# The Work of Richard Ewen Borcherds

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## 1 Introduction

Richard Borcherds has used the study of certain exceptional and exotic algebraic structures to motivate the introduction of important new algebraic concepts: vertex algebras and generalized Kac-Moody algebras, and he has demonstrated their power by using them to prove the "moonshine conjectures" of Conway and Norton about the Monster Group and to find whole new families of automorphic forms.

A central thread in his research has been a particular Lie algebra, now known as the Fake Monster Lie algebra, which is, in a certain sense, the simplest known example of a generalized Kac-Moody algebra which is not finite-dimensional or affine (or a sum of such algebras). As the name might suggest, this algebra *appears* to have something to do with the Monster group, *i.e.* the largest sporadic finite simple group.

The story starts with the observation that the Leech lattice can be interpreted as the Dynkin diagram for a Kac-Moody algebra, $\mathcal{L}_\infty$. But $\mathcal{L}_\infty$ is difficult to handle; its root multiplicities are not known explicitly. Borcherds showed how to enlarge it to obtain the more amenable Fake Monster Lie algebra. In order to construct this algebra, Borcherds introduced the concept of a vertex algebra, in the process establishing a comprehensive algebraic approach to (two-dimensional) conformal field theory, a subject of major importance in theoretical physics in the last thirty years.

To provide a general context for the Fake Monster Lie algebra, Borcherds has developed the theory of generalized Kac-Moody algebras, proving, in particular, generalizations of the Kac-Weyl character and denominator formulae. The denominator formula for the Fake Monster Lie algebra motivated Borcherds to construct a "real" Monster Lie algebra, which he used to prove the moonshine conjectures. The results for the Fake Monster Lie algebra also motivated Borcherds to explore the properties of the denominator formula for other generalized Kac-Moody algebras, obtaining remarkable product expressions for modular functions, results on the moduli spaces of certain complex surfaces and much else besides.

## 2 The Leech Lattice and the Kac-Moody Algebra $\mathcal{L}_\infty$

We start by recalling that a finite-dimensional simple complex Lie algebra, $\mathcal{L}$, can be expressed in terms of generators and relations as follows. There is a nonsingular invariant bilinear form $(\,,)$ on $\mathcal{L}$ which induces such a form on the $\operatorname{rank}\mathcal{L}$ dimensional space spanned by the roots of $\mathcal{L}$. Suppose $\{\alpha_i : 1 \le i \le \operatorname{rank}\mathcal{L}\}$ is a basis of simple roots for $\mathcal{L}$. Then the numbers $a_{ij} = (\alpha_i, \alpha_j)$ have the following properties:

$$a_{ii} > 0, \quad (1)$$

$$a_{ij} = a_{ji}, \quad (2)$$

$$a_{ij} \leq 0 \quad \text{if } i \neq j, \quad (3)$$

$$2a_{ij}/a_{ii} \in \mathbb{Z}. \quad (4)$$

The symmetric matrix $A = (a_{ij})$ obtained in this way is positive definite.

The algebra $\mathcal{L}$ can be reconstructed from the matrix $A$ by the system of generators and relations used to define $\mathcal{L}_\infty$,

$$[e_i, f_i] = h_i, \qquad [e_i, f_j] = 0 \quad \text{for } i \neq j, \quad (5)$$

$$[h_i, e_j] = a_{ij} e_j, \qquad [h_i, f_j] = -a_{ij} f_j, \quad (6)$$

$$\mathrm{Ad}(e_i)^{n_{ij}}(e_j) = \mathrm{Ad}(f_i)^{n_{ij}}(f_j) = 0, \quad \text{for } n_{ij} = 1 - 2a_{ij}/a_{ii}. \quad (7)$$

These relations can be used to define a Lie algebra, $\mathcal{L}_A$, for any matrix $A$ satisfying the conditions (1-4). $\mathcal{L}_A$ is called a (symmetrizable) Kac-Moody algebra. If $A$ is positive definite, $\mathcal{L}_A$ is semi-simple and, if $A$ is positive semi-definite, $\mathcal{L}_A$ is a sum of affine and finite-dimensional algebras.

Although Kac and Moody only explicitly considered the situation in which the number of simple roots was finite, the theory of Kac-Moody algebras applies to algebras which have a infinite number of simple roots. Borcherds and others [1] showed how to construct such an algebra with simple roots labelled by the points of the Leech lattice, $\Lambda_L$. We can conveniently describe $\Lambda_L$ as a subset of the unique even self-dual lattice, $\mathrm{II}_{25,1}$, in 26-dimensional Lorentzian space, $\mathbb{R}^{25,1}$. $\mathrm{II}_{25,1}$ is the set of points whose coordinates are all either integers or half odd integers which have integral inner product with the vector $(\frac{1}{2}, \dots, \frac{1}{2}; \frac{1}{2}) \in \mathbb{R}^{25,1}$, where the norm of $x = (x_1, x_2, \dots, x_{25}; x_0)$ is $x^2 = x_1^2 + x_2^2 + \dots + x_{25}^2 - x_0^2$.

The vector $\rho = (0, 1, 2, \dots, 24; 70) \in \mathrm{II}_{25,1}$ has zero norm, $\rho^2 = 0$; the Leech lattice can be shown to be isomorphic to the set $\{x \in \mathrm{II}_{25,1} : x \cdot \rho = -1\}$ modulo displacements by $\rho$. We can take the representative points for the Leech lattice to have norm 2 and so obtain an isometric correspondence between $\Lambda_L$ and

$$\{r \in \mathrm{II}_{25,1} : r \cdot \rho = -1, r^2 = 2\}. \quad (8)$$

Then, with each point $r$ of the Leech lattice, we can associate a reflection $x \mapsto \sigma_r(x) = x - (r \cdot x)r$ which is an automorphism of $\mathrm{II}_{25,1}$. Indeed these reflections $\sigma_r$ generate a Weyl group, $W$, and the whole automorphism group of $\mathrm{II}_{25,1}$ is the semi-direct product of $W$ and the automorphism group of the affine Leech lattice, which is the Dynkin/Coxeter diagram of the Weyl group $W$. To this Dynkin diagram can be associated an infinite-dimensional Kac-Moody algebra, $\mathcal{L}_\infty$, generated by elements $\{e_r, f_r, h_r : r \in \Lambda_L\}$ subject to the relations (5-7). Dividing by the linear combinations of the $h_r$ which are in the centre reduces its rank to 26.

The point about Kac-Moody algebras is that they share many of the properties enjoyed by semi-simple Lie algebras. In particular, we can define a Weyl group,

$W$, and for suitable (*i.e.* lowest weight) representations, there is a straightforward generalization of the Weyl character formula. For a representation with lowest weight $\lambda$, this generalization, the Weyl-Kac character formula, states

$$\chi_\lambda = \sum_{w\in W} \det(w) w(e^{\rho+\lambda}) \Big/ e^\rho \prod_{\alpha>0} (1-e^\alpha)^{m_\alpha}, \tag{9}$$

where $\rho$ is the Weyl vector, with $\rho \cdot r = -r^2/2$ for all simple roots $r$, $m_\alpha$ is the multiplicity of the root $\alpha$, the sum is over the elements $w$ of the Weyl group $W$, and the product is over positive roots $\alpha$, that is roots which can be expressed as the sum of a subset of the simple roots with positive integral coefficients.

Considering even just the trivial representation, for which $\lambda = 0$ and $\chi_0 = 1$, yields a potentially interesting relation from (9),

$$\sum_{w\in W} \det(w) w(e^\rho) = e^\rho \prod_{\alpha>0} (1-e^\alpha)^{m_\alpha}. \tag{10}$$

Kac showed that this denominator identity produces the Macdonald identities in the affine case. Kac-Moody algebras, other than the finite-dimensional and affine ones, would seem to offer the prospect of new identities generalizing these but the problem is that in other cases of Kac-Moody algebras, although the simple roots are known (as for $\mathcal{L}_\infty$), which effectively enables the sum over the Weyl group to be evaluated, the root multiplicities, $m_\alpha$, are not known, so that the product over positive roots cannot be evaluated.

No general simple explicit formula is known for the root multiplicities of $\mathcal{L}_\infty$ but, using the "no-ghost" theorem of string theory, I. Frenkel established the bound

$$m_\alpha \le p_{24}(1 - \frac{1}{2}\alpha^2), \tag{11}$$

where $p_k(n)$ is the number of partitions of $n$ using $k$ colours. This bound is saturated for some of the roots of $\mathcal{L}_\infty$ and, where it is not, there is the impression that that is because something is missing. What seems to be missing are some simple roots of zero or negative norm. In Kac-Moody algebras all the simple roots are specified by (1) to be of positive norm, even though some of the other roots they generate may not be.

## 3 Vertex Algebras

Motivated by Frenkel's work, Borcherds introduced in [3] the definition of a vertex algebra, which could in turn be used to define Lie algebras with root multiplicities which are explicitly calculable. A vertex algebra is a graded complex vector space, $V = \bigoplus_{n\in\mathbb{Z}} V_n$, together with a "vertex operator", $a(z)$, for each $a \in V$, which is a formal power series in the complex variable $z$,

$$a(z) = \sum_{m\in\mathbb{Z}} a_m z^{-m-n}, \qquad \text{for } a \in V_n, \tag{12}$$

where the operators $a_m$ map $V_n \to V_{n-m}$ and satisfy the following properties:

1. $a_n b = 0$ for $n > N$ for some integer $N$ dependent on $a$ and $b$;
2. there is an operator (derivation) $D : V \to V$ such that $[D, a(z)] = \frac{d}{dz}a(z)$;
3. there is a vector $\mathbf{1} \in V_0$ such that $\mathbf{1}(z) = 1$, $D\mathbf{1} = 0$;
4. $a(0)\mathbf{1} = a$;
5. $(z-\zeta)^N (a(z)b(\zeta) - b(\zeta)a(z)) = 0$ for some integer $N$ dependent on $a$ and $b$.

[We may define vertex operators over other fields or over the integers with more effort if we wish but the essential features are brought out in the complex case.]

The motivation for these axioms comes from string theory, where the vertex operators describe the interactions of "strings" (which are to be interpreted as models for elementary particles). Condition (5) states that $a(z)$ and $b(\zeta)$ commute apart from a possible pole at $z = \zeta$, *i.e.* they are local fields in the sense of quantum field theory. A key result is that, in an appropriate sense,

$$(a(z-\zeta)b)(\zeta) = a(z)b(\zeta) = b(\zeta)a(z). \tag{13}$$

More precisely

$$\int_0 d\zeta \int_\zeta dz\ (a(z-\zeta)b)(\zeta)f = \int_0 dz \int_0 d\zeta\ a(z)b(\zeta)f - \int_0 d\zeta \int_0 dz\ b(\zeta)a(z)f. \tag{14}$$

where $f$ is a polynomial in $z$, $\zeta$, $z-\zeta$ and their inverses, and the integral over $z$ is a circle about $\zeta$ in the first integral, one about $\zeta$ and the origin in the second integral and a circle about the origin excluding the $\zeta$ in the third integral. The axioms originally proposed by Borcherds [2] were somewhat more complicated in form and follow from those given here from the conditions generated by (14).

We can associate a vertex algebra to any even lattice $\Lambda$, the space $V$ then having the structure of the tensor product of the complex group ring $\mathbb{C}(\Lambda)$ with the symmetric algebra of a sum $\bigoplus_{n>0} \Lambda_n$ of copies $\Lambda_n, n \in \mathbb{Z}$, of $\Lambda$. In terms of string theory, this is the Fock space describing the (chiral) states of a string moving in a space-time compactified into a torus by imposing perodicity under displacements by the lattice $\Lambda$.

The first triumph of vertex algebras was to provide a natural setting for the Monster group, $M$. $M$ acts on a graded infinite-dimensional space $V^\natural$, constructed by Frenkel, Lepowsky and Meurman, where $V^\natural = \oplus_{n\geq -1} V_n^\natural$, and the dimensions of $\dim V_n^\natural$ is the coefficent, $c(n)$ of $q^n$ in the elliptic modular function,

$$j(\tau) - 744 = \sum_{n=-1}^{\infty} c(n) q^n = q^{-1} + 196884q + 21493760q^2 + \dots, \qquad q = e^{2\pi i \tau}. \tag{15}$$

A first thought might have been that the Monster group should be related to the space $V_{\Lambda_L}$, the vertex algebra directly associated with the Leech lattice, but $V_{\Lambda_L}$ has a grade 0 piece of dimension 24 and the lowest non-trivial representation of the Monster is of dimension 196883. $V^\natural$ is related to $V_{\Lambda_L}$ but is a sort of twisted version of it; in string theory terms it corresponds to the string moving on an orbifold rather than a torus.

The Monster group is precisely the group of automorphisms of the vertex algebra $V^\natural$,

$$ga(z)g^{-1} = (ga)(z), \qquad g \in M. \tag{16}$$

This characterizes $M$ in a way similar to the way that two other sporadic simple finite groups, Conway's group $Co_1$ and the Mathieu group $M_{24}$, can be characterized as the automorphism groups of the Leech lattice (modulo $-1$) and the Golay Code, respectively.

## 4 Generalized Kac-Moody Algebras

In their famous moonshine conjectures, Conway and Norton went far beyond the existence of the graded representation $V^\natural$ with dimension given by $j$. Their main conjecture was that, for each element $g \in M$, the Thompson series

$$T_g(q) = \sum_{n=-1}^{\infty} \mathrm{Trace}(g|V_n^\natural)q^n \tag{17}$$

is a Hauptmodul for some genus zero subgroup, $G$, of $\mathrm{SL}_2(\mathbb{R})$, *i.e.*, if

$$H = \{\tau : \mathrm{Im}(\tau) > 0\} \tag{18}$$

denotes the upper half complex plane, $G$ is such that the closure of $H/G$ is a compact Riemann surface, $\overline{H/G}$, of genus zero with a finite number of points removed and $T_g(q)$ defines an isomorphism of $\overline{H/G}$ onto the Riemann sphere.

To attack the moonshine conjectures it is necessary to introduce some Lie algebraic structure. For any vertex algebra, $V$, we can introduce [2, 4] a Lie algebra of operators

$$L(a) = \frac{1}{2\pi i}\oint a(z)dz = a_{-h+1}, \qquad a \in V_h\,. \tag{19}$$

Closure $[L(a), L(b)] = L(L(a)b)$ follows from (14), but this does not define a Lie algebra structure directly on $V$ because $L(a)b$ is not itself antisymmetric in $a$ and $b$. However, $DV$ is in the kernel of the map $a \mapsto L(a)$ and $L(a)b = -L(b)a$ in $V/DV$, so it does define a Lie algebra $\mathcal{L}^0(V)$ on this quotient [2], but this is not the most interesting Lie algebra associated with $V$.

Vertex algebras of interest come with an additional structure, an action of the Virasoro algebra, a central extension of the Lie algebra of polynomial vector fields on the circle, spanned by $L_n, n \in \mathbb{Z}$ and 1,

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}m(m^2-1)\delta_{m,-n}, \qquad [L_n, c] = 0, \tag{20}$$

with $L_{-1} = D$ and $L_0 a = ha$ for $a \in V_h$. For $V_\Lambda$, $c = \dim\Lambda$, and for $V^\natural$, $c = 24$. The Virasoro algebra plays a central role in string theory. The space of "physical states" of the string is defined by the Virasoro conditions: let

$$P^k(V) = \{a \in V : L_0 a = ka; L_n a = 0, n > 0\}, \tag{21}$$

the space of physical states is $P^1(V)$. The space $P^1(V)/L_{-1}P^0(V)$ has a Lie algebra structure defined on it (because $L_{-1}V \cap P^1(V) \subset L_{-1}P^0(V)$). This can be reduced in size further using a contravariant form (which it possesses naturally for lattice theories). The "no-ghost" theorem states that the space of physical states $P^1(V)$ has lots of null states and is positive semi-definite for $V_\Lambda$, where $\Lambda$ is a Lorentzian lattice with $\dim \Lambda \le 26$. So we can quotient $P^1(V)/L_{-1}P^0(V)$ further by its null space with the respect to the contravariant form to obtain a Lie algebra $\mathcal{L}(V)$.

The results of factoring by the null space are most dramatic when $c = 26$. The vertex algebra $V_L$ has a natural grading by the lattice $L$ and the "no-ghost" theorem states that the dimension of the subspace of $\mathcal{L}(V)$ of non-zero grade $\alpha$ is $p_{24}(1 - \frac{1}{2}\alpha^2)$ if $\Lambda$ is a Lorentzian lattice of dimension 26 but $p_{k-1}(1 - \alpha^2/2) - p_{k-1}(\alpha^2/2)$ if $\dim \Lambda = k \ne 26$, $k > 2$. Thus the algebra $\mathcal{L}'_M = \mathcal{L}(V_{\mathrm{II}_{25,1}})$ saturates Frenkel's bound, and Borcherds initially named it the "Monster Lie algebra" because it appeared to be directly connected to the Monster; it is now known as the "Fake Monster Lie algebra."

Borcherds [4] had the great insight not only to construct the Fake Monster Lie algebra, but also to see how to generalize the definition of a Kac-Moody algebra effectively in order to bring $\mathcal{L}'_M$ within the fold. What was required was to relax the condition (1), requiring roots to have positive norm, and to allow them to be either zero or negative norm. The condition (4) then needs modification to apply only in the space-like case $a_{ii} > 0$ and the same applies to the condition (7) on the generators. The only condition which needs to be added is that

$$[e_i, e_j] = [f_i, f_j] = 0 \qquad \text{if } a_{ij} = 0. \tag{22}$$

The closeness of these conditions to those for Kac-Moody algebras means that most of the important structural results carry over; in particular there is a generalization of the Weyl-Kac character formula for representations with highest weight $\lambda$,

$$\chi_\lambda = \sum_{w \in W} \det(w) w \left( e^\rho \sum_\mu \epsilon_\lambda(\mu) e^{\mu+\lambda} \right) \Big/ e^\rho \prod_{\alpha > 0} (1 - e^\alpha)^{m_\alpha}, \tag{23}$$

where the second sum in the numerator is over vectors $\mu$ and $\epsilon_\lambda(\mu) = (-1)^n$ if $\mu$ can be expressed as the sum of $n$ pairwise orthogonal simple roots with non-positive norm, all orthogonal to $\lambda$, and 0 otherwise. Of course, putting $\lambda = 0$ and $\chi_\lambda = 1$ again gives a denominator formula.

The description of generalized Kac-Moody algebras in terms of generators and relations enables the theory to be taken over rather simply from that of Kac-Moody algebras but it is not so convenient as a method of recognising them in practice, *e.g.* from amongst the algebras $\mathcal{L}(V)$ previously constructed by Borcherds. But Borcherds [3] gave an alternative characterization of them as graded algebras with an "almost postitive definite" contravariant bilinear form. More precisely, he showed that a graded Lie algebra, $\mathcal{L} = \bigoplus_{n \in \mathbb{Z}} \mathcal{L}_n$, is a generalized Kac-Moody algebra if the following conditions are satisfied:

1. $\mathcal{L}_0$ is abelian and $\dim \mathcal{L}_n$ is finite if $n \neq 0$;
2. $\mathcal{L}$ possesses an invariant bilinear form such that $(\mathcal{L}_m, \mathcal{L}_n) = 0$ if $m \neq n$;
3. $\mathcal{L}$ possesses an involution $\omega$ which is $-1$ on $\mathcal{L}_0$ and such that $\omega(\mathcal{L}_m) \subset \mathcal{L}_{-m}$;
4. the contravariant bilinear form $\langle L, M\rangle = -(L, \omega(M))$ is positive definite on $\mathcal{L}_m$ for $m \neq 0$ ;
5. $\mathcal{L}_0 \subset [\mathcal{L}, \mathcal{L}]$.

This characterization shows that the Fake Monster Lie algebra, $\mathcal{L}'_M$, is a generalised Kac-Moody algebra, and its root multiplicities are known to be given by $p_{24}(1-\frac{1}{2}\alpha^2)$, but Borcherds' theorem establishing the equivalence of his two definitions does not give a constructive method of finding the simple roots. As we remarked in the context of Kac-Moody algebras, if we knew both the root multiplicities and the simple roots, the denominator formula

$$\sum_{w\in W} \det(w) w\left(e^\rho \sum_\mu \epsilon_\mu(\alpha) e^\mu\right) = e^\rho \prod_{\alpha>0}(1-e^\alpha)^{m_\alpha} \tag{24}$$

might provide an interesting identity. Borcherds solved [4] the problem of finding the simple roots, or rather proving that the obvious ones were all that there were, by inverting this argument. The positive norm simple roots can be identified with the Leech lattice as for $\mathcal{L}_\infty$. Writing $\mathrm{II}_{25,1} = \Lambda_L \oplus \mathrm{II}_{1,1}$, which follows by uniqueness or the earlier comments, the 'real' or space-like simple roots are $\{(\lambda, 1, \frac{1}{2}\lambda^2 - 1) : \lambda \in \Lambda_L\}$. (Here we are using we are writing $\mathrm{II}_{1,1} = \{(m,n) : m,n \in \mathbb{Z}\}$ with $(m,n)$ having norm $-2mn$.) Light-like simple roots are quite easily seen to be $n\rho$, where $n$ is a positive integer and $\rho = (0,0,1)$. The denominator identity is then used to prove that there are no other light-like and that there are no time-like simple roots.

The denominator identity provides a remarkable relation between modular functions (apparently already known to some of the experts in the subject) which is the precursor of other even more remarkable identities. If we restrict attention to vectors $(0,\sigma,\tau) \in \mathrm{II}_{25,1} \otimes \mathbb{C}$, with $\mathrm{Im}(\sigma) > 0$, $\mathrm{Im}(\tau) > 0$, it reads

$$p^{-1} \prod_{m>0, n\in\mathbb{Z}} (1-p^m q^n)^{c'(mn)} = \Delta(\sigma)\Delta(\tau)(j(\sigma) - j(\tau)) \tag{25}$$

where $c'(0) = 24$, $c'(n) = c(n)$ if $n \neq 0$, $p = e^{2\pi i \sigma}$, $q = e^{2\pi i \tau}$, and

$$\Delta(\tau)^{-1} = q^{-1} \prod_{n>1} (1-q^n)^{-24} = \sum_{n\geq 0} p_{24}(n) q^{n-1}. \tag{26}$$

## 5 MOONSHINE, THE MONSTER LIE ALGEBRA AND AUTOMORPHIC FORMS

The presence of $j(\sigma)$ in (25) suggests a relationship to the moonshine conjectures and Borcherds used [5, 6] this as motivation to construct the "real" Monster Lie Algebra, $\mathcal{L}_M$ as one with denominator identity obtained by multiplying each side of (25) by $\Delta(\sigma)\Delta(\tau)$, to obtain the simpler formula

$$p^{-1} \prod_{m>0, n\in\mathbb{Z}} (1-p^m q^n)^{c(mn)} = j(\sigma) - j(\tau). \tag{27}$$

This looks like the denominator formula for a generalised Kac-Moody algebra which is graded by $\mathrm{II}_{1,1}$ and is such that the dimension of the subspace of grade $(m,n) \neq (0,0)$ is $c(mn)$, the dimension of $V^\natural_{mn}$. It is not difficult to see that this can be constructed by using the vertex algebra which is the tensor product $V^\natural \otimes V_{\mathrm{II}_{1,1}}$ and defining $\mathcal{L}_M$ to be the generalised Lie algebra, $\mathcal{L}(V^\natural \otimes V_{\mathrm{II}_{1,1}})$, constructed from the physical states.

Borcherds used [5, 6] twisted forms of the denominator identity for $\mathcal{L}_M$ to prove the moonshine conjectures. The action of $M$ on $V^\natural$ provides an action on $V = V^\natural \otimes V_{\mathrm{II}_{1,1}}$ induces an action on the physical state space $P^1(V)$ and on its quotient, $\mathcal{L}_M = \mathcal{L}(V)$, by its null space. The "no-ghost" theorem implies that the part of $\mathcal{L}_M$ of grade $(m,n)$, $(\mathcal{L}_M)_{(m,n)}$, is isomorphic to $V^\natural_{mn}$ as an $M$ module. Borcherds adapted the argument he used to establish the denominator identity to prove the twisted relation

$$p^{-1}\exp\left(-\sum_{N>0}\sum_{m>0,n\in\mathbb{Z}}\mathrm{Tr}(g^N|V^\natural_{mn})p^{mN}q^{nN}/N\right) = \sum_{m\in\mathbb{Z}}\mathrm{Tr}(g|V^\natural_m)p^m - \sum_{n\in\mathbb{Z}}\mathrm{Tr}(g|V^\natural_n)q^n. \quad (28)$$

These relations on the Thompson series are sufficient to determine them from their first few terms and to establish that they are modular functions of genus 0.

Returning to the Fake Monster Lie Algebra, the denominator formula given in (25) was restricted to vectors of the form $v = (0,\sigma,\tau)$ but we consider it for more general $v \in \mathrm{II}_{25,1} \otimes \mathbb{C}$, giving the denominator function

$$\Phi(v) = \sum_{w\in W} \det(w) e^{2\pi i(w(\rho),v)} \prod_{n>0}\left(1 - e^{2\pi i n(w(\rho),v)}\right)^{24}. \quad (29)$$

This expression converges for $\mathrm{Im}(v)$ inside a certain cone (the positive light cone). Using the explicit form for $\Phi(v)$ when $v = (0,\sigma,\tau)$, the known properties of $j$ and $\Delta$ and the fact that $\Phi(v)$ manifestly satisfies the wave equation, Borcherds [6, 7, 9] establishes that $\Phi(v)$ satisfies the functional equation

$$\Phi(2v/(v,v)) = -((v,v)/2)^{12}\Phi(v). \quad (30)$$

It also has the properties that

$$\Phi(v+\lambda) = \Phi(v) \quad \text{for } \lambda \in \mathrm{II}_{25,1} \quad (31)$$

and

$$\Phi(w(v)) = \det(w)\Phi(v) \quad \text{for } w \in \mathrm{Aut}(\mathrm{II}_{25,1})^+, \quad (32)$$

the group of automorphisms of the lattice $\mathrm{II}_{25,1}$ which preserve the time direction. These transformations generate a discrete subgroup of the group of conformal transformations on $\mathbb{R}^{25,1}$, which is itself isomorphic to $O_{26,2}(\mathbb{R})$; in fact the discrete group is isomorphic to $\mathrm{Aut}(\mathrm{II}_{26,2})^+$. The denominator function for the Fake Monster Lie algebra defines in this way an automorphic form of weight 12 for the discrete subgroup $\mathrm{Aut}(\mathrm{II}_{26,2})^+$ of $O_{26,2}(\mathbb{R})^+$. This result once obtained is seen not to depend essentially on the dimension 26 and Borcherds has developed this approach of obtaining representations of modular functions as infinite products from denominator formulae for generalized Kac-Moody algebras to obtain a plethora of beautiful formulae [7, 9, 11], *e.g.*

$$j(\tau) = q^{-1}\prod_{n>0}(1-q^n)^{c_0(n^2)} = q^{-1}(1-q)^{-744}(1-q^2)^{80256}(1-q^3)^{-12288744}\dots, \tag{33}$$

where $f_0(\tau) = \sum_n c_0(n)q^n$ is the unique modular form of weight $\frac{1}{2}$ for the group $\Gamma_0(4)$ which is such that $f_0(\tau) = 3q^{-3} + \mathcal{O}(q)$ at $q = 0$ and $c_0(n) = 0$ if $n \equiv 2$ or 3 mod 4. He has also used these denominator functions to establish results about the moduli spaces of Enriques surfaces and and families of K3 surfaces [8, 10].

Displaying penetrating insight, formidable technique and brilliant originality, Richard Borcherds has used the beautiful properties of some exceptional structures to motiviate new algebraic theories of great power with profound connections with other areas of mathematics and physics. He has used them to establish outstanding conjectures and to find new deep results in classical areas of mathematics. This is surely just the beginning of what we have to learn from what he has created.

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| | |
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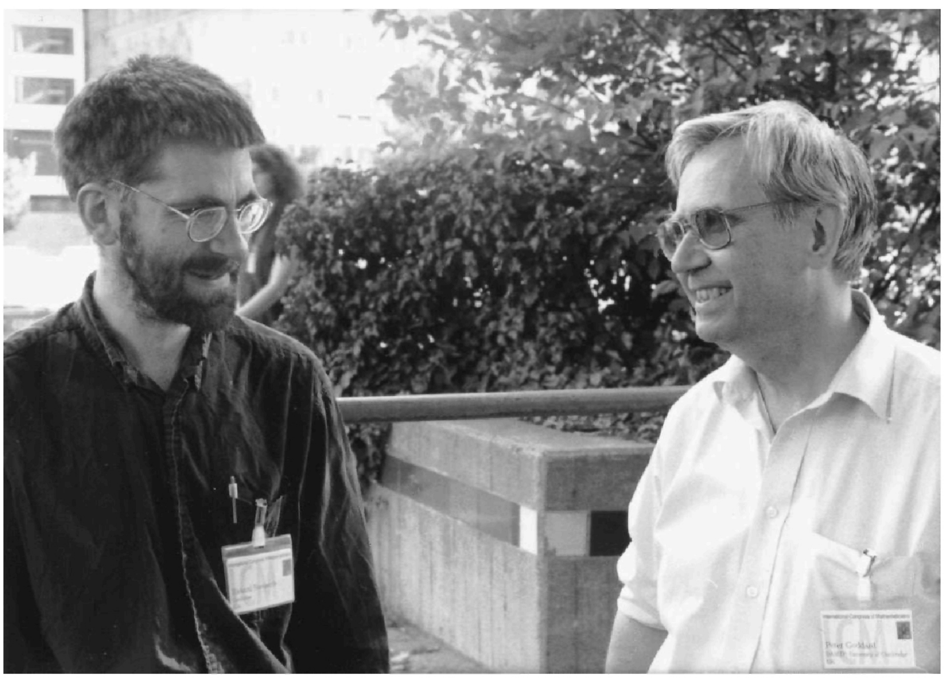

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