.1 and 11.2, and from the configuration of the Stokes rays $R_{ij} := \{-r\sqrt{-1}(\overline{x_i} - \overline{x_j}) : r \in \mathbb{R}_+\}$: these are given by

$$R_{12} = \{\arg z = \pi\}, \qquad R_{13} = \{\arg z = 2.36573\},$$
$$R_{14} = \{\arg z = 1.88197\}, \quad R_{23} = \{\arg z = 0.775863\},$$
$$R_{24} = \{\arg z = 1.25962\}, \quad R_{34} = \left\{\arg z = \frac{\pi}{2}\right\},$$

see Figure 11.2. ∎

Remark 11.7.5. Notice that, according to Proposition 11.6.4, the function Σ_3 has the expected exponential growth $\exp(zx_2)$ in the sector in which this is minimal with respect to the dominance relation, i.e. in which it is dominated by any other exponential $\exp(zx_1), \exp(zx_3), \exp(zx_4)$. Hence, we expect that $\lambda_3 = 0$.

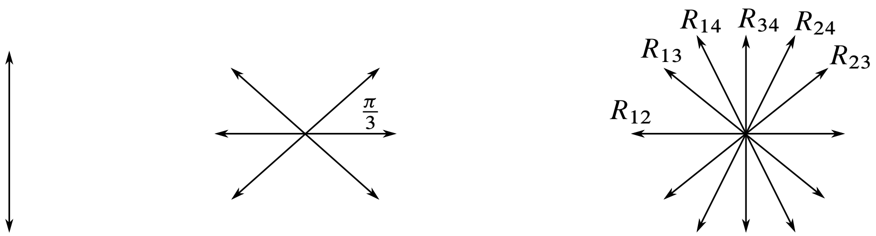

Figure 11.2. From the left to the right: Stokes rays corresponding to the origin of the quantum cohomology of $\mathbb{P}^1$, $\mathbb{P}^2$, and $\mathbb{F}_1$, respectively.

11.8 Computation of the central connection and Stokes matrices

Denote by $\mathcal{H}_0''$ the system of differential equations $\mathcal{H}_0^{\mathrm{od}}$ specialized at $0 \in QH^\bullet(\mathbb{F}_1)$. Consider the fundamental system of solutions of $\mathcal{H}_0''$

$$\Xi_\Upsilon(z) := \begin{pmatrix} z\Upsilon_1(z) & z\Upsilon_2(z) & z\Upsilon_3(z) & z\Upsilon_4(z) \\ \vdots & \vdots & \vdots & \vdots \end{pmatrix},$$

reconstructed from the basis $(\Upsilon_1, \Upsilon_2, \Upsilon_3, \Upsilon_4)$ of the qDE of $\mathbb{F}_1$ (see Corollary 11.5.2) by formulas (11.2.2)–(11.2.5).

Proposition 11.8.1. *We have*

$$\Xi_\Upsilon(z) = \Xi_{\mathrm{top}}(z) \cdot C_0,$$

where

$$C_0 := \begin{pmatrix} \frac{1}{18} & 0 & 0 & 0 \\ -\frac{\gamma}{18} & \frac{1}{2} & 0 & 0 \\ -\frac{\gamma}{18} & -\frac{1}{2} & \frac{1}{3} & 0 \\ \frac{6\gamma^2+\pi^2}{72} & -\frac{\gamma}{2} & -\frac{\gamma}{3} & 1 \end{pmatrix}. \tag{11.8.1}$$

Proof. From Lemmata 11.4.8 and 11.4.11, we can compute the asymptotic expansions of $\Upsilon_i(z)$ for $z \to 0$. We have

$$\begin{aligned}
\Upsilon_1(z) = {} & \frac{1}{72}\left(16\log^2(z) - 20\gamma\log(z) + 6\gamma^2 + \pi^2\right) + \frac{1}{18}z(\log(z) - \gamma - 2) \\
& + \frac{1}{72}z^2\left(16\log^2(z) - 20\gamma\log(z) - 17\log(z) + 6\gamma^2 + \pi^2 + 13\gamma + 2\right) \\
& + \frac{z^3}{1944}\left(432\log^2(z) - 540\gamma\log(z) - 750\log(z) + 162\gamma^2\right. \\
& \qquad \left. + 27\pi^2 + 426\gamma + 311\right) + \cdots, \\
\Upsilon_2(z) = {} & \frac{1}{2}(\log(z) - \gamma) - \frac{z}{2} + \frac{1}{8}z^2(4\log(z) - 4\gamma + 5) \\
& + \frac{1}{36}z^3(18\log(z) - 18\gamma - 37) \\
& + \frac{1}{192}z^4(24\log(z) - 24\gamma + 13) + \cdots, \\
\Upsilon_3(z) = {} & -\frac{\gamma}{3} + \frac{2\log(z)}{3} + \frac{z}{3} + \frac{1}{12}z^2(8\log(z) - 4\gamma - 9) \\
& + \frac{1}{54}z^3(36\log(z) - 18\gamma - 17) \\
& + \frac{1}{288}z^4(48\log(z) - 24\gamma - 49) + \cdots, \\
\Upsilon_4(z) = {} & 1 + z^2 + z^3 + \frac{z^4}{4} + \cdots.
\end{aligned}$$

After some computations, one finds the first terms of the asymptotic expansion of $\Xi_\Upsilon(z)$ for $z \to 0$:

$$\Xi_\Upsilon(z) = \begin{pmatrix} \frac{z(16\log^2(z)-20\gamma\log(z)+6\gamma^2+\pi^2)}{72} & \frac{z(\log(z)-\gamma)}{2} & \frac{z(2\log(z)-\gamma)}{3} & z \\ \frac{\log(z)}{6}-\frac{\gamma}{9} & 0 & \frac{1}{3} & 0 \\ \frac{\log(z)}{9}+\frac{z(\log(z)-\gamma-1)}{18}-\frac{\gamma}{18} & \frac{1}{2}-\frac{z}{2} & \frac{z}{3} & 0 \\ \frac{z(2\log(z)-\gamma-1)}{18}+\frac{1}{18z} & \frac{z}{2} & 0 & 0 \end{pmatrix} + \text{h.o.t.}$$

The leading term of the asymptotic expansion of $\Xi_{\text{top}}(z)$ for $z \to 0$ is

$$\begin{aligned} \Xi_{\text{top}}(z) &= \eta z^\mu z^R + \text{h.o.t.} \\ &= \begin{pmatrix} 4z\log^2(z) & 3z\log(z) & 2z\log(z) & z \\ 3\log(z) & 1 & 1 & 0 \\ 2\log(z) & 1 & 0 & 0 \\ \frac{1}{z} & 0 & 0 & 0 \end{pmatrix} + \text{h.o.t.}, \end{aligned}$$

where $\mu = \operatorname{diag}(-1, 0, 0, 1)$ and R is the operator of $\cup$-multiplication by $c_1(\mathbb{F}_1)$ on $H^\bullet(\mathbb{F}_1, \mathbb{C})$, that is,

$$R = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 3 & 2 & 0 \end{pmatrix}.$$

By comparison of the leading terms of the asymptotic expansions of Ξ_Υ and Ξ_{top}, one obtains the matrix C_0 in formula (11.8.1). ■

Theorem 11.8.2. *The central connection and Stokes matrices at* $0 \in QH^\bullet(\mathbb{F}_1)$*, computed with respect to an admissible oriented line of slope* $\varepsilon > 0$ *sufficiently small, equal*

$$C = \begin{pmatrix} \frac{1}{2\pi} & -\frac{1}{2\pi} & \frac{1}{2\pi} & -\frac{1}{2\pi} \\ \frac{\gamma}{\pi} & -\frac{\gamma}{\pi} & i+\frac{\gamma}{\pi} & -i-\frac{\gamma}{\pi} \\ \frac{1}{2}\left(-i+\frac{\gamma}{\pi}\right) & -\frac{\gamma+i\pi}{2\pi} & \frac{1}{2}\left(-i+\frac{\gamma}{\pi}\right) & -\frac{\gamma+i\pi}{2\pi} \\ \gamma\left(-i+\frac{2\gamma}{\pi}\right) & \gamma\left(-i-\frac{2\gamma}{\pi}\right) & \frac{2\gamma(\gamma+i\pi)}{\pi} & -\frac{2(\gamma+i\pi)^2}{\pi} \end{pmatrix}, \tag{11.8.2}$$

$$S = \begin{pmatrix} 1 & 2 & -1 & -3 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{11.8.3}$$

Proof. Denote

- by $\Xi_{\boldsymbol{\lambda}}$ the fundamental system of solutions of $\mathscr{H}_0''$ constructed from the basis $\boldsymbol{\Sigma}_{\boldsymbol{\lambda}}$ of Proposition 11.7.4,

- by Ξ_{Σ} the fundamental system of solutions of $\mathscr{H}_0''$ constructed from the basis $\boldsymbol{\Sigma}$ of Corollary 11.7.3.

We have

$$\Xi_{\boldsymbol{\lambda}} = \Xi_{\Sigma}\cdot\begin{pmatrix}0&0&0&\lambda_7\\ \lambda_1&\lambda_3&\lambda_6&\lambda_{10}\\ 0&\lambda_2&\lambda_5&\lambda_9\\ 0&0&\lambda_4&\lambda_8\end{pmatrix} = \Xi_{\Upsilon}\Pi^T C_1^T\begin{pmatrix}0&0&0&\lambda_7\\ \lambda_1&\lambda_3&\lambda_6&\lambda_{10}\\ 0&\lambda_2&\lambda_5&\lambda_9\\ 0&0&\lambda_4&\lambda_8\end{pmatrix},$$

where C_1 is as in Remark 11.7.2 and

$$\Pi := \begin{pmatrix}0&1&0&0\\ 1&0&0&0\\ 0&0&1&0\\ 0&0&0&1\end{pmatrix}.$$

Thus, we obtain

$$\Xi_{\boldsymbol{\lambda}} = \Xi_{\mathrm{top}}C_{\boldsymbol{\lambda}},\quad C_{\boldsymbol{\lambda}} := C_0\Pi^T C_1^T\begin{pmatrix}0&0&0&\lambda_7\\ \lambda_1&\lambda_3&\lambda_6&\lambda_{10}\\ 0&\lambda_2&\lambda_5&\lambda_9\\ 0&0&\lambda_4&\lambda_8\end{pmatrix},$$

where C_0 is given by (11.8.1). In order to determine the values of $\boldsymbol{\lambda}$ for which $\Xi_{\boldsymbol{\lambda}}$ is the Stokes basis, let us compute the product

$$C_{\boldsymbol{\lambda}}^T\eta e^{\pi i\mu}e^{\pi iR}C_{\boldsymbol{\lambda}}. \tag{11.8.4}$$

If $\Xi_{\boldsymbol{\lambda}}$ is the Stokes basis, then the matrix above is the inverse of the Stokes matrix S, by equation (4.4.2): in particular, it is an upper triangular matrix with ones along the main diagonal. An explicit computation gives the following result: the columns of (11.8.4) are

$$\begin{pmatrix}-576\pi^4\lambda_1^2\\ -576\pi^4\lambda_1\lambda_3\\ -576\pi^4\lambda_1\lambda_6\\ -576\pi^4\lambda_1(3\lambda_7+\lambda_{10})\end{pmatrix},$$

$$\begin{pmatrix}576\pi^4\lambda_1(2\lambda_2-\lambda_3)\\ -576\pi^4(\lambda_2-\lambda_3)^2\\ -576\pi^4(\lambda_3\lambda_6+\lambda_2(\lambda_4+\lambda_5-2\lambda_6))\\ 576\pi^4(\lambda_2(\lambda_7-\lambda_8-\lambda_9+2\lambda_{10})-\lambda_3(3\lambda_7+\lambda_{10}))\end{pmatrix},$$

$$\begin{pmatrix}-576\pi^4\lambda_1(\lambda_4-2\lambda_5+\lambda_6)\\ -576\pi^4(\lambda_2\lambda_5+\lambda_3(\lambda_4-2\lambda_5+\lambda_6))\\ -576\pi^4(\lambda_4^2+(\lambda_5+\lambda_6)\lambda_4+(\lambda_5-\lambda_6)^2)\\ -576\pi^4(\lambda_6(3\lambda_7+\lambda_{10})+\lambda_4(5\lambda_7+\lambda_8+\lambda_{10})+\lambda_5(-\lambda_7+\lambda_8+\lambda_9-2\lambda_{10}))\end{pmatrix},$$

$$\begin{pmatrix} 576\pi^4\lambda_1(6\lambda_7-\lambda_8+2\lambda_9-\lambda_{10}) \\ -576\pi^4(\lambda_2(6\lambda_7+\lambda_9)+\lambda_3(-6\lambda_7+\lambda_8-2\lambda_9+\lambda_{10})) \\ -576\pi^4(\lambda_5(6\lambda_7+\lambda_9)+\lambda_4(6\lambda_7+\lambda_8+\lambda_9)+\lambda_6(-6\lambda_7+\lambda_8-2\lambda_9+\lambda_{10})) \\ -576\pi^4(13\lambda_7^2+(11\lambda_8+5\lambda_9-3\lambda_{10})\lambda_7+\lambda_8^2+(\lambda_9-\lambda_{10})^2+\lambda_8(\lambda_9+\lambda_{10})) \end{pmatrix}.$$

The matrix (11.8.4) is upper triangular with ones along the diagonal if and only if

$$\lambda_1^2=-\frac{1}{576\pi^4},\quad \lambda_2^2=-\frac{1}{576\pi^4},\quad \lambda_3=0,\quad \lambda_4^2=-\frac{1}{576\pi^4},\quad \lambda_5=-\lambda_4,$$
$$\lambda_6=0,\quad \lambda_7^2=-\frac{1}{576\pi^4},\quad \lambda_8=-2\lambda_7,\quad \lambda_9=-3\lambda_7,\quad \lambda_{10}=-3\lambda_7.$$

For the choice $\lambda_1=\lambda_2=\lambda_4=\lambda_7=-\frac{i}{24\pi^2}$, we obtain the central connection and Stokes matrices (11.8.2) and (11.8.3). ■

Theorem 11.8.3. *The central connection matrix of $QH^\bullet(\mathbb{F}_{2k+1})$, computed with respect to an oriented line of slope $\varepsilon>0$ sufficiently small, and a suitable choice of the branch of the Ψ-matrix, equals*

$$C_k=\begin{pmatrix} \frac{1}{2\pi} & -\frac{1}{2\pi} & \frac{1}{2\pi} & -\frac{1}{2\pi} \\ \frac{\gamma}{\pi} & -\frac{\gamma}{\pi} & i+\frac{\gamma}{\pi} & -i-\frac{\gamma}{\pi} \\ \frac{\gamma-2\gamma k-i\pi}{2\pi} & -\frac{\gamma-2\gamma k+i\pi}{2\pi} & \frac{-2\gamma k-i(2\pi k+\pi)+\gamma}{2\pi} & \frac{(2k-1)(\gamma+i\pi)}{2\pi} \\ \gamma\left(-i+\frac{2\gamma}{\pi}\right) & \gamma\left(-i-\frac{2\gamma}{\pi}\right) & \frac{2\gamma(\gamma+i\pi)}{\pi} & -\frac{2(\gamma+i\pi)^2}{\pi} \end{pmatrix}. \tag{11.8.5}$$

This is the matrix associated with the morphism

$$\mathcal{D}^-_{\mathbb{F}_{2k+1}}\colon K_0(\mathbb{F}_{2k+1})_{\mathbb{C}}\to H^\bullet(\mathbb{F}_{2k+1},\mathbb{C}),$$
$$[\mathscr{F}]\mapsto \frac{1}{2\pi}\widehat{\Gamma}^-_{\mathbb{F}_{2k+1}}\cup e^{-\pi i c_1(\mathbb{F}_{2k+1})}\cup \mathrm{Ch}(\mathscr{F}),$$

with respect to

- *an exceptional basis $\mathfrak{E}:=(E_i)_{i=1}^4$ of $K_0(\mathbb{F}_{2k+1})_{\mathbb{C}}$,*
- *the basis $(T_{i,2k+1})_{i=0}^3$ of $H^\bullet(\mathbb{F}_{2k+1},\mathbb{C})$.*

The exceptional basis $\mathfrak{E}$ mutates to the exceptional basis

$$\left([\mathcal{O}],[\mathcal{O}(\Sigma_2^{2k+1})],[\mathcal{O}(\Sigma_4^{2k+1})],[\mathcal{O}(\Sigma_2^{2k+1}+\Sigma_4^{2k+1})]\right), \tag{11.8.6}$$

by application of the following natural transformations:

(1) *action of the braid $\beta_3\beta_2\beta_1\beta_3\beta_2$,*
(2) *action of the element $\widetilde{J}_k\in(\mathbb{Z}/2\mathbb{Z})^4$*

$$\widetilde{J}_k:=\begin{cases}(-1,-1,(-1)^p,(-1)^{p+1}) & \text{if } k=2p,\\ (-1,-1,(-1)^{p+1},(-1)^{p+1}) & \text{if } k=2p+1,\end{cases}$$

(3) *action of the element β_3^k.*

Proof. Equations (9.3.11) and Proposition 4.5.1 imply equation (11.8.5). The matrix associated to $\varPi^{-}_{\mathbb{F}_{2k+1}}$ with respect to the basis (11.8.6) is

$$E_k := \begin{pmatrix} \frac{1}{2\pi} & \frac{1}{2\pi} & \frac{1}{2\pi} & \frac{1}{2\pi} \\ -i + \frac{\gamma}{\pi} & -i + \frac{\gamma}{\pi} & \frac{\gamma}{\pi} & \frac{\gamma}{\pi} \\ \frac{(1-2k)(\gamma - i\pi)}{2\pi} & \frac{-2\gamma k + i(2\pi k + \pi) + \gamma}{2\pi} & \frac{(1-2k)(\gamma - i\pi)}{2\pi} & \frac{-2\gamma k + i(2\pi k + \pi) + \gamma}{2\pi} \\ \frac{2(\gamma - i\pi)^2}{\pi} & \frac{2\gamma(\gamma - i\pi)}{\pi} & \gamma\left(-i + 2ik + \frac{2\gamma}{\pi}\right) & \gamma\left(i + 2ik + \frac{2\gamma}{\pi}\right) \end{pmatrix}.$$

Set $C'_k := C_k^{\beta_3\beta_2\beta_1\beta_3\beta_2}$. We have

$$(C'_k)^{-1} E_k = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1-k & -k \\ 0 & 0 & -k & -k-1 \end{pmatrix}.$$

It is now easy to see that this is the matrix representing the action of the element $(\widetilde{J}_k, \beta_3^k) \in (\mathbb{Z}/2\mathbb{Z})^4 \rtimes \mathcal{B}_4$: the argument is the same as in Step 3 of the proof of Theorem 10.3.3. ∎