Local Index Theory and Higher Analytic Torsion

Jean-Michel Bismut[1]

Abstract. In this paper, we report on the construction of secondary invariants in connection with the Atiyah-Singer index theorem for families, and the theorem of Riemann-Roch-Grothendieck. The local families index theorem plays an important role in the construction.

In complex geometry, the corresponding objects are the analytic torsion forms and the analytic torsion currents. These objects exhibit natural functorial properties with respect to composition of maps. Gillet and Soulé have used these objects to prove a Riemann-Roch theorem in Arakelov geometry.

Also we state a Riemann-Roch theorem for flat vector bundles, and report on the construction of corresponding higher analytic torsion forms.

1991 Mathematics Subject Classification: 32L10,57R20,58G10
Keywords and Phrases: Sheaves and cohomology of sections of holomorphic vector bundles. Characteristic classes and numbers. Index theory and related fixed point theory.

The purpose of this paper is to report on the construction of certain secondary invariants which appear in connection with the families index theorem of Atiyah-Singer [4] and the Riemann-Roch-Grothendieck theorem [7]. These invariants are refinements of the η invariant of Atiyah-Patodi-Singer [2], and of the Ray-Singer analytic torsion for de Rham and Dolbeault complexes [50], [51], which are spectral invariants of the considered manifolds.

Progress in this area was made possible by the development of several related tools:

- The discovery by Quillen [48] of superconnections.
- A better understanding of local index theory (Getzler [31]) and the proof of a local families index theorem by the author [9], and of related results by Berline-Vergne [6], Berline-Getzler-Vergne [5].
- Progress on the theory of determinant bundles, by Quillen [49], Freed and the author [16], and Gillet, Soulé and the author [17].
- The development of adiabatic limit techniques to study the behaviour of certain spectral invariants (like the η-invariants of Atiyah-Patodi-Singer [2]) under degenerations, by Cheeger and the author [15], Mazzeo-Melrose [44], and Dai [29].

[1]Supported by the Institut Universitaire de France (I.U.F.) and by C.N.R.S., URA 1169.

Algebraic geometry gave an essential impetus to the above developments. Extending earlier work by Arakelov and Faltings, Gillet and Soulé [33],[34] developed an algebraic formalism which could use as an input results coming from analysis, and invented the adequate Riemann-Roch-Grothendieck theorem.

Our starting point is the local families index theorem [9], [5]. Let $\pi : X \to S$ be a fibration with compact even dimensional oriented Riemannian spin fibre Z. Let E be a complex vector bundle on X . Let $(D^Z_s)_{s\in S}$ be the associated family of Dirac operators [3] acting along the fibres Z. Let $\mathrm{Ind}(D^Z_+) \in K(S)$ be the corresponding index bundle. In [4], Atiyah and Singer proved the index theorem for families,

$$\mathrm{ch}(\mathrm{Ind}(D^Z_+)) = \pi_* \left[\widehat{A}(TZ)\mathrm{ch}(E)\right] \mathrm{in}\, H(S, \mathbf{Q}). \tag{0.1}$$

In [9], starting from natural geometric data, connections were introduced on the vector bundles appearing in (0.1), so that by Chern-Weil theory, we can represent the cohomology classes in (0.1) by differential forms. Using a special case of a Quillen superconnection [48], the Levi-Civita superconnection [9], a "natural" family of closed differential forms $\alpha_{t|t\in\mathbf{R}_+}$ on S was produced, which interpolates between the differential forms representing the right-hand side of (0.1) (for $t \to 0$) and the left-hand side of (0.1) (for $t \to +\infty$, by [6], [5]). Moreover, following earlier work by Quillen [49], Freed and the author [16] proved a curvature theorem for smooth determinant bundles associated to a family of Dirac operators. Also extending earlier work in [16], [27], Cheeger and the author [15] constructed an odd form on S, $\tilde{\eta}$, which transgresses equation (0.1) at the level of differential forms. These forms $\tilde{\eta}$ were used to evaluate the "adiabatic" limit of η-invariants [16], [27], [15].

Let $f : X \to S$ be a proper holomorphic map of complex quasiprojective manifolds, and let E be a holomorphic vector bundle on X. By Riemann-Roch-Grothendieck [7],

$$\mathrm{Td}(TS)\mathrm{ch}(f_*E) = f_*[\mathrm{Td}(TX)\mathrm{ch}(E)]\, \mathrm{in}\, H(S, \mathbf{Q}). \tag{0.2}$$

Assume that $\pi : X \to S$ is a holomorphic fibration with compact fibre Z. Let E be a holomorphic vector bundle on X. Let $(\Omega(Z, E_{|Z}), \overline{\partial}^Z)$ be the family of relative Dolbeault complexes along the fibres Z. Let ω^X be a closed (1,1)-form on X restricting to a Kähler metric g^{TZ} along the fibres Z, and let g^E be a Hermitian metric on E. Recall that a holomorphic Hermitian vector bundle is naturally equipped with a unitary connection, which can be used to calculate Chern-Weil forms. Assume that $R\pi_*E$ is locally free. Let $g^{R\pi_*E}$ be the L_2 metric on $R\pi_*E$ one obtains via Hodge theory. In work by Gillet, Soulé and the author [17], and by Köhler and the author [20], a sum of real (p, p) forms on S was constructed, the analytic torsion forms $T(\omega^X, g^E)$, such that the following refinement of (0.2) holds,

$$\frac{\overline{\partial}\partial}{2i\pi} T(\omega^X, g^E) = \mathrm{ch}(R\pi_*E, g^{R\pi_*E}) - \pi_* \left[\mathrm{Td}(TZ, g^{TZ})\mathrm{ch}(E, g^E)\right]. \tag{0.3}$$

The forms $T(\omega^X, g^E)$ also refine the forms $\tilde{\eta}$ of [15]. The component of degree 0 of $T(\omega^X, g^E)$ is the fibrewise holomorphic Ray-Singer torsion [51] of the considered

Dolbeault complex, a spectral invariant of the Hodge Laplacians along the fibres. It was used by Quillen [49] to construct a metric on $(\det(R\pi_*E))^{-1}$, whose properties were studied by Quillen [49], and by Gillet, Soulé and the author [17].

At the same time, Gillet and Soulé were pursuing their effort to construct an intersection theory on arithmetic varieties, in order to formulate a Riemann-Roch-Grothendieck in Arakelov geometry. In [33], [34], they constructed refined Chow groups $\widehat{CH}$, and Hermitian K-theory groups $\widehat{K}$. They used the analytic torsion forms $T(\omega^X, g^E)$ to define a direct image in $\widehat{K}$. From a computation with Zagier [35] of the analytic torsion of $\mathbf{P^N}$ equipped with the Fubini-Study metric, they conjectured a Riemann-Roch-Grothendieck theorem in Arakelov geometry, where the additive genus associated to an exotic power series $R(x)$ appears as a correction to the Todd genus $\widehat{\mathrm{Td}}$.

In [11], a secondary characteristic class for short exact sequences of holomorphic vector bundles was constructed, which was evaluated in terms of the R class.

In [10], [18], the analogue of the above construction for submersions was carried out for immersions. Namely, let $i : Y \to X$ be an embedding of complex manifolds, let F be a holomorphic vector bundle on Y, and let (E, v) be a resolution of i_*F by a complex of holomorphic vector bundles on X. Under natural compatibility assumptions on Hermitian metrics $g^E, g^F, g^{N_{Y/X}}$, analytic torsion currents $T(E, g^E)$ were constructed on X, such that

$$\frac{\overline{\partial}\partial}{2i\pi} T(E, g^E) = \mathrm{Td}^{-1}(N_{Y/X}, g^{N_{Y/X}}) \mathrm{ch}(F, g^F)\delta_Y - \mathrm{ch}(E, g^E). \tag{0.4}$$

Again, (0.4) refines (0.2) at the level of currents. The functoriality of these constructions was established in work by Gillet, Soulé and the author [19].

In [21], using [11], Lebeau and the author calculated the behaviour of Quillen metrics under resolutions. Then Gillet and Soulé [36] gave a proof of their Riemann-Roch formula for the first Chern class. In [30], Faltings provided an alternative strategy to a proof of the Riemann-Roch theorem of Gillet-Soulé, by using deformation to the normal cone. In [13], the author extended his previous result with Lebeau [21]. Namely, in the case of the composition of an embedding and a submersion, a natural combination of analytic torsion forms is expressed in terms of analytic torsion currents. When combined with the arguments of Gillet and Soulé [36], this leads to a proof of the Riemann-Roch-Grothendieck theorem of Gillet and Soulé in the general case. A remaining mystery of the theory was the fact that the genus R seemed to appear twice in the theory: through the explicit spectral computations in [35] of the analytic torsion of $\mathbf{P}_n$, and also in the evaluation of certain characteristic classes in [11]. The mystery was solved by Bost [24] and Roessler [53]. They show in particular that the evaluation in [35] of the analytic torsion of $\mathbf{P}_n$ can be obtained as a consequence of [11],[21].

In [22], Lott and the author extended the formalism of higher analytic torsion to de Rham theory. Assume that $\pi : X \to S$ is a fibration of real manifolds with compact fibre Z. Let F be a complex flat vector bundle on X. Then $R\pi_*F$ is a flat vector bundle on S. The differential characters of Cheeger-Simons [28] produce Chern classes of flat vector bundles on a manifold M, with values in $H^{\mathrm{odd}}(M, \mathbf{C}/\mathbf{Z})$. In [22], a Riemann-Roch-Grothendieck formula was established for the real part of these classes, and corresponding real higher analytic torsion forms were introduced, whose part of degree 0 is just the Ray-Singer torsion of [50]. From these torsion forms, one can produce certain even cohomology classes on S. In degree 0, the Ray-Singer conjecture, proved by Cheeger [26] and Müller [45], shows that, for unitarily flat vector bundles, the Ray-Singer torsion coincides with the Reidemeister torsion [52]. In positive degree, the evaluation of the higher analytic torsion forms of [22] is still mysterious, although some evidence suggests they might possibly be related to constructions by Igusa and Klein [39] using Borel regulators.

This paper is organized as follows. In Section 1, we state the local families index theorem. In Section 2, we introduce the higher analytic torsion forms. In Section 3, we describe the analytic torsion currents. In Section 4, we give a compatibility result between analytic torsion forms and analytic torsion currents, and we state the Riemann-Roch theorem of Gillet-Soulé. Finally, in Section 5, we state a Riemann-Roch theorem for flat vector bundles.

For a more detailed survey on the analytic aspects of this paper, we refer the reader to [14].

1. The local families index theorem

1.1. The local index theorem. Let Z be a compact even dimensional oriented spin manifold. Let g^{TZ} be a Riemannian metric on TZ. Let $S^{TZ} = S_+^{TZ} \oplus S_-^{TZ}$ be the $\mathbf{Z_2}$-graded hermitian vector bundle of (TZ, g^{TZ}) spinors. Let ∇^{TZ} be the Levi-Civita connection on (TZ, g^{TZ}). Let $\nabla^{S^{TZ}} = \nabla^{S_+^{TZ}} \oplus \nabla^{S_-^{TZ}}$ be the corresponding unitary connection on $S^{TZ} = S_+^{TZ} \oplus S_-^{TZ}$. Let (E, g^E, ∇^E) be a complex Hermitian vector bundle on Z, equipped with a unitary connection ∇^E.

Let $c(TZ)$ be the bundle of Clifford algebras of (TZ, g^{TZ}). Then $S^{TZ} \otimes E$ is a Clifford module for the Clifford algebra $c(TZ)$. If $X \in TZ$, let $c(X)$ denote the action of $X \in c(TZ)$ on $S^{TZ} \otimes E$. Put

$$H = C^\infty(Z, S^{TZ} \otimes E),\ H_\pm = C^\infty(Z, S_\pm^{TZ} \otimes E). \tag{1.1}$$

Let $e_1, \cdots, e_n$ be an orthonormal basis of TZ.

Let D^Z be the Dirac operator acting on H,

$$D^Z = \sum_1^n c(e_i) \nabla_{e_i}^{S^{TZ} \otimes E}. \tag{1.2}$$

Let $D_\pm^Z$ be the restriction of D^Z to $H_\pm$, so that

$$D^Z = \begin{bmatrix} 0 & D_-^Z \\ D_+^Z & 0 \end{bmatrix}. \tag{1.3}$$

The elliptic operator D_+^Z is Fredholm. Its index $\mathrm{Ind}(D_+^Z) \in \mathbf{Z}$ is given by

$$\mathrm{Ind}(D_+^Z) = \dim(\ker D_+^Z) - \dim(\ker D_-^Z). \tag{1.4}$$

Let $\widehat{A}$ be the multiplicative genus associated to the power series

$$\widehat{A}(x) = \frac{x/2}{\sinh(\mathrm{x}/2)}. \tag{1.5}$$

The Atiyah-Singer index theorem [3] asserts that

$$\operatorname{Ind}(D_+^Z) = \int_Z \widehat{A}(TZ)\operatorname{ch}(E). \tag{1.6}$$

If $F = F_+ \oplus F_-$ is a $\mathbf{Z_2}$-graded vector space, let $\tau = \pm 1$ on $F_\pm$ define the grading. If $A \in \operatorname{End}(F)$, let $\operatorname{Tr_s}[A]$ be the supertrace of A, i.e. $\operatorname{Tr_s}[A] = \operatorname{Tr}[\tau A]$. Now we state the McKean-Singer formula [42].

Proposition 1.1. *For any $t > 0$,*

$$\operatorname{Ind}(D_+^Z) = \operatorname{Tr_s}[\exp(-tD^{Z,2})]. \tag{1.7}$$

Let $P_t(x,y)$ be the smooth kernel of $\exp(-tD^{Z,2})$ with respect to the volume element dy, so that (1.7) can be written as

$$\operatorname{Ind}(D_+^Z) = \int_Z \operatorname{Tr_s}[P_t(x,x)]dx. \tag{1.8}$$

In Patodi [46], Gilkey [32], Atiyah-Bott-Patodi [1], it was proved that, as conjectured in [42], "fantastic cancellations" occur in the asymptotic expansion of $\operatorname{Tr_s}[P_t(x,x)]$, so that as $t \to 0$,

$$\operatorname{Tr_s}[P_t(x,x)] \to \{\widehat{A}(TZ, \nabla^{TZ})\operatorname{ch}(E, \nabla^E)\}^{\max}. \tag{1.9}$$

Another proof of (1.9) by Getzler [31] has considerably improved our geometric understanding of the above cancellations. Equation (1.9) is known as a local index theorem. From (1.8), (1.9), one recovers the index formula (1.6).

1.2. Quillen's superconnections. Here we follow Quillen [48]. Let $E = E_+ \oplus E_-$ be a $\mathbf{Z_2}$-graded vector bundle on a manifold S.

Definition 1.2. A superconnection is an odd first order differential operator A acting on $C^\infty(S, \Lambda(T^*S)\widehat{\otimes}E)$ such that if $\omega \in C^\infty(S, \Lambda(T^*S)), s \in C^\infty(S,E)$,

$$A(\omega s) = d\omega s + (-1)^{\deg \omega}\omega A s. \tag{1.10}$$

By definition, the curvature of A is $A^2 \in C^\infty(S, (\Lambda(T^*S)\widehat{\otimes}\operatorname{End}(E))^{\mathrm{even}})$. Let $\varphi : \omega \in \Lambda(T^*S) \to \varphi\omega = (2i\pi)^{-\deg\omega/2}\omega \in \Lambda(T^*S)$.

Definition 1.3. Let $\operatorname{ch}(E,A)$ be the even form on S,

$$\operatorname{ch}(E,A) = \varphi\operatorname{Tr_s}[\exp(-A^2)]. \tag{1.11}$$

Theorem 1.4. *The even form $\operatorname{ch}(E,A)$ is closed, and its cohomology class $[\operatorname{ch}(E,A)]$ is given by*

$$[\operatorname{ch}(E,A)] = \operatorname{ch}(E_+) - \operatorname{ch}(E_-). \tag{1.12}$$

Remark 1.5. Observe the striking algebraic similarity of the right-hand sides of (1.7) and (1.11) with the density $\exp(-x^2)$ of the gaussian distribution on $\mathbf{R}$.

1.3. LOCAL FAMILIES INDEX THEOREM AND ADIABATIC LIMITS. Let $\pi : X \to S$ be a submersion of smooth manifolds with even dimensional compact fibre Z. We assume that TZ is oriented and spin. Let g^{TZ} be a Riemannian metric on TZ. Let (E, g^E, ∇^E) be a Hermitian vector bundle on X with unitary connection. Let $(D^Z_s)_{s\in S}$ be the family of Dirac operators acting fibrewise along the fibres Z on $H_s = H_{+s} \oplus H_{-s}$. Then to the family of Fredholm operators $(D^Z_{+,s})_{s\in S}$, there is an associated virtual vector bundle $\mathrm{Ind}(D^Z_+) \in K(S)$. The families index theorem of Atiyah-Singer [4] asserts in particular that

$$\mathrm{ch}(\mathrm{Ind}(D^Z_+)) = \pi_*[\widehat{A}(TZ)\mathrm{ch}(E)] \,\mathrm{in}\, H^{\mathrm{even}}(S, \mathbf{Q}). \tag{1.13}$$

Assume temporarily that X and S are even dimensional oriented compact spin manifolds. Let g^{TX}, g^{TS} be Riemannian metrics on TX, TS. For $\varepsilon > 0$, put

$$g^{TX}_\varepsilon = g^{TX} + \frac{1}{\varepsilon}\pi^* g^{TS}. \tag{1.14}$$

Letting ϵ tend to 0 is often described as taking an adiabatic limit. Let D^X_ϵ be the Dirac operator associated to $(g^{TX}_\epsilon, \nabla^E)$.

Let ∇^{TX}_ε and ∇^{TS} be the Levi-Civita connections on (TX, g^{TX}_ϵ) and (TS, g^{TS}). Let $T^H X$ be the orthogonal bundle to TZ in TX with respect to g^{TX}. If $U \in TS$, let $U^H \in T^H X$ be the lift of U in $T^H X$. Let P^{TZ} be the projection $TX = T^H X \oplus TZ \to TZ$. Let ∇^{TZ} be the connection on (TZ, g^{TZ}),

$$\nabla^{TZ} = P^{TZ}\nabla^{TX}_\varepsilon, \tag{1.15}$$

which does not depend on $\varepsilon > 0$. A trivial calculation shows that as $\varepsilon \to 0$,

$$\widehat{A}(TX, \nabla^{TX}_\varepsilon) \to \pi^*[\widehat{A}(TS, \nabla^{TS})]\widehat{A}(TZ, \nabla^{TZ}). \tag{1.16}$$

Let $P^\varepsilon_t(x, y)$ be the smooth kernel of $\exp(-tD^{X,2}_\varepsilon)$. Then by (1.9),

$$\mathrm{Tr_s}[P^\varepsilon_t(x, x)] \to \{\widehat{A}(TX, \nabla^{TX}_\varepsilon)\mathrm{ch}(E, \nabla)\}^{\max}. \tag{1.17}$$

We change our notation slightly, and temporarily assume that g^{TX}_ϵ is given by $g^{TX}_\epsilon = \pi^* \frac{g^{TS}}{\epsilon} \oplus g^{TZ}$. If $U, V \in TS$, put

$$T(U, V) = -P^{TZ}[U^H, V^H]. \tag{1.18}$$

If $U \in TS$, let $\mathrm{div}_Z(U^H)$ be the divergence of U^H with respect to the vertical volume form dv_Z. Let $(e_1, \ldots, e_n)$ and $(f_1, \ldots, f_m)$ be orthogonal bases of (TZ, g^{TZ}) and (TS, g^{TS}). If S^{TX}_ϵ is the vector bundle of (TX, g^{TX}_ϵ) spinors,

$$S^{TX}_\varepsilon = \pi^* S^{TS}\widehat{\otimes}S^{TZ}. \tag{1.19}$$

Put

$$D^H = \sum_1^m c(f_\alpha)(\nabla^{\pi^* S^{TS}\widehat{\otimes}S^{TZ}\otimes E}_{f_\alpha} + \frac{1}{2}\mathrm{div}_Z(f^H_\alpha)). \tag{1.20}$$

Then by [15],

$$D^{X,\varepsilon} = \sqrt{\varepsilon}D^H + D^Z - \frac{\varepsilon}{8}c(f_\alpha)c(f_\beta)c(T(f_\alpha, f_\beta)). \tag{1.21}$$

Put

$$H = C^\infty(Z, (S^{TZ} \otimes E)_{|Z}). \tag{1.22}$$

Then $H = H_+ \oplus H_-$ is an infinite dimensional $\mathbf{Z_2}$-graded vector bundle on S, and $C^\infty(M, \pi^* S^{TS} \otimes E) = C^\infty(S, S^{TS} \widehat{\otimes} H)$.

DEFINITION 1.6. Let ∇^H be the connection on H, such if $U \in TS, s \in C^\infty(S, H)$,

$$\nabla^H_U s = \nabla^{S^{TZ} \otimes E}_{U^H} s + \frac{1}{2} \mathrm{div}_Z(U^H) s \tag{1.23}$$

Then D^H is the Dirac operator action on $C^\infty(S, S^{TS} \widehat{\otimes} H)$ associated to (g^{TS}, ∇^H). Following [9], we formally replace $c(f_\alpha)$ by $f^\alpha \wedge .$ in (1.21).

DEFINITION 1.7. For $t > 0$, put

$$A_t = \nabla^H + \sqrt{t} D^Z - \frac{1}{8\sqrt{t}} f^\alpha f^\beta c(T(f_\alpha, f_\beta)). \tag{1.24}$$

Then A_t is a superconnection on H, the Levi-Civita superconnection associated to $(T^H X, g^{TZ}, \nabla^E)$.

For $t > 0$, let α_t be the even form on S

$$\alpha_t = \varphi \mathrm{Tr_s}[\exp(-A_t^2)]. \tag{1.25}$$

Now we state the local families index theorem [9], [6], [5].

THEOREM 1.8. *The form α_t is real, even and closed. Moreover*

$$[\alpha_t] = \mathrm{ch}(\mathrm{Ind} D^Z_+) \in H^{\mathrm{even}}(B, \mathbf{Q}). \tag{1.26}$$

As $t \to 0$,

$$\alpha_t = \pi_*[\widehat{A}(TZ, \nabla^{TZ}) \mathrm{ch}(E, \nabla^E)] + \mathcal{O}(t). \tag{1.27}$$

If $\ker D^Z \subset H$ is a vector bundle, and $\nabla^{\ker D^Z}$ is the orthogonal projection of ∇^H on $\ker D^Z$, as $t \to +\infty$,

$$\alpha_t = \mathrm{ch}(\ker D^Z, \nabla^{\ker D^Z}) + \mathcal{O}(\frac{1}{\sqrt{t}}). \tag{1.28}$$

Remark 1.9. Equations (1.26) and (1.27) were proved by the author in [9], and equation (1.28) by Berline-Vergne [6], Berline-Getzler-Vergne [5]. Equation (1.27) is known as the local families index theorem. It extends the local index formula given in (1.9).

2. COMPLEX GEOMETRY AND HIGHER ANALYTIC TORSION FORMS

2.1. THE ANALYTIC TORSION FORMS OF A HOLOMORPHIC COMPLEX. Here we follow [17]. Let S be a complex manifold, and let

$$(E, v) : 0 \to E_m \xrightarrow{v} E_{m-1} \ldots \xrightarrow{v} E_0 \to 0 \tag{2.1}$$

be a holomorphic complex of vector bundles on S. Put

$$E_+ = \bigoplus_{i\,\mathrm{even}} E_i,\ E_- = \bigoplus_{i\,\mathrm{odd}} E_i. \tag{2.2}$$

Then $E = E_+ \oplus E_-$ is $\mathbf{Z_2}$-graded. Let $g^E = \bigoplus_{i=0}^m g^{E_i}$ be a Hermitian metric on $E = \bigoplus_{i=0}^m E_i$. Let $\nabla^E = \bigoplus_{i=0}^m \nabla^{E_i}$ be the corresponding holomorphic Hermitian connection. Let v^* be the adjoint of v. Set

$$V = v + v^*. \tag{2.3}$$

For $t > 0$, set

$$\begin{aligned} C_t^{''} &= \nabla^{E''} + \sqrt{t}v,\ C_t^{'} = \nabla^{E'} + \sqrt{t}v^*, \\ C_t &= C_t'' + C_t'. \end{aligned} \tag{2.4}$$

Let N be the number operator of E, which acts on E_k by multiplication by k.

PROPOSITION 2.1. *The following identities hold*

$$\begin{gathered} {C''}_t^2 = 0, {C'}_t^2 = 0, \\ \tfrac{\partial C_t''}{\partial t} = \tfrac{1}{2t}[C_t'', N],\ \tfrac{\partial C'}{\partial t} = -\tfrac{1}{2t}[C_t', N]. \end{gathered} \tag{2.5}$$

DEFINITION 2.2. Let P^S be the set of smooth real forms on S, which are sums of forms of type (p,p). Let $P^{S,0}$ be the set of $\alpha \in P^S$ which can be written as $\alpha = \overline{\partial}\beta + \partial\gamma$, with β and γ smooth.

DEFINITION 2.3. For $t > 0$, put

$$\alpha_t = \varphi \mathrm{Tr_s}[\exp(-C_t^2)],\ \gamma_t = \varphi \mathrm{Tr_s}[N \exp(-C_t^2)]. \tag{2.6}$$

The following result is obtained in [17] as an easy consequence of Proposition 2.1.

PROPOSITION 2.4. *The forms α_t and γ_t lie in P^S. Also*

$$\frac{\partial \alpha_t}{\partial t} = \frac{\overline{\partial}\partial}{2i\pi}\frac{\gamma_t}{t}. \tag{2.7}$$

Assume now that $H(E,v)$ is of locally constant dimension. Then $H(E,v)$ is a holomorphic $\mathbf{Z}$-graded vector bundle. By finite dimensional Hodge theory, $H(E,v) \simeq \ker V$ inherits a Hermitian metric $g^{H(E,v)}$. Set

$$\mathrm{ch}'(E, g^E) = \sum_{i=0}^m (-1)^i i \,\mathrm{ch}(E, g^E). \tag{2.8}$$

By [6], [5], as $t \to +\infty$,

$$\begin{aligned} \alpha_t &= \mathrm{ch}(H(E,v), g^{H(E,v)})) + \mathcal{O}(\tfrac{1}{\sqrt{t}}), \\ \gamma_t &= \mathrm{ch}'(H(E,v), g^{H(E,v)}) + \mathcal{O}(\tfrac{1}{\sqrt{t}}). \end{aligned} \tag{2.9}$$

DEFINITION 2.5. For $s \in \mathbf{C}, 0 < \mathrm{Re(s)} < 1/2$, set

$$\begin{aligned} R(E,g)^E)(s) &= \frac{1}{\Gamma(s)} \int_0^{+\infty} t^{s-1}(\gamma_t - \gamma_\infty)dt, \\ T(E, g^E) &= \frac{\partial}{\partial s} R(E, g^E)(0). \end{aligned} \tag{2.10}$$

As the notation suggests, by (2.9), $R(E,g^E)(s)$ extends to a holomorphic function of s near $s = 0$, so that $T(E, g^E)$ is well defined.

Proposition 2.6. *The form $T(E, g^E)$ lies in $P^{S,0}$, and*

$$\frac{\overline{\partial}\partial}{2i\pi}T(E, g^E) = \mathrm{ch}(H(E, v), g^{H(E,v)}) - \mathrm{ch}(E, g^E). \tag{2.11}$$

2.2. Bott-Chern classes. Let E be a holomorphic vector bundle on a complex manifold S. Let $g^E, {g'}^E$ be two Hermitian metrics on E. Then by Bott and Chern [25], and by [17], there is a uniquely defined class $\widetilde{\mathrm{ch}}(E, g^E, {g'}^E) \in P^S/P^{S,0}$ such that

- If $g^E = {g'}^E$, $\widetilde{\mathrm{ch}}(E, g^E, {g'}^E) = 0$.
- The class $\widetilde{\mathrm{ch}}(E, g^E, {g'}^E)$ is functorial.
- The following equation holds

$$\frac{\overline{\partial}\partial}{2i\pi}\widetilde{\mathrm{ch}}(E, g^E, {g'}^E) = \mathrm{ch}(E, {g'}^E) - \mathrm{ch}(E, g^E). \tag{2.12}$$

The above classes are called Bott-Chern classes. The same construction applies to classes like $\widetilde{\mathrm{Td}}(E, g^E, {g'}^E)$. The class of forms $T(E, g^E) \in P^S/P^{S,0}$ constructed in Definition 2.5 is also a Bott-Chern class.

2.3. The higher analytic torsion forms associated to a holomorphic submersion. Following work by Gillet, Soulé and the author [17], we will extend the arguments of Section 2.1 to an infinite dimensional situation.

Let $\pi : X \to S$ be a holomorphic submersion with compact fibre Z. Let E be a holomorphic vector bundle on X, and let $R\pi_* E$ be the direct image of E. In the sequel $TX, TZ = TX/S \ldots$ denote the corresponding holomorphic tangent bundles. Let ω^X be a real closed $(1, 1)$ form on X which restricts to a fibrewise Kähler form on $TZ = TX/S$, so that if $J^{T_{\mathbf{R}}Z}$ is the complex structure of $T_{\mathbf{R}}Z$, $\omega(J^{T_{\mathbf{R}}Z}.,.)$ is a Hermitian product g^{TZ} on TZ . Let g^E be a Hermitian metric on E. Let $T^H X$ be the orthogonal bundle to TZ in TX with respect to ω^X. Let $(\Omega(Z, E_{|Z}), \overline{\partial}^Z)$ be the family of relative Dolbeault complexes along the fibres Z. Then $\Omega(Z, E_{|Z})$ can be equipped with the L_2 metric

$$< s, s' >= \int_Z < s, s' >_{\Lambda(T^{*(0,1)}Z)\otimes E} \frac{dv_Z}{(2\pi)^{\mathrm{dim} Z}}. \tag{2.13}$$

Let $\overline{\partial}^{Z*}$ be the adjoint of $\overline{\partial}^Z$. Put

$$D^Z = \overline{\partial}^Z + \overline{\partial}^{Z*}. \tag{2.14}$$

Definition 2.7. Let $\nabla^{\Omega(Z,E_{|Z})}$ be the connection on $\Omega(Z, E_{|Z})$, such that if $U \in T_{\mathbf{R}}S$, if s is a smooth section of $\Lambda(T^{*(0,1)}Z) \otimes E$,

$$\nabla_U^{\Omega(Z,E_{|Z})} s = \nabla_{U^H}^{\Lambda(T^{*(0,1)}Z)\otimes E} s. \tag{2.15}$$

Let T be the tensor defined in (1.18) associated to $(g^{TZ}, T^H X)$. Then T is of type $(1, 1)$. Let N be the number operator of $\Omega(Z, E_{|Z})$. Let $\omega^{X,H}$ be the restriction of ω^X to $T_{\mathbf{R}}^H X$. Then $\omega^{X,H}$ is a smooth section of $\pi^*\Lambda^{(1,1)}(T_{\mathbf{R}}^* S)$. Finally recall that $\Lambda(T^{*(0,1)}S) \otimes E$ is a Clifford module for the Clifford algebra of $(T_{\mathbf{R}}Z, g^{T_{\mathbf{R}}Z})$.

DEFINITION 2.8. For $t > 0$, put

$$B_t^{''} = \sqrt{t}\overline{\partial}^Z + \nabla^{\Omega(Z,E_{|Z})"} - \frac{c(T^{(1,0)})}{2\sqrt{2t}}, \tag{2.16}$$
$$B_t^{'} = \sqrt{t}\overline{\partial}^{Z*} + \nabla^{\Omega(Z,E_{|Z})'} - \frac{c(T^{(0,1)})}{2\sqrt{2t)}},$$
$$B_t = B_t^{''} + B_t^{'},\, N_t = N + i\frac{\omega^{X,H}}{t}.$$

Then one can show that, in (2.16), the superconnection B_t is a form of the Levi-Civita superconnection $A_{t/2}$ considered in (1.24). Also, by [17], an obvious analogue of Proposition 2.1 holds, with C_t'', C_t' replaced by B_t'', B_t', and N replaced by N_t.

DEFINITION 2.9. For $t > 0$, set

$$\alpha_t = \varphi \mathrm{Tr_s}[\exp(-B_t^2)], \gamma_t = \varphi \mathrm{Tr_s}[N_t \exp(-B_t^2)]. \tag{2.17}$$

THEOREM 2.10. *For $t > 0$, the form α_t and γ_t lie in P^S, the form α_t is closed and*

$$[\alpha_t] = \mathrm{ch}(R\pi_* E) \,\mathrm{in}\, H^{\mathrm{even}}(S, \mathbf{Q}), \tag{2.18}$$
$$\frac{\partial \alpha_t}{\partial t} = -\frac{\overline{\partial}\partial}{2i\pi}\frac{\gamma_t}{t}.$$

Furthermore, as $t \to 0$, there are forms $C_{-1}, C_0 \in P^S$ such that

$$\alpha_t = \pi_*[\mathrm{Td}(TZ, g^{TZ})\mathrm{ch}(E, g^E)] + \mathcal{O}(t), \tag{2.19}$$
$$\gamma_t = \frac{C_{-1}}{t} + C_0 + \mathcal{O}(t).$$

Observe that the first equation in (2.19) is a consequence of the local families index theorem of [9] stated in (1.27)

Assume that $R\pi_* E$ is locally free. Then the holomorphic vector bundle $R\pi_* E \simeq \ker D^Z$ inherits a metric $g^{R\pi_* E}$. By [5], as $t \to +\infty$,

$$\alpha_t = \mathrm{ch}(R\pi_* E, g^{R\pi_* E}) + \mathcal{O}(\frac{1}{\sqrt{t}}), \tag{2.20}$$
$$\gamma_t = \mathrm{ch}'(R\pi_* E, g^{R\pi_* E}) + \mathcal{O}(\frac{1}{\sqrt{t}}).$$

DEFINITION 2.11. For $s \in \mathbf{C}, 0 < \mathrm{Re(s)} < 1/2$, put

$$R(\omega^X, g^E)(s) = -\frac{1}{\Gamma(s)}\int_0^{+\infty} t^{s-1}(\gamma_t - \gamma_\infty)dt, \tag{2.21}$$
$$T(\omega^X, g^E) = \frac{\partial}{\partial s}R(\omega^X, g^E)(0).$$

In fact, by equations (2.19), (2.20), $R(\omega^X, g^E)(s)$ extends to a holomorphic function of s near $s = 0$, so that $T(\omega^X, g^E)$ is well-defined. The forms $T(\omega^X, g^E)$ are called higher analytic torsion forms. The following result was established in work by Gillet-Soulé and the author [17], and Köhler and the author [20].

Theorem 2.12. *The form $T(\omega^X, g^E)$ lies in P^S. Moreover*

$$\frac{\overline{\partial}\partial}{2i\pi}T(\omega^X, g^E) = \mathrm{ch}(R\pi_*E, g^{R\pi_*E}) - \pi_*[\mathrm{Td}(TZ, g^{TZ})\mathrm{ch}(E, g^E)]. \tag{2.22}$$

Remark 2.13. Clearly (2.22) refines (0.2) at the level of differential forms. Köhler and the author [20] showed that $T(\omega^X, g^E) \in P^S/P^{S,0}$ depends on ω^X, g^E via Bott-Chern classes. This result was proved before in degree 0 in [17]. A consequence of [20] is that $T(\omega^X, g^E) \in P^S/P^{S,0}$ depends on ω^X only via g^{TZ}. Let $P^{\ker D^Z}$ be the orthogonal projection of $\Omega(Z, E_{|Z})$ on $\ker D^Z$. Set $P^{\ker D^Z,\perp} = 1 - P^{\ker D^Z}$. For $s \in \mathbf{C}, \mathrm{Re}(s) >> 0$, put

$$\theta(s) = -\mathrm{Tr_s}[N(D^{Z,2})^{-s}P^{\ker D^Z,\perp}] \tag{2.23}$$

Then

$$T(\omega^X, g^E)^{(0)} = \frac{\partial\theta}{\partial s}(0). \tag{2.24}$$

Also $\exp(-\frac{1}{2}\frac{\partial\theta}{\partial s}(0))$ is called the Ray-Singer analytic torsion [51] of the complex $\Omega(Z, E_{|Z})$. The Ray-Singer torsion is an alternate product of generalized determinants of Laplacians.

By [17], the odd form $\tilde{\eta} = \frac{1}{4i\pi}(\overline{\partial} - \partial)T(\omega^X, g^E)$ coincides with the form constructed by Cheeger and the author in [15].

2.4. Quillen metrics. Assume temporarily that S is a point. Put

$$\lambda = (\det H^{\cdot}(Z, E_{|Z}))^{-1}. \tag{2.25}$$

Then λ is a complex line, the inverse of the determinant of the cohomology of E. Let $|\ \ |_\lambda$ be the metric on λ induced by the fibrewise L_2 metric on $g^{H(Z,E_{|Z})}$, which we obtain by identifying $H(Z, E_{|Z})$ to the corresponding harmonic forms.

Definition 2.14. The Quillen metric $\|\ \ \|_\lambda$ on λ is defined by

$$\|\ \ \|_\lambda = |\ \ |_\lambda \exp(-\frac{1}{2}\frac{\partial\theta}{\partial s}(0)). \tag{2.26}$$

In the general case where S is not a point, we still assume the existence of a form ω^X taken as in Section 2.3. Let g^{TZ} be an arbitrary fibrewise Kähler metric on TZ. Let g^E be a Hermitian metric on E. We no longer assume $R\pi_*E$ to be locally free. Put

$$\lambda(E) = (\det R\pi_*E)^{-1}. \tag{2.27}$$

Then by Knudsen-Mumford [40], $\lambda(E)$ is a holomorphic line bundle on S, and for any $s \in S$, there is a canonical isomorphism.

$$\lambda(E)_s \simeq (\det(H(Z_s, E_{|Z_s})))^{-1}. \tag{2.28}$$

By Definition 2.14, the fibres $\lambda(E)_s$ are equipped with the Quillen metric $\|\ \ \|_{\lambda(E)_s}$. The following result was established by Quillen [49] in the case where the fibres Z are a fixed Riemann surface, and by Gillet, Soulé and the author [17], following earlier work by Freed and the author [16] on smooth determinant bundles.

THEOREM 2.15. *The Quillen metric is smooth on $\lambda(E)$. Moreover*

$$c_1(\lambda(E), \| \quad \|_{\lambda(E)}) = -\pi_*[\mathrm{Td}(TZ, g^{TZ})\mathrm{ch}(E, g^E)]^{(2)}. \tag{2.29}$$

Remark 2.16. Theorem 2.15 is a consequence of (2.22), and also of anomaly formulas [17], describing the variation of Quillen metrics when g^{TZ}, g^E themselves vary. These anomaly formulas extend the Polyakov anomaly formulas for generalized determinants on Riemann surfaces [47].

2.5. FUNCTORIALITY OF THE ANALYTIC TORSION FORMS WITH RESPECT TO COMPOSITION OF SUBMERSIONS. Let

$$\begin{tikzcd} Z \arrow[r] \arrow[d, "\pi_{Z/Y}"'] & W \arrow[d, "\pi_{W/V}"'] \arrow[dr, "\pi_{W/S}"] & \\ Y \arrow[r] & V \arrow[r, "\pi_{V/S}"'] & S \end{tikzcd} \tag{2.30}$$

be a diagram of submersions $\pi_{W/S}, \pi_{V/S}, \pi_{W/V}$, with compact fibres Z, Y, X. Let ω^W, ω^V be closed (1,1) forms on W, V as in Section 2.3 . Let (E, g^E) be a holomorphic Hermitian vector bundle on W, such that $R\pi_{W/S*}E, R\pi_{W/V*}E$, $R\pi_{W/S*}R_{\pi_{W/V}*}E$ are locally free. Let $T_{W/V}(\omega^W, g^E), T_{W/S}(\omega^W, g^E)$, $T_{V/S}(\omega^V, g^{R\pi_{W/V*}E})$ be the analytic torsion forms which are associated to the maps in the above diagram. Then in work by Berthomieu and the author [8] and by Ma [41], using the adiabatic limit techniques of Cheeger and the author [15], Mazzeo-Melrose [44] and Dai [29], these forms were shown to be naturally compatible, i.e. they verify a relation which refines the functoriality of Riemann-Roch with respect to the composition of submersions. Namely, let $\widetilde{\mathrm{Td}}(TZ, TY, g^{TZ}, g^{TY}) \in P^W/P^{W,0}$ be the Bott-Chern class such that

$$\frac{\overline{\partial}\partial}{2i\pi}\widetilde{\mathrm{Td}}(TZ, TY, g^{TZ}, g^{TY}) = \mathrm{Td}(TZ, g^{TZ}) - \pi^*_{W/V}\left[\mathrm{Td}(TY, g^{TY})\right]\mathrm{Td}(TX, g^{TX}). \tag{2.31}$$

Under suitable assumptions, Ma [41] has constructed a Bott-Chern class $\alpha \in P^S/P^{S,0}$ such that

$$\begin{aligned} \tfrac{\overline{\partial}\partial}{2i\pi}\alpha = {} & \mathrm{ch}(R\pi_{V/S*}R\pi_{W/V*}E, g^{R\pi_{V/S*}R\pi_{W/V*}E}) \\ & -\mathrm{ch}(R\pi_{W/S*}E, g^{R\pi_{W/S*}E}), \end{aligned} \tag{2.32}$$

for which the following result holds.

THEOREM 2.17. *The following identity holds*

$$\begin{aligned} T_{W/S}(\omega^W, g^E) = {} & T_{V/S}(\omega^V, g^{R\pi_{W/V*}E}) + \pi_{W/S*}[\mathrm{Td}(TY, g^{TY})T_{W/V}(\omega^W, g^E)] \\ & +\alpha - \pi_{W/S*}[\widetilde{\mathrm{Td}}(TZ, TY, g^{TZ}, g^{TY})\mathrm{ch}(E, g^E)] \,\mathrm{in}\, \mathrm{P^S}/\mathrm{P^{S,0}}. \end{aligned} \tag{2.33}$$

Remark 2.18. The case where S is a point was considered in [8].

3. The analytic torsion currents associated to an embedding

3.1. Construction of the analytic torsion currents. Let $i : Y \to X$ be an embedding of complex manifolds. Let $N_{Y/X}$ be the normal bundle to Y in X. Let F be a holomorphic vector bundle on Y. Let

$$(E, v) : 0 \to E_m \overset{v}{\to} E_{m-1} \ldots \overset{v}{\to} E_0 \to 0 \tag{3.1}$$

be a holomorphic complex of vector bundles on X, which, together with a holomorphic restriction map: $r : E_{0|Y} \to F$, provides a resolution of the sheaf $i_* \mathcal{O}_Y(F)$. In particular (E, v) is acyclic on $X \setminus Y$. By [10], $H(E, v)_{|Y}$ is a holomorphic vector bundle on Y. Move precisely, if $U \in TX_{|Y}$, let $\partial_U v$ be the derivative of v in any holomorphic trivialization of (E, v) near Y. Then by $\partial_U v$ only depends on the image $z \in N_{Y/X}$ of U, and $(\partial_z v)^2 = 0$. Let $\pi : N_{Y/X} \to Y$ be the canonical projection. Then there is a canonical isomorphism

$$(\pi^* H((E, v)_{|Y}), \partial_z v) \simeq (\pi^*(\Lambda(N_{Y/X}) \otimes F), \sqrt{-1} i_z). \tag{3.2}$$

Let $g^E = \oplus_{i=0}^m g^{E_i}, g^{N_{Y/X}}, g^F$ be Hermitian metrics on $E = \oplus_{i=0}^m E_i, N_{Y/X}, F$. As in (2.3), put $V = v + v^*$. Then $H(E, v)_{|Y} \simeq \ker V_{|Y} \subset E_{|Y}$. Let $g^{H(E,v)}$ be the corresponding metric on $H(E, v)$.

We will say that g^E verifies assumption (A) with respect to $g^{N_{Y/X}}, g^F$ if (3.2) is an isometry. By [10], given $g^{N_{Y/X}}, g^F$, there exists $g^E = \oplus_{i=0}^m g^{E_i}$ such that assumption (A) is verified. From now on, we assume that (A) holds. For $t > 0$, we define $\alpha_t, \gamma_t \in P^X$ as in (2.6). Let δ_Y be the current of integration on Y. The following result was proved in [10], using formulas of Mathai and Quillen [43].

Theorem 3.1. *As $t \to +\infty$,*

$$\alpha_t = T\mathrm{d}^{-1}(N_{Y/X}, g^{N_{Y/X}}) \mathrm{ch}(F, g^F) \delta_Y + \mathcal{O}(\frac{1}{\sqrt{t}}), \tag{3.3}$$

where $\mathcal{O}(\frac{1}{\sqrt{t}})$ is taken in the suitable Sobolev space.

Remark 3.2. Using (1.12), we find that (3.3) refines the theorem of Riemann-Roch-Grothendieck [7] stated in (0.2) at the level of currents.

By (3.3), one can construct a current $T(E, g^E)$ on X as in (2.10). Let P_Y^X be the set of real currents which are sum of currents of type (p, p), whose front set is included in $N^*_{Y/X,\mathbf{R}}$. We define $P_Y^{X,0}$ as in Definition 2.2. The following result was proved in [18].

Theorem 3.3. *The current $T(E, g^E)$ lies in P_Y^X. Moreover*

$$\frac{\overline{\partial}\partial}{2i\pi} T(E, g^E) = \mathrm{Td}^{-1}(N_{Y/X}, g^{N_{Y/X}}) \mathrm{ch}(F, g^F) \delta_Y - \mathrm{ch}(E, g^E). \tag{3.4}$$

Remark 3.4. Harvey and Lawson [38] have also constructed currents related to smooth versions of Riemann-Roch-Grothendieck for embeddings.

3.2. Functoriality of the analytic torsion currents with respect to the composition of embeddings. Let $i' : Y' \to X, F', (E', v')$ be another set of data similar to the above data. Assume that Y and Y' intersect transversally. Put $Y'' = Y \cap Y'$. Then $(E \widehat{\otimes} E', v \widehat{\otimes} 1 + 1 \widehat{\otimes} v')$ is a resolution of $(F_{|Y''} \widehat{\otimes} F'_{|Y''})$.

Let $(g^E, g^{N_{Y/X}}, g^F)$ and $(g^{E'}, g^{N_{Y'/X}}, g^{F'})$ be metrics verifying (A). Recall that $N_{Y''/X} = N_{Y/X|Y''} \oplus N_{Y'/X|Y''}$. Then $(g^E \widehat{\otimes} g^{E'}, g^{N_{Y/X}}_{|Y''} \oplus g^{N_{Y'/X}}_{|Y''}, (g^F_{Y''} \widehat{\otimes} g^{F'}_{Y''}))$ also verify (A). Let $P^X_{Y \cup Y'}, P^{X,0}_{Y \cup Y'}$ be the obvious analogues of $P^X_Y, P^{X,0}_Y$, when replacing Y by $Y \cup Y'$. The following result was proved by Gillet, Soulé and the author in [19].

Theorem 3.5. *The following identity holds*

$$
\begin{aligned}
T(E \widehat{\otimes} E', g^{E \widehat{\otimes} E'}) = {} & T(E, g^E) \mathrm{ch}(E', g^{E'}) + \\
& \mathrm{Td}^{-1}(N_{Y/X}, g^{N_{Y/X}}) \mathrm{ch}(F, g^F) T(E', g^{E'}) \delta_Y \\
& \text{in } P^X_{Y \cup Y'} / P^{X,0}_{Y \cup Y'}.
\end{aligned}
\tag{3.5}
$$

Remark 3.6. In [19], Theorem 3.5 is used to evaluate the currents $T(E, g^E)$ in terms of the arithmetic characteristic classes of Gillet and Soulé [33], [34].

4. Analytic torsion forms and analytic torsion currents

4.1. Composition of an embedding and a submersion. Let $i : W \to V$ be an embedding of complex manifolds, and let S be a complex manifold. Let $\pi_{W/S}, \pi_{V/S}$ be holomorphic submersions of W, V onto S, with compact fibres X, Y, so that $\pi_{V/S} i = \pi_{W/S}$. Then we have the diagram

$$
\begin{tikzcd}
Y \arrow[r] \arrow[d, "i"'] & W \arrow[d, "i"'] \arrow[dr, "\pi_{W/S}"] & \\
X \arrow[r] & V \arrow[r, "\pi_{V/S}"'] & S
\end{tikzcd}
\tag{4.1}
$$

Let F be a holomorphic vector bundle on W. Let (E, v) be a complex of holomorphic vector bundles on V as in (3.1), which together with a restriction map $r : E_{0|V} \to F$, provides a resolution of $i_* F$. In the sequel we assume that $R\pi_{W/S*} F$ is locally free. Let $R\pi_{V/S*} E$ be the direct image of E. Tautologically, $R\pi_{V/S*} E \simeq R\pi_{W/S*} F$. Let ω^V, ω^W be (1,1) closed forms on V, W which restrict to Kähler forms on the fibres X, Y. Note that $N_{W/V} \simeq N_{Y/X}$. Let $g^{N_{Y/X}}, g^F$ be Hermitian metrics on $N_{Y/X}, F$. Let $g^E = \oplus_{i=0}^m g^{E_i}$ be a Hermitian metric on $E = \oplus_{i=0}^m E_i$, which verifies (A) with respect to $g^{N_{Y/X}}, g^F$.

4.2. Functoriality of the analytic torsion objects with respect to the composition of an embedding and a submersion. Let $\zeta(s) = \sum_{n=1}^{+\infty} \frac{1}{n^s}$ be the Riemann zeta function. Now we introduce the power series R of Gillet-Soulé [35].

Definition 4.1. Let R be the formal power series

$$R(x) = \sum_{\substack{n\geq 1 \\ n \text{ odd}}} \left(2\frac{\zeta'(-n)}{\zeta(-n)} + \sum_{j=1}^{n} \frac{1}{j} \right) \zeta(-n) \frac{x^n}{n!}. \tag{4.2}$$

We identify $R(x)$ with the corresponding additive genus. The power series R was obtained by Gillet-Soulé and Zagier by an explicit computation of the analytic torsion of $\mathbf{P}_n$, as a correction to the Todd genus $\widehat{\mathrm{Td}}$ of Gillet-Soulé's theory, which would fit into a conjectural form of Riemann-Roch-Grothendieck in Arakelov geometry.

Let $\widetilde{\mathrm{Td}}(TX_{|Y}, g^{TY}, g^{TX}_{|Y}, g^{N_{Y/X}}) \in P^W/P^{W,0}$ be the Bott-Chern class such that

$$\begin{aligned} \frac{\overline{\partial}\partial}{2i\pi} \widetilde{\mathrm{Td}}(TX_{|Y}, g^{TY}, g^{TX_{|Y}}, g^{N_{Y/X}}) &= \mathrm{Td}(TX_{|Y}, g^{TX_{|Y}}) \\ &-\mathrm{Td}(TY, g^{TY})\mathrm{Td}(N_{Y/X}, g^{N_{Y/X}}). \end{aligned} \tag{4.3}$$

Let $T(\omega^V, g^E) \in P^S$ be the analytic torsion forms associated to the family of double complexes $(\Omega(X, E_{|X}), (\overline{\partial}^X + v))$. Observe that $R\pi_{V/S*}E \simeq R\pi_{W/S*}F$ is now equipped with twoL_2 metrics $g^{R\pi_{V/S*}E}$ and $g^{R\pi_{W/S*}F}$. The following result was proved by Lebeau and the author [21] in the case where S is a point, and extended by the author in [13] to the general case.

Theorem 4.2. *The following identity holds*

$$\begin{aligned} &\widetilde{\mathrm{ch}}(R\pi_{W/S*}F, g^{R\pi_{W/S*}E}, g^{R\pi_{V/S*}F}) - T(\omega^W, g^F) + T(\omega^V, g^E) \\ &-\pi_{V/S*}[\mathrm{Td}(TX, g^{TX})T(E, g^E)] + \pi_{W/S*}\left[\frac{\widetilde{\mathrm{Td}}(TX_{|W}, g^{TY}, g^{TX_{|W}}, g^{N_{Y/X}})}{\mathrm{Td}(N_{Y/X}, g^{N_{Y/X}})}\mathrm{ch}(F, g^F)\right] \\ &-\pi_{V/S*}\left[\mathrm{Td}(TX)R(TX)\mathrm{ch}(E)\right] + \pi_{W/S*}[\mathrm{Td}(TY)R(TY)\mathrm{ch}(F)] = 0 \,\text{in}\, P^S/P^{S,0}. \end{aligned} \tag{4.4}$$

Remark 4.3. The main result of [21] is formulated as a formula of comparison of Quillen metrics on the determinant lines $\lambda(E) \simeq \lambda(F)$. An important idea in [21],[13] is to replace v by Tv, with $T > 0$, and to study the behaviour of the corresponding analytic torsion forms as $T \to +\infty$. Then one has to describe the behaviour of the associated harmonic forms, and also the full spectrum of the corresponding Laplacians In [21], [13], the appearance of the additive genus R is related to the evaluation in [11] of a characteristic class, the higher analytic torsion forms associated to a short exact sequence of holomorphic vector bundles. The evaluation of this class involves computations on a harmonic oscillator. The coincidence of this class of forms with the genus evaluated by Gillet and Soulé [35] remained unexplained until Bost [24] and Roessler [53] showed that the evaluation of the analytic torsion of $\mathbf{P}_n$ given in [35] can be obtained as a consequence of [21]. Of course, Theorems 2.17, 3.5 and 4.2 are compatible. In [12], the main result of [21] was interpreted as an excess intersection formula for Bott-Chern currents in infinite dimensions.

4.3. The Riemann-Roch theorem of Gillet and Soulé. Let X be an arithmetic variety, i.e. a regular flat scheme over $\mathrm{Spec}(\mathbf{Z})$. In [33], [34], Gillet-Soulé constructed an arithmetic Chow group $\widehat{CH}(X)$. By definition, $\widehat{CH}(X) = \widehat{Z}(X)/\widehat{R}(X)$, where $\widehat{Z}(X)$ is the group of arithmetic cycles (Z, g_Z) , with Z an algebraic cycle, and g_Z is a Green current on $X_{\mathbf{C}}$, i.e. it is a sum of real currents of type (p,p), smooth on $X_{\mathbf{C}} \backslash Z_{\mathbf{C}}$, such that $\frac{\overline{\partial}\partial}{2i\pi} g_Z + \delta_Z = \omega_Z$ is a smooth form on X, and $\widehat{R}(X)$ is an equivalence relation which refines linear equivalence.

Let (E, g^E) be an arithmetic vector bundle on X. Namely E is an algebraic vector bundle on X, g^E is a Hermitian metric on $X_{\mathbf{C}}$. Then Gillet and Soulé constructed arithmetic characteristic classes of (E, g^E) with values in $\widehat{CH}(X)_{\mathbf{Q}}$. More precisely they constructed a Grothendieck group $\widehat{K}_0(X)$ with contains equivalence classes of vector bundles (E, g^E), and also classes of forms of the type $P^X/P^{X,0}$, and a Chern character map $\widehat{\mathrm{ch}} : \widehat{K}_0(X) \to \widehat{CH}(X)_{\mathbf{Q}}$.

Let now $\pi : X \to S$ be a projective flat morphism of arithmetic varieties. Suppose that $\pi : X_{\mathbf{Q}} \to Y_{\mathbf{Q}}$ is smooth. Let ω^X be a smooth real $(1,1)$ form on $X_{\mathbf{Q}}$ as in Section 2.3. Let $(E, g^E) \in \widehat{K}_0(X)$ be such that $R^i\pi_* E = 0$ for $i > 0$. In [35], Gillet and Soulé defined $\pi_!(E, g^E) \in \widehat{K}_0(S)$ by the formula

$$\pi_!(E, g^E) = (R\pi_* E, g^{R\pi_* E}) - T(\omega^X, g^E). \tag{4.5}$$

This definition is then extended to arbitrary $(E, g^E)) \in \widehat{K}_0(X)$. Put

$$\mathrm{Td}^A(TX/S, g^{TX/S}) = \widehat{\mathrm{Td}}(TX/S, g^{TX/S})(1 - R(TX/S)). \tag{4.6}$$

The following result was conjectured by Gillet and Soulé in [35] and proved in [36], [37], using Theorem 4.2.

Theorem 4.4. *The following identity holds*

$$\widehat{\mathrm{ch}}(\pi_!(E, g^E)) = \pi_*[\mathrm{Td}^A(TX/S, g^{TX/S})\widehat{\mathrm{ch}}(E, g^E)] \text{ in } \widehat{CH}(S)_{\mathbf{Q}}. \tag{4.7}$$

Remark 4.5. Assume that $S = \mathrm{Spec}(\mathbf{Z})$. Then (4.7) is an equality in $\mathbf{R}$. It expresses the Arakelov degree of $\det(R\pi_* E)$ in terms of arithmetic characteristic classes.

In [30], Faltings has indicated an alternative strategy to the proof of the Gillet-Soulé theorem, based on the technique of deformation to the normal cone. Then one has to study the behaviour of the analytic torsion forms, as smooth fibres are deformed to the union of two smooth fibres intersecting transversally.

5. Higher analytic torsion and flat vector bundles

Let X be a smooth manifold, and let F be a complex flat vector bundle on X. Then by [28], the bundle F has Chern classes $c(F) \in H^{\mathrm{odd}}(X, \mathbf{C}/\mathbf{Z})$. For $\mathrm{Re}(c)(F) \in H(X, \mathbf{R})$, there is a corresponding Chern-Weil theory. In fact let ∇^F be the flat connection on F. Let g^F be a Hermitian metric on F. Put $\theta = (g^F)^{-1}\nabla^F g^F$. Then for k odd, $\mathrm{Re}(c_k)(F, g^F) = (2i\pi)^{-(k-1)/2} 2^{-k} \mathrm{Tr}[\theta^k]$ is a closed form which represents $\mathrm{Re}(c_k)(F) \in H^k(X, \mathbf{R})$.

Let $\pi : X \to S$ be a submersion of smooth manifolds, with compact fibre Z. Then $R\pi_* F$ is a $\mathbf{Z}$-graded flat vector bundle on S. Let $e(TZ) \in H(X, \mathbf{Q})$ be the Euler class of TZ.

Now we state a result by Lott and the author [22], which was proved using flat superconnections.

Theorem 5.1. *For any $k \in \mathbf{N}$, k odd,*

$$\operatorname{Re}(c_k)(R\pi_* F) = \pi_*[e(TZ)\operatorname{Re}(c_k)(F)]. \tag{5.1}$$

Given a metric g^F and a Euclidean connection ∇^{TZ}, let $g^{R\pi_* F}$ be the L_2 Hermitian metric on $R\pi_* F$ which is obtained via fibrewise Hodge theory. In [22], higher analytic torsion forms $T(g^F, \nabla^{TZ})$ are constructed such that

$$dT(g^F, \nabla^{TZ}) = \pi_*[e(TZ, \nabla^{TZ})\operatorname{Re}(c_\cdot)(F, g^F)] - \operatorname{Re}(c_\cdot)(R\pi_* F, g^{R\pi_* F}). \tag{5.2}$$

In degree 0, $T(g^F, \nabla^{TZ})$ is the Ray-Singer analytic torsion of [50]. The Ray-Singer conjecture, proved by Cheeger [26] and Müller [45] says that for unitarily flat vector bundles, the Ray-Singer analytic torsion coincides with a geometrically defined invariant of the manifold, the Reidemeister torsion [52]. In higher degree, the interpretation of $T(g^F, \nabla^{TZ})$ is still mysterious. There is a possible link with work by Igusa and Klein [39] on Borel regulators. For related results in an algebraic context, we refer to Bloch and Esnault [23].

Acknowledgements *The author is indebted to J.-B. Bost, E. Getzler and C. Soulé for their comments and suggestions on a preliminary version of this paper.*

References

[1] Atiyah, M.F., Bott, R., Patodi, V.K. : On the heat equation and the index theorem. Invent. Math. 19, 279-330 (1973).

[2] Atiyah, M.F., Patodi, V.K., Singer I.M. : Spectral asymmetry and Riemannian geometry, I. Math. Proc. Cambridge Philos. Soc. 77, 43-69 (1975).

[3] Atiyah, M.F., Singer, I.M. : The index of elliptic operators. III, Ann. of Math. 87, 546-604 (1968).

[4] Atiyah, M.F., Singer, I.M. : The index of elliptic operators. IV, Ann. of Math. 93, 119-138 (1971).

[5] Berline, N., Getzler E., Vergne, M. : Heat kernels and the Dirac operator. Grundl. der Math. Wiss. Band 298. Berlin-Heidelberg-New-York: Springer 1992.

[6] Berline N., Vergne M. : A proof of Bismut local index theorem for a family of Dirac operators. Topology 26, 435-463 (1987).

[7] Berthelot, P., Grothendieck, A., Illusie, L. : Théorie des intersections et Théorème de Riemann-Roch. Lecture Notes in Math. 225. Berlin-Heidelberg-New-York: Springer 1971

[8] Berthomieu A., Bismut, J.-M. : Quillen metrics and higher analytic torsion forms. J. reine angew. Math. 457, 85-184 (1994).

[9] Bismut, J.-M. : The index Theorem for families of Dirac operators : two heat equation proofs. Invent. Math. 83, 91-151 (1986).

[10] Bismut, J.-M. : Superconnection currents and complex immersions. Invent. Math. 99, 59-113 (1990).

[11] Bismut, J.-M. : Koszul complexes, harmonic oscillators and the Todd class. J.A.M.S. 3, 159-256 (1990).

[12] Bismut, J.-M.: On certain infinite dimensional aspects of Arakelov intersection theory. Comm. Math. Phys. 148, 217-248 (1992).

[13] Bismut, J.-M. : Holomorphic families of immersions and higher analytic torsion forms. Astérisque 244. Paris SMF 1997.

[14] Bismut, J.-M.: Local index theory, eta invariants and holomorphic torsion: a survey. J. Diff. Geom. To appear.

[15] Bismut, J.-M., Cheeger, J. : η-invariants and their adiabatic limits. J.A.M.S. 2, 33-70 (1989).

[16] Bismut, J.-M., Freed D. : The analysis of elliptic families. I. Metrics and connections on determinant bundles. Comm. Math. Phys. 106, 159-176 (1986). II. Eta invariants and the holonomy theorem. Comm. Math. Phys. 107, 103-163 (1986).

[17] Bismut, J.-M., Gillet, H., Soulé, C. : Analytic torsion and holomorphic determinant bundles, I. Comm. Math. Phys. 115, 49-78 (1988).II. Comm. Math. Phys. 115, 79-126 (1988). III. Comm. Math. Phys. 115, 301-351 (1988).

[18] Bismut, J.-M., Gillet, H., Soulé, C. : Bott-Chern currents and complex immersions. Duke Math. Journal 60, 255-284 (1990).

[19] Bismut, J.-M., Gillet, H., Soulé, C. : Complex immersions and Arakelov geometry. The Grothendieck Festschrift, P. Cartier et al. ed. Vol. I, pp. 249-331. Progress in Math. n° 86. Boston : Birkhaüser 1990.

[20] Bismut, J.-M., Köhler, K. : Higher analytic torsion forms for direct images and anomaly formulas, J. Alg. Geom. 1, 647-684 (1992).

[21] Bismut, J.-M., Lebeau, G. : Complex immersions and Quillen metrics, Publ. Math. IHES, 74, 1-297 (1991).

[22] Bismut, J.-M., Lott, J. : Flat vector bundles, direct images, and higher real analytic torsion. J.Am. Math. Soc. 8, 291-363 (1995).

[23] Bloch, S., Esnault, H. : A Riemann-Roch theorem for flat bundles, with values in the algebraic Chern-Simons theory, preprint 1998.

[24] Bost, J.-M. : Analytic torsion of projective spaces and compatibility with immersions of Quillen metrics. I.M.R.N. 8, 427-435 (1998).

[25] Bott, R., Chern, S.S. : Hermitian vector bundles and the equidistribution of the zeroes of their holomorphic sections. Acta Math. 114, 71-112 (1965).

[26] Cheeger J.: Analytic torsion and the heat equation. Ann. of Math. 109, 259-322 (1979).

[27] Cheeger J.: η-invariants, the adiabatic approximation and conical singularities. J. Diff. Geom. 26, 175-221 (1987).

[28] Cheeger, J., Simons, J. : Differential characters and geometric invariants. In Geometry and Topology, J. Alexander and J. Harer ed., Lect. Notes in Math. n° 1167, p. 50-80. Berlin-Heidelberg-New-York: Springer 1985.

[29] Dai, X. : Adiabatic limits, nonmultiplicativity of signature and Leray spectral sequence, J.A.M.S. 4, 265-321 (1991).

[30] Faltings G. : Lectures on the arithmetic Riemann-Roch theorem. Ann. Math. Studies 127. Princeton: Princeton University Press 1992.

[31] Getzler, E. : A short proof of the Atiyah-Singer Index Theorem, Topology 25, 111-117 (1986).

[32] Gilkey, P. : Curvature and the eigenvalues of the Laplacian for elliptic complexes. Adv. Math. 10, 344-382 (1973).

[33] Gillet, H., Soulé, C. : Arithmetic Intersection Theory. Publ. Math. IHES, 72, 93-174 (1990).

[34] Gillet, H., Soulé, C. : Characteristic classes for algebraic vector bundles with hermitian metrics. Ann. Math. I. 131, 163-203 (1990). II. 131, 205-238 (1990).

[35] Gillet, H., Soulé, C. : Analytic torsion and the arithmetic Todd genus. Topology, 30, 21-54 (1991).

[36] Gillet, H. Soulé, C. : An arithmetic Riemann-Roch theorem. Inv. Math., 110, 473-543 (1992).

[37] Gillet, H., Soulé C. : To appear.

[38] Harvey, F.R., Lawson, H.B. : A theory of characteristic currents associated with a singular connection. Astérisque 213. Paris: S.M.F. (1993).

[39] Igusa, K., Klein J.: The Borel regulator map on pictures II: An example from Morse theory. J. of K-theory, 7, 225-267 (1993)

[40] Knudsen, F.F., Mumford, D. : The projectivity of the moduli space of stable curves, I: Preliminaries on "det" and "div". Math Scand. 39, 19-55 (1976).

[41] Ma, X. : Formes de torsion analytique et familles de submersion. C.R.A.S. 324 (Série I), 205-210 (1997).

[42] McKean, H., Singer, I.M. : Curvature and the eigenforms of the Laplacian. J. Diff. Geom. 1, 43-69 (1967).

[43] Mathai, V., Quillen, D. : Superconnections, Thom classes, and equivariant differential forms. Topology 25, 85-110 (1986).

[44] Mazzeo, R., Melrose, R. : The adiabatic limit, Hodge cohomology and Leray's spectral sequence of a fibration. J. Diff. Geom. 31, 185-213 (1990).

[45] Müller W. : Analytic torsion and R-torsion of Riemannian manifolds. Adv. in Math. 28, 233-305 (1978).

[46] Patodi, V.K. : Curvature and the eigenforms of the Laplacian. J. Diff. Geom. 5, 233-249 (1971).

[47] Polyakov, A.M.: Quantum geometry of bosonic strings. Phys. Letters B, 103, 207-210 (1981).

[48] Quillen, D. : Superconnections and the Chern character. Topology 24, 89-95 (1985).

[49] Quillen, D. : Determinants of Cauchy-Riemann operators over a Riemannian surface. Funct. Anal. Appl. 19, 31-34 (1985).

[50] Ray, D.B., Singer, I.M. : R-torsion and the Laplacian on Riemannian manifolds. Adv. in Math. 7, 145-210 (1971).

[51] Ray, D.B., Singer, I.M. : Analytic torsion for complex manifolds. Ann. of Math. 98, 154-177 (1973).

[52] Reidemeister K. : Homotopieringe und Linsenräume. Hamburger Abhandl. 11, 102-109 (1935).

[53] Roessler D.: An Adams-Riemann-Roch theorem in Arakelov geometry. To appear in Duke Math. Journal (1999).

Jean-Michel Bismut
Département de Mathématique
Université Paris-Sud
Bâtiment 425
91405 Orsay Cedex
France
bismut@topo.math.u-psud.fr