Appendix A

Proof of Theorem 5.1.2

We need some preliminary results.

Lemma A.1. *For $n \geqslant 0$, and $\delta \in H^2(X, \mathbb{C})$, we have*

$$\langle\langle \tau_n T_\alpha, 1\rangle\rangle_0(\delta) = \frac{1}{(n+1)!}\left(\int_X T_\alpha \cup \delta^{n+1}\right) + \sum_{\beta\neq 0}\sum_{\nu\geqslant 0} \frac{\mathbf{Q}^\beta e^{\int_\beta \delta}}{\nu!}\langle \tau_{n-\nu} T_\alpha \cup \delta^\nu, 1\rangle^X_{0,2,\beta}.$$

Proof. We have

$$\langle\langle \tau_n T_\alpha, 1\rangle\rangle_0(\delta) = \frac{\partial}{\partial t^{\alpha,n}}\frac{\partial}{\partial t^{0,0}}\mathcal{F}_0^X\bigg|_\delta = \sum_{k=0}^{\infty}\sum_\beta \frac{\mathbf{Q}^\beta}{k!}\langle \tau_n T_\alpha, 1, \delta\ \ldots, \delta\rangle^X_{0,k+2,\beta}.$$

Two cases occur:

- If $\beta \neq 0$, then for $k \geqslant 0$ we have

$$\langle \tau_n T_\alpha, 1, \delta\ \ldots, \delta\rangle^X_{0,k+2,\beta} = \sum_{\mu+\nu=k}\frac{k!}{\mu!\nu!}\left(\int_\beta \delta\right)^\mu \langle \tau_{n-\nu} T_\alpha \cup \delta^\nu, 1\rangle^X_{0,2,\beta},$$

  by the divisor axiom of Gromov–Witten invariants. Here any invariant with τ_{-r} with $r > 0$ is vanishing.
- If $\beta = 0$, then for $k > 0$ by the divisor axiom we have[1]

$$\langle \tau_n T_\alpha, 1, \delta\ \ldots, \delta\rangle^X_{0,k+2,0} = \langle \tau_{n-k+1} T_\alpha \cup \delta^k, 1, \delta\rangle_{0,3,0} = \left(\int_X T_\alpha \cup \delta^k\right)\delta_{k,n+1}.$$

So, we obtain

$$\begin{aligned}\langle\langle \tau_n T_\alpha, 1\rangle\rangle_0(\delta) &= \frac{1}{(n+1)!}\left(\int_X T_\alpha \cup \delta^{n+1}\right)\\ &\quad + \sum_{\beta\neq 0}\sum_{k\geqslant 0}\frac{\mathbf{Q}^\beta}{k!}\sum_{\mu+\nu=k}\frac{k!}{\mu!\nu!}\left(\int_\beta \delta\right)^\mu \langle \tau_{n-\nu} T_\alpha \cup \delta^\nu, 1\rangle^X_{0,2,\beta}\\ &= \frac{1}{(n+1)!}\left(\int_X T_\alpha \cup \delta^{n+1}\right)\\ &\qquad + \sum_{\beta\neq 0}\sum_{\nu\geqslant 0}\frac{\mathbf{Q}^\beta e^{\int_\beta \delta}}{\nu!}\langle \tau_{n-\nu} T_\alpha \cup \delta^\nu, 1\rangle^X_{0,2,\beta}.\end{aligned}$$

■

[1] Here, we use the fact that $\mathcal{L}_1$ is trivial on $\overline{\mathcal{M}}_{0,3}(X, 0)$ and hence has zero Chern class. This follows from the fact that $\overline{\mathcal{M}}_{0,3}(X, 0) \cong X$, and the forgetful morphism $\overline{\mathcal{M}}_{0,4}(X, 0) \to \overline{\mathcal{M}}_{0,3}(X, 0)$ is the projection $X \times \overline{\mathcal{M}}_{0,4} \to X$.

Lemma A.2. *Let $\delta \in H^2(X,\mathbb{C})$. We have*

$$J_X(\delta) = e^{\frac{\delta}{\hbar}} + \sum_{\alpha}\sum_{\beta\neq 0}\sum_{n=0}^{\infty}\sum_{k+p=n} \hbar^{-(n+1)}\frac{\mathbf{Q}^{\beta} e^{\int_\beta \delta}}{p!}\langle \tau_k T_\alpha \cup \delta^p, 1\rangle_{0,2,\beta}^{X} T^{\alpha}.$$

Proof. By Lemma A.1, we have

$$\begin{aligned} J_X(\delta) &= 1 + \sum_{\alpha}\sum_{n=0}^{\infty}\frac{\hbar^{-(n+1)}}{(n+1)!}\left(\int_X T_\alpha \cup \delta^{n+1}\right)T^{\alpha} \\ &\quad + \sum_{\alpha}\sum_{\beta\neq 0}\sum_{n=0}^{\infty}\sum_{k+p=n} \hbar^{-(n+1)}\frac{\mathbf{Q}^{\beta} e^{\int_\beta \delta}}{p!}\langle \tau_k T_\alpha \cup \delta^p, 1\rangle_{0,2,\beta}^{X} T^{\alpha} \\ &= e^{\frac{\delta}{\hbar}} + \sum_{\alpha}\sum_{\beta\neq 0}\sum_{n=0}^{\infty}\sum_{k+p=n} \hbar^{-(n+1)}\frac{\mathbf{Q}^{\beta} e^{\int_\beta \delta}}{p!}\langle \tau_k T_\alpha \cup \delta^p, 1\rangle_{0,2,\beta}^{X} T^{\alpha}. \end{aligned}$$

■

Lemma A.3. *For $\delta \in H^2(X,\mathbb{C})$, we have*

$$\begin{aligned} Z_{\mathrm{top}}(\delta,z)T_\alpha &= e^{z\delta}\cup z^{\mu}z^{c_1(X)}T_\alpha \\ &\quad + \sum_{\beta\neq 0}\sum_{\lambda} e^{\int_\beta \delta}\left\langle \frac{z e^{z\delta}}{1-z\psi}\cup z^{\mu}z^{c_1(X)}T_\alpha, T_\lambda\right\rangle_{0,2,\beta}^{X} T^{\lambda}. \end{aligned} \tag{A.1}$$

Proof. For $\boldsymbol{\tau} \in H^\bullet(X,\mathbb{C})$, we have

$$\begin{aligned} \Theta(\boldsymbol{\tau},z)T_\alpha &= \sum_{\varepsilon}\Theta(\boldsymbol{\tau},z)_{\alpha}^{\varepsilon}T_\varepsilon \\ &= \sum_{\lambda}\left.\frac{\partial\theta_\alpha}{\partial t^{\lambda}}\right|_{(\boldsymbol{\tau},z)} T^{\lambda} \\ &= \sum_{\lambda}\sum_{p=0}^{\infty} z^{p}\langle\langle \tau_p T_\alpha, 1, T_\lambda\rangle\rangle_0(\boldsymbol{\tau})|_{\mathbf{Q}=1}T^{\lambda} \\ &= \sum_{\lambda}\sum_{p=0}^{\infty}\sum_{k=0}^{\infty}\sum_{\beta}\frac{z^{p}}{k!}\langle \tau_p T_\alpha, 1, T_\lambda, \boldsymbol{\tau},\dots,\boldsymbol{\tau}\rangle_{0,3+k,\beta}^{X} T^{\lambda}. \end{aligned}$$

Consider the contribution coming from the fact that $(k,\beta)=(0,0)$: by the mapping-to-point axiom of Gromov–Witten invariants, we have[2]

$$\begin{aligned} \sum_{\lambda}\sum_{p=0}^{\infty} z^{p}\langle \tau_p T_\alpha, 1, T_\lambda,\rangle_{0,3,0}^{X} T^{\lambda} &= \sum_{\lambda}\sum_{p=0}^{\infty} z^{p}\left(\int_X T_\alpha\cup T_\lambda\right)\delta_{0,p}T^{\lambda} \\ &= T_\alpha. \end{aligned}$$

[2]Also here, we use the fact that $\mathcal{L}_1$ is trivial on $\overline{\mathcal{M}}_{0,3}(X,0)$.

By the fundamental class axiom, instead, the contribution from $(k,\beta) \neq (0,0)$ can be rewritten as

$$\sum_{\lambda}\sum_{p=0}^{\infty}\sum_{k=1}^{\infty}\sum_{\beta\neq 0}\frac{z^p}{k!}\langle\tau_{p-1}T_\alpha, T_\lambda, \boldsymbol{\tau}, \dots, \boldsymbol{\tau}\rangle^X_{0,2+k,\beta}T^\lambda.$$

Thus, we have recovered the formula

$$\Theta(\boldsymbol{\tau}, z) = \mathrm{Id} + \sum_{\lambda}\sum_{p=0}^{\infty} z^{p+1}\langle\!\langle\tau_p(-), T_\lambda\rangle\!\rangle_0(\boldsymbol{\tau})|_{\mathbf{Q}=1}T^\lambda,$$

which was used in [23, Proposition 7.1] to define Θ. At this point the proof is known, and can be found in [27, Proposition 10.2.3]: the parameter $\hbar$ of [27] has to be replaced by our z, and pre-composition with $z^\mu z^{c_1(X)}$ has to be taken into account in order to obtain formula (A.1). ■

We are now ready for the proof of Theorem 5.1.2.

Proof of Theorem 5.1.2. Let us compute the entries of the first row of the matrix

$$\eta\Theta(\delta, z)z^\mu z^{c_1(X)}.$$

By Lemma A.3, we have

$$\begin{aligned}
&\left[\eta\Theta(\delta,z)z^\mu z^{c_1(X)}\right]^1_\alpha\\
&\quad= \eta\big(1, \Theta(\delta,z)z^\mu z^{c_1(X)}T_\alpha\big)\\
&\quad= \eta\Bigg(1, e^{z\delta}\cup z^\mu z^{c_1(X)}T_\alpha\\
&\qquad\qquad + \sum_{\beta\neq 0}\sum_{\lambda} e^{\int_\beta\delta}\left\langle\frac{ze^{z\delta}}{1-z\psi}\cup z^\mu z^{c_1(X)}T_\alpha, T_\lambda\right\rangle^X_{0,2,\beta}T^\lambda\Bigg)\\
&\quad= \eta(1, e^{z\delta}\cup z^\mu z^{c_1(X)}T_\alpha)\\
&\qquad\qquad + \eta\left(1, \sum_{\beta\neq 0}\sum_{\lambda} e^{\int_\beta\delta}\left\langle\frac{ze^{z\delta}}{1-z\psi}\cup z^\mu z^{c_1(X)}T_\alpha, T_\lambda\right\rangle^X_{0,2,\beta}T^\lambda\right).
\end{aligned}$$

Using the identity of endomorphisms of $H^\bullet(X,\mathbb{C})$

$$z^{-\mu}\circ(h^k\cup)\circ z^\mu = z^{-k}(h^k\cup), \quad h\in H^2(X,\mathbb{C}), \quad k\in\mathbb{N},$$

and the η-skew-symmetry of μ, we can rewrite the first summand as

$$\begin{aligned}
\eta(1, e^{z\delta}\cup z^\mu z^{c_1(X)}T_\alpha) &= \eta(1, z^\mu e^\delta z^{c_1(X)}T_\alpha)\\
&= \eta(z^{-\mu}(1), e^\delta z^{c_1(X)}T_\alpha)\\
&= z^{\frac{\dim_{\mathbb{C}} X}{2}}\int_X e^\delta z^{c_1(X)}T_\alpha.
\end{aligned}$$

For the second summand, notice that

(1) the only nonzero contribution comes from $\lambda = 0$,

(2) for any $\varphi \in H^\bullet(X, \mathbb{C})$ we have

$$\frac{ze^{z\delta}}{1 - z\psi} \cup \varphi = \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{z^{n+1}}{(n-k)!} \psi^k \delta^{n-k} \varphi,$$

(3) we have

$$z^\mu z^{c_1(X)} T_\alpha = \sum_{\ell=0}^{\infty} \frac{(\log z)^\ell}{\ell!} z^{\frac{2\ell + \deg T_\alpha - \dim X}{2}} c_1(X)^\ell T_\alpha,$$

(4) the Gromov–Witten invariant

$$\langle \tau_k \delta^{n-k} c_1(X)^\ell T_\alpha, 1\rangle^X_{0,2,\beta}$$

is nonzero only if

$$2k + 2(n-k) + 2\ell + \deg T_\alpha = 2\dim_{\mathbb{C}} X + 2\int_\beta c_1(X) - 2.$$

So, we obtain

$$\begin{aligned}
&\left\langle \frac{ze^{z\delta}}{1 - z\psi} \cup z^\mu z^{c_1(X)} T_\alpha, 1 \right\rangle^X_{0,2,\beta} \\
&= \sum_{n=0}^{\infty} \sum_{k=0}^{n} \sum_{\ell=0}^{\infty} \frac{(\log z)^\ell}{\ell!(n-k)!} z^{n+1+\frac{2\ell + \deg T_\alpha - \dim X}{2}} \langle \tau_k \delta^{n-k} c_1(X)^\ell T_\alpha, 1\rangle^X_{0,2,\beta} \\
&= z^{\frac{\dim X}{2}} z^{\int_\beta c_1(X)} \sum_{h=0}^{\infty} \sum_{m+\ell+k=h} \frac{(\log z)^\ell}{\ell! m!} \langle \tau_k \delta^m c_1(X)^\ell T_\alpha, 1\rangle^X_{0,2,\beta} \\
&= z^{\frac{\dim X}{2}} z^{\int_\beta c_1(X)} \sum_{h=0}^{\infty} \sum_{k+p=h} \frac{1}{p!} \langle \tau_k (\delta + \log z \cdot c_1(X))^p T_\alpha, 1\rangle^X_{0,2,\beta}.
\end{aligned}$$

Putting this all together, we obtain

$$\begin{aligned}
&\left[\eta \Theta(\delta, z) z^\mu z^{c_1(X)}\right]^1_\alpha \\
&= z^{\frac{\dim X}{2}} \Bigg(\int_X e^\delta z^{c_1(X)} T_\alpha \\
&\qquad + \sum_{\beta \neq 0} e^{\int_\beta \delta} z^{\int_\beta c_1(X)} \sum_{h=0}^{\infty} \sum_{k+p=h} \frac{1}{p!} \langle \tau_k (\delta + \log z \cdot c_1(X))^p T_\alpha, 1\rangle^X_{0,2,\beta} \Bigg) \\
&= z^{\frac{\dim X}{2}} \int_X T_\alpha \cup J_X(\delta + \log z \cdot c_1(X)) \Big|_{\substack{\mathbf{Q}=1 \\ \hbar = 1}}.
\end{aligned}$$

The last equality follows by Lemma A.2. This completes the proof. ∎