# Some Analogies Between Number Theory and Dynamical Systems on Foliated Spaces

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Abstract. In this article we describe what a cohomology theory related to zeta and $L$-functions for algebraic schemes over the integers should look like. We then point out some striking analogies with the leafwise reduced cohomology of certain foliated dynamical systems.

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## 1 Introduction

For the arithmetic study of varieties over finite fields powerful cohomological methods are available which in particular shed much light on the nature of the corresponding zeta functions. These investigations culminated in Deligne's proof of an analogue of the Riemann conjecture for such zeta functions. This had been the hardest part of the Weil conjectures. For algebraic schemes over $\operatorname{Spec}\mathbb{Z}$ and in particular for the Riemann zeta function no cohomology theory has yet been developed that could serve similar purposes. For a long time it had even been a mystery how such a theory could look like even formally. In this article following [D1–D4] we first describe the shape that a cohomological formalism for algebraic schemes over the integers should take. We then discuss how it would relate to the many conjectures on arithmetic zeta- and $L$-functions and indicate a couple of consequences of the formalism that can be proved using standard methods. As it turns out there is a large class of dynamical systems on foliated manifolds whose reduced leafwise cohomology has many of the expected structural properties of the desired cohomology for algebraic schemes. Comparing the arithmetic and dynamical pictures leads to some insight into the basic geometric structures

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that dynamical systems relevant for $L$-functions of varieties over number fields should have. There is also a very interesting recent approach by Connes [C] to the Riemann conjecture for Hecke $L$-series which bears some formal similarities to the preceding considerations. It seems to be closer in spirit to the theory of automorphic $L$-functions though.

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## 2 GEOMETRIC ZETA- AND $L$-FUNCTIONS

Consider the Riemann zeta function

$$\zeta(s) = \prod_p (1 - p^{-s})^{-1} = \sum_{n=1}^{\infty} n^{-s} \quad \text{for } \operatorname{Re} s > 1 \; .$$

It has a holomorphic continuation to $\mathbb{C} \setminus \{1\}$ with a simple pole at $s = 1$. To its finite Euler factors

$$\zeta_p(s) = (1 - p^{-s})^{-1}$$

we add an Euler factor corresponding to the archimedean place $p = \infty$ of $\mathbb{Q}$

$$\zeta_\infty(s) = 2^{-1/2}\, \pi^{-s/2}\, \Gamma(s/2)$$

and introduce the completed zeta function

$$\hat{\zeta}(s) = \zeta(s)\zeta_\infty(s) \; .$$

It is holomorphic in $\mathbb{C} \setminus \{0, 1\}$ with simple poles at $s = 0, 1$ and satisfies the functional equation:

$$\hat{\zeta}(1 - s) = \hat{\zeta}(s) \; .$$

Its zeroes are the so called non-trivial zeroes of $\zeta(s)$, i.e. those in the critical strip $0 < \operatorname{Re} s < 1$. The famous Riemann conjecture asserts that they all lie on the line $\operatorname{Re} s = 1/2$.
Apart from its zeroes, the special values of $\zeta(s)$, i.e. the numbers $\zeta(n)$ for integers $n \geq 2$, have received a great deal of attention. Recently, as a special case of the Bloch–Kato conjectures, it has been possible to express them entirely in terms of cohomological invariants of $\mathbb{Q}$; c.f. [BK], [HW]. Together with the theory of $\zeta$-functions of curves over finite fields this suggests that the Riemann zeta function should be cohomological in nature. The rest of this article will be devoted to a thorough discussion of this hypothesis in a broader context.

A natural generalization of the Riemann zeta function to the context of arithmetic geometry is the Hasse–Weil zeta function $\zeta_{\mathcal{X}}(s)$ of an algebraic scheme $\mathcal{X}/\mathbb{Z}$

$$\zeta_{\mathcal{X}}(s) = \prod_{x \in |\mathcal{X}|} (1 - N(x)^{-s})^{-1} \; , \; \operatorname{Re} s > \dim \mathcal{X}$$

where $|\mathcal{X}|$ is the set of closed points of $\mathcal{X}$ and $N(x)$ is the number of elements in the residue field of $x$. For $\mathcal{X} = \operatorname{Spec}\mathbb{Z}$ we recover $\zeta(s)$, and for $\mathcal{X} = \operatorname{Spec}\mathfrak{o}_k$, where $\mathfrak{o}_k$ is the ring of integers in a number field $k$, the Dedekind zeta function of $k$. It is expected that $\zeta_{\mathcal{X}}(s)$ has a meromorphic continuation to $\mathbb{C}$ and, if $\mathcal{X}$ is regular, that

$$\hat{\zeta}_{\mathcal{X}}(s) = \zeta_{\mathcal{X}}(s)\zeta_{\mathcal{X}_\infty}(s)$$

has a simple functional equation with respect to the substitution of $s$ by $\dim\mathcal{X} - s$. Here $\zeta_{\mathcal{X}_\infty}(s)$ is a certain product of $\Gamma$-factors depending on the Hodge structure on the cohomology of $\mathcal{X}_\infty = \mathcal{X}\otimes\mathbb{R}$. This is known if $\mathcal{X}$ is equicharacteristic, i.e. an $\mathbb{F}_p$-scheme for some $p$, by using the Lefschetz trace formula and Poincaré duality for $l$-adic cohomology.
The present strategy for approaching $\zeta_{\mathcal{X}}(s)$ was first systematically formulated by Langlands. He conjectured that every Hasse–Weil zeta function is up to finitely many Euler factors the product of automorphic $L$-functions. One could then apply the theory of these $L$-functions which is quite well developed in important cases although by no means in general. For $\mathcal{X}$ with generic fibre related to Shimura varieties this Langlands program has been achieved in very interesting examples. Another spectacular instance was Wiles' proof with Taylor of modularity for most elliptic curves over $\mathbb{Q}$.
The strategy outlined in section 3 of the present article is completely different and much closer to the cohomological methods in characteristic $p$.
By the work of Deligne [De], it is known that for proper regular $\mathcal{X}/\mathbb{F}_p$ the zeroes (resp. poles) of $\hat{\zeta}_{\mathcal{X}}(s) = \zeta_{\mathcal{X}}(s)$ have real parts equal to $\nu/2$ for odd (resp. even) integers $0 \le \nu \le 2\dim\mathcal{X}$, and one may expect the same for the completed Hasse Weil zeta function $\hat{\zeta}_{\mathcal{X}}(s)$ of an arbitrary proper and regular scheme $\mathcal{X}/\mathbb{Z}$.
As for the orders of vanishing at the integers, a conjecture of Soulé [So] asserts that for $\mathcal{X}/\mathbb{Z}$ regular, quasiprojective connected and of dimension $d$, we have the formula

$$\operatorname{ord}_{s=d-n}\zeta_{\mathcal{X}}(s) = \sum_{i=0}^{2n}(-1)^{i+1}\dim\operatorname{Gr}^n_\gamma(K_{2n-i}(\mathcal{X})\otimes\mathbb{Q}) \ . \tag{1}$$

Here the associated graded spaces are taken with respect to the $\gamma$-filtration on algebraic $K$-theory. Unfortunately it is not even known, except in special cases, whether the dimensions on the right hand side are finite.

For a (mixed) motive $M$ over $\mathbb{Q}$ – intuitively a "piece" in the total cohomology of a variety $X$, such as $H^w(X)$ – analogy with the function field case leads to the following definition of the $L$-function:

$$L(M,s) = \prod_p L_p(M,s) \quad \text{where} \quad L_p(M,s) = \det\nolimits_{\mathbb{Q}_l}(1 - p^{-s}\mathrm{Fr}^*_p \mid M_l^{I_p})^{-1} \ .$$

Here $M_l$ is the $l$-adic realization of $M$ for any $l \neq p$ and $\mathrm{Fr}_p, I_p$ are the inverse of a Frobenius automorphism in $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ and an inertia group at $p$, respectively. For example, the $l$-adic realization of $M = H^w(X)$ is the $w$-th $l$-adic cohomology of $X \otimes \overline{\mathbb{Q}}$. Rationality and independence of $l$ of the characteristic polynomial of $\mathrm{Fr}_p^*$ are expected for all $p$, known in many cases and assumed in the following.

If $\mathcal{X}$ is proper and flat over $\operatorname{Spec}\mathbb{Z}$ with smooth generic fibre $X = \mathcal{X} \otimes \mathbb{Q}$, then up to finitely many Euler factors we have:

$$\zeta_{\mathcal{X}}(s) = \prod_{w=0}^{2\dim X} L(H^w(X), s)^{(-1)^w}$$

Adding a suitable product of $\Gamma$-factors $L_\infty(M,s)$ defined in [Se] and [F-PR] III which depends only on the real Hodge realization $M_B$ over $\mathbb{R}$ we obtain the completed $L$-function of the motive

$$\hat{L}(M,s) = L(M,s) L_\infty(M,s) \; .$$

In terms of the filtration $\mathcal{V}$ on $M_B$ introduced in [D3] § 6, we have

$$L_\infty(M,s) = \prod_{n\in\mathbb{Z}} \zeta_\infty(s-n)^{d_n}$$

where $d_n = \dim \operatorname{Gr}_{\mathcal{V}}^n M_B$.
Define $\hat{L}_S(M,s)$ by omitting the Euler factors corresponding to a finite set of places $S$.
For later purposes we recall the following definition due to Scholl. A motive over $\mathbb{Q}$ is called integral at $p$ if the weight filtration on the $l$-adic realization for $l \neq p$ splits as a module under the inertia group at $p$. For a finite set $S$ of prime numbers let $\mathcal{M}_{\mathbb{Z}_S}$ be the category of motives over $\mathbb{Q}$ which are integral at all $p \notin S$.

The following conjectures are a great challenge to arithmetic geometry. Except for the fourth they have been confirmed in many cases after first identifying the $L$-function of a motive with a product of automorphic $L$-functions.

CONJECTURES 2.1 Let $M$ be a (mixed) motive over $\mathbb{Q}$.

1. $L(M,s)$ and hence $\hat{L}(M,s)$ have a meromorphic continuation to $\mathbb{C}$ and there is a functional equation

$$\hat{L}(M,s) = \varepsilon(M,s)\hat{L}(M^*, 1-s)$$

where $\varepsilon(M,s) = a\, e^{bs}$ for some real $a, b$.

2. $\hat{L}(M,s) = \hat{L}_1(M,s)\hat{L}_{02}(M,s)^{-1}$
where $\hat{L}_1(M,s)$ is entire of genus one and $\hat{L}_{02}(M,s)$ is a polynomial in $s$ whose zeroes are integers.

3. (Artin) If $M$ is simple and not a Tate motive $\mathbb{Q}(n)$, the $L$-function $L(M,s)$ has no poles.

4. (Riemann) If $M$ is pure of weight $w$, e.g. $M = H^w(X)$ for a smooth proper variety $X/\mathbb{Q}$, then the zeroes of $\hat{L}(M,s)$ lie on the line $\operatorname{Re} s = \frac{w+1}{2}$.

5. (Deligne, Beilinson, Scholl) For $M$ in $\mathcal{M}_{\mathbb{Z}}$

$$\operatorname{ord}_{s=0} L(M,s) = \dim \operatorname{Ext}^1_{\mathcal{M}_{\mathbb{Z}}}(\mathbb{Q}(0), M^*(1)) - \dim \operatorname{Hom}_{\mathcal{M}_{\mathbb{Z}}}(\mathbb{Q}(0), M^*(1)) \; .$$

## 3 The conjectural cohomological formalism

In this section we interpret many of the conjectures about zeta- and $L$-functions in terms of an as yet speculative infinite dimensional cohomology theory. We also describe a number of consequences of this very rigid formalism that can be proved directly. Among these there is a formula which expresses the Riemann $\zeta$-function as a zeta-regularized product. After giving the definition of regularized determinants in a simple algebraic setting we first discuss the formalism in the case of the Riemann zeta function and then generalize to Hasse–Weil zeta functions and motivic $L$-series.

Given a $\mathbb{C}$-vector space $H$ with an endomorphism $\Theta$ such that $H$ is the countable sum of finite dimensional $\Theta$-invariant subspaces $H_\alpha$, the spectrum $\mathrm{sp}\,(\Theta)$ is defined as the union of the spectra of $\Theta$ on $H_\alpha$, the eigenvalues being counted with their algebraic multiplicities. The (zeta-)regularized determinant $\det_\infty(\Theta \,|\, H)$ of $\Theta$ is defined to be zero if $0 \in \mathrm{sp}\,(\Theta)$, and by the formula

$$\det{}_\infty(\Theta \,|\, H) := \prod_{\alpha \in \mathrm{sp}\,(\Theta)} \alpha := \exp(-\zeta'_\Theta(0)) \tag{2}$$

if $0 \notin \mathrm{sp}\,(\Theta)$. Here

$$\zeta_\Theta(z) = \sum_{0 \neq \alpha \in \mathrm{sp}\,(\Theta)} \alpha^{-z} \;, \quad \text{where} \quad -\pi < \arg \alpha \leq \pi \;,$$

is the spectral zeta function of $\Theta$. For (2) to make sense we require that $\zeta_\Theta$ be convergent in some right half plane, with meromorphic continuation to $\mathrm{Re}\, z > -\varepsilon$, for some $\varepsilon > 0$, holomorphic at $z = 0$. For an endomorphism $\Theta_0$ on a real vector space $H_0$, such that $\Theta = \Theta_0 \otimes \mathrm{id}$ on $H = H_0 \otimes \mathbb{C}$ satisfies the above requirements, we set

$$\det{}_\infty(\Theta_0 \,|\, H_0) = \det{}_\infty(\Theta \,|\, H) \;.$$

On a finite dimensional vector space $H$ we obtain the ordinary determinant of $\Theta$. As an example of a regularized determinant, consider an endomorphism $\Theta$ whose spectrum consists of the number $1, 2, 3, \ldots$ with multiplicities one. Then

$$\det{}_\infty(\Theta \,|\, H) = \prod_{\nu=1}^{\infty} \nu = \sqrt{2\pi} \quad \text{since} \quad \zeta'(0) = -\log \sqrt{2\pi} \;.$$

The regularized determinant plays a role for example in Arakelov theory and in string theory. In our context it allows us to write the different Euler factors of zeta- and $L$-functions in a uniform way as we will first explain for the Riemann zeta function.

Let $\mathcal{R}_p$ for $p \neq \infty$ be the $\mathbb{R}$-vector space of real valued finite Fourier series on $\mathbb{R}/(\log p)\mathbb{Z}$ and set

$$\mathcal{R}_\infty = \mathbb{R}[\exp(-2y)] \quad \text{for } p = \infty \;.$$

These spaces carry a natural $\mathbb{R}$-action $\sigma^t$ via $(\sigma^t f)(y) = f(y+t)$ with infinitesimal generator $\Theta = d/dy$. The eigenvalues of $\Theta$ on $\mathcal{C}_p = \mathcal{R}_p \otimes \mathbb{C}$ are just the poles of $\zeta_p(s)$.

PROPOSITION 3.1 *We have* $\zeta_p(s) = \det_\infty \left(\frac{1}{2\pi}(s-\Theta) \,|\, \mathcal{R}_p\right)^{-1}$ *for* $\quad p \le \infty$.

This is easily proved by applying a classical formula of Lerch for the derivative of the Hurwitz zeta function at zero [D3] 2.7.
In a sense $\overline{\operatorname{Spec}\mathbb{Z}} = \operatorname{Spec}\mathbb{Z} \cup \infty$ is analogous to a projective curve over a finite field. The Grothendieck Lefschetz trace formula in characteristic $p$ together with the proposition, suggest that a formula of the following type might hold:

$$\hat{\zeta}(s) = \prod_{i=0}^{2} \det_\infty \left(\tfrac{1}{2\pi}(s-\Theta) \,|\, H^i(\text{“}\overline{\operatorname{Spec}\mathbb{Z}}\text{”}, \mathcal{R})\right)^{(-1)^{i+1}} . \tag{3}$$

Here $H^i(\text{“}\overline{\operatorname{Spec}\mathbb{Z}}\text{”}, \mathcal{R})$ would be some real cohomology vector space equipped with a canonical endomorphism $\Theta$ associated to some space “$\overline{\operatorname{Spec}\mathbb{Z}}$” corresponding to $\overline{\operatorname{Spec}\mathbb{Z}}$. As recalled earlier $\hat{\zeta}(s)$ has poles only at $s = 0, 1$ and these are of first order. Moreover the zeroes of $\hat{\zeta}(s)$ are just the non-trivial zeroes of $\zeta(s)$. If we assume that the eigenvalues of $\Theta$ on $H^i(\text{“}\overline{\operatorname{Spec}\mathbb{Z}}\text{”}, \mathcal{R})$ are distinct for $i = 0, 1, 2$ it follows therefore that

- $H^0(\text{“}\overline{\operatorname{Spec}\mathbb{Z}}\text{”}, \mathcal{R}) = \mathbb{R}$ with trivial action of $\Theta$, i.e. $\Theta = 0$,
- $H^1(\text{“}\overline{\operatorname{Spec}\mathbb{Z}}\text{”}, \mathcal{R})$ is infinite dimensional, the spectrum of $\Theta$ consisting of the non-trivial zeroes $\rho$ of $\zeta(s)$ with their multiplicities,
- $H^2(\text{“}\overline{\operatorname{Spec}\mathbb{Z}}\text{”}, \mathcal{R}) \cong \mathbb{R}$ but with $\Theta = \mathrm{id}$.
- For $i > 2$ the cohomologies $H^i(\text{“}\overline{\operatorname{Spec}\mathbb{Z}}\text{”}, \mathcal{R})$ should vanish.

Formula (3) implies that

$$\xi(s) := \frac{s}{2\pi}\frac{(s-1)}{2\pi}\hat{\zeta}(s) = \prod_{\rho} \frac{1}{2\pi}(s-\rho) .$$

This formula turned out to be true [D2], [SchS]. Earlier a related formula had been observed in [K].
If $H$ is some space with an endomorphism $\Theta$ let us write $H(\alpha)$ for $H$ equipped with the twisted endomorphism $\Theta_{H(\alpha)} = \Theta - \alpha\,\mathrm{id}$. With this notation we expect a canonical “trace”-isomorphism:

$$\mathrm{tr} : H^2(\text{“}\overline{\operatorname{Spec}\mathbb{Z}}\text{”}, \mathcal{R}) \xrightarrow{\sim} \mathbb{R}(-1) .$$

In our setting the cup product pairing

$$\cup : H^i(\text{“}\overline{\operatorname{Spec}\mathbb{Z}}\text{”}, \mathcal{R}) \times H^{2-i}(\text{“}\overline{\operatorname{Spec}\mathbb{Z}}\text{”}, \mathcal{R}) \longrightarrow H^2(\text{“}\overline{\operatorname{Spec}\mathbb{Z}}\text{”}, \mathcal{R}) \cong \mathbb{R}(-1)$$

induces a pairing for every $\alpha$ in $\mathbb{C}$:

$$\cup : H^i(\text{“}\overline{\operatorname{Spec}\mathbb{Z}}\text{”}, \mathcal{C})^{\Theta\sim\alpha} \times H^{2-i}(\text{“}\overline{\operatorname{Spec}\mathbb{Z}}\text{”}, \mathcal{C})^{\Theta\sim 1-\alpha} \longrightarrow H^2(\text{“}\overline{\operatorname{Spec}\mathbb{Z}}\text{”}, \mathcal{C})^{\Theta\sim 1} \cong \mathbb{C} .$$

Here $\Theta \sim \alpha$ denotes the subspace of

$$H^i(\text{“}\overline{\operatorname{Spec}\mathbb{Z}}\text{”}, \mathcal{C}) = H^i(\text{“}\overline{\operatorname{Spec}\mathbb{Z}}\text{”}, \mathcal{R}) \otimes \mathbb{C}$$

of elements annihilated by some power of $\Theta - \alpha$. We expect Poincaré duality in the sense that these pairings should be non-degenerate for all $\alpha$. This is compatible with the functional equation of $\hat{\zeta}(s)$. For the precise relation see [D3] 7.19.
In the next section we will have more to say on the type of cohomology theory that might be expected for $H^i(\text{“}\overline{\operatorname{Spec}\mathbb{Z}}\text{”}, \mathcal{R})$. But first let us note a nice consequence our approach would have. Consider the linear flow $\lambda^t = \exp t\Theta$ on $H^i(\text{“}\overline{\operatorname{Spec}\mathbb{Z}}\text{”}, \mathcal{R})$. It is natural to expect that it is the flow induced on cohomology by a flow $\phi^t$ on the underlying space “$\overline{\operatorname{Spec}\mathbb{Z}}$”, i.e. $\lambda^t = (\phi^t)^*$. This implies that $\lambda^t$ would respect cup product and that $\Theta$ would behave as a derivation. Now assume that as in the case of compact Riemann surfaces there is a Hodge $*$-operator:

$$* : H^1(\text{“}\overline{\operatorname{Spec}\mathbb{Z}}\text{”}, \mathcal{R}) \xrightarrow{\sim} H^1(\text{“}\overline{\operatorname{Spec}\mathbb{Z}}\text{”}, \mathcal{R}) \; ,$$

such that

$$\langle f, f' \rangle = \operatorname{tr}(f \cup (*f')) \quad \text{for } f, f' \text{ in } H^1(\text{“}\overline{\operatorname{Spec}\mathbb{Z}}\text{”}, \mathcal{R}) \; ,$$

is positive definite, i.e. a scalar product on $H^1(\text{“}\overline{\operatorname{Spec}\mathbb{Z}}\text{”}, \mathcal{R})$. It is natural to assume that $(\phi^t)^*$ and hence $\Theta$ commutes with $*$ on $H^1(\text{“}\overline{\operatorname{Spec}\mathbb{Z}}\text{”}, \mathcal{R})$. From the equality:

$$f_1 \cup f_2 = \Theta(f_1 \cup f_2) = \Theta f_1 \cup f_2 + f_1 \cup \Theta f_2$$

for $f_1, f_2$ in $H^1(\text{“}\overline{\operatorname{Spec}\mathbb{Z}}\text{”}, \mathcal{R})$ we would thus obtain the formula

$$\langle f_1, f_2 \rangle = \langle \Theta f_1, f_2 \rangle + \langle f_1, \Theta f_2 \rangle \; ,$$

and hence that $\Theta = \frac{1}{2} + A$ where $A$ is a skew-symmetric endomorphism of $H^1(\text{“}\overline{\operatorname{Spec}\mathbb{Z}}\text{”}, \mathcal{R})$. Hence the Riemann conjecture would follow.
The formula $\Theta = \frac{1}{2} + A$ is also in accordance with numerical investigations on the fluctuations of the spacings between consecutive non-trivial zeroes of $\zeta(s)$. It was found that their statistics resembles that of the fluctuations in the spacings of consecutive eigenvalues of random real skew symmetric matrices, as opposed to the different statistics for random real symmetric matrices; see [Sa] for a full account of this story. In fact the comparison was made between hermitian and symmetric matrices, but as pointed out to me by M. Kontsevich, the statistics in the hermitian and real skew symmetric cases agree.
The completion of $H^1(\text{“}\overline{\operatorname{Spec}\mathbb{Z}}\text{”}, \mathcal{R})$ with respect to $\langle , \rangle$, together with the unbounded operator $\Theta$ would be the space that Hilbert was looking for, and that Berry [B] suggested to realize in a quantum physical setting.

The following considerations are necessary for comparison with the dynamical picture.
Formula (3) is closely related to a reformulation of the explicit formulas in analytic number theory using the conjectural cohomology theory above, see [I] Kap. 3 and [JL] for the precise relationship. Set $\mathbb{R}^+ = (0, \infty)$.

PROPOSITION 3.2 *For a test function $\varphi \in \mathcal{D}(\mathbb{R}^+) = C_0^\infty(\mathbb{R}^+)$ define an entire function $\Phi(s)$ by the formula*

$$\Phi(s) = \int_{\mathbb{R}} \varphi(t) e^{ts}\, dt \, .$$

*Then we have the "explicit formula":*

$$\Phi(0) - \sum_{\hat{\zeta}(\rho)=0} \Phi(\rho) + \Phi(1) = \sum_p \log p \sum_{k=1}^{\infty} \varphi(k \log p) + \int_0^\infty \frac{\varphi(t)}{1-e^{-2t}}\, dt \, .$$

We wish to interpret this well known formula along the lines of [P] § 3. For this we require the following elementary notion of a distributional trace. Consider a real or complex vector space $H$ with a linear $\mathbb{R}$-action

$$\lambda : \mathbb{R} \times H \to H \, , \quad \lambda(t,h) = \lambda^t(h) \, ,$$

which decomposes into a countable direct sum of finite dimensional invariant subspaces $H_n$. Let $\mathrm{Tr}(\lambda \,|\, H_n)_{\mathrm{dis}}$ be the distribution on $\mathbb{R}^+$ associated to the function $t \mapsto \mathrm{Tr}(\lambda^t \,|_{H_n})$, and set

$$\mathrm{Tr}(\lambda \,|\, H)_{\mathrm{dis}} = \sum_n \mathrm{Tr}(\lambda \,|\, H_n)_{\mathrm{dis}} \tag{4}$$

if the sum converges in the space of distributions $\mathcal{D}'(\mathbb{R}^+)$. By assumption $\lambda$ can be written as $\lambda^t = \exp t\Theta$ with an endomorphism $\Theta$ of $H$, and we have

$$\mathrm{Tr}(\lambda \,|\, H)_{\mathrm{dis}} = \sum_{\alpha \in \mathrm{sp}\,(\Theta)} \langle e^{t\alpha} \rangle \quad \text{in } \mathcal{D}'(\mathbb{R}^+)$$

if the series converges. Here $\langle f \rangle \in \mathcal{D}'(\mathbb{R}^+)$ denotes the distribution associated to a locally integrable function $f$ on $\mathbb{R}^+$. Thus

$$\langle \mathrm{Tr}(\lambda \,|\, H)_{\mathrm{dis}}, \varphi \rangle = \sum_{\alpha \in \mathrm{sp}\,(\Theta)} \int_{\mathbb{R}} \varphi(t) e^{t\alpha}\, dt = \sum_{\alpha \in \mathrm{sp}\,(\Theta)} \Phi(\alpha)$$

for any test function $\varphi$ in the Schwartz space $\mathcal{D}(\mathbb{R}^+)$. Conjecturally (3.2) can thus be reformulated as the following identity of distributions

$$\sum_i (-1)^i \mathrm{Tr}(\phi^* \,|\, H^i(\text{“}\overline{\mathrm{Spec}\,\mathbb{Z}}\text{”}, \mathcal{R}))_{\mathrm{dis}} = \sum_p \log p \sum_{k=1}^{\infty} \delta_{k \log p} + \langle (1-e^{-2t})^{-1} \rangle \, . \tag{5}$$

Using the Poisson summation formula one sees that

$$\mathrm{Tr}(\sigma \,|\, \mathcal{R}_p)_{\mathrm{dis}} = \log p \sum_{k=1}^{\infty} \delta_{k \log p} \quad \text{for finite } p \, .$$

A direct calculation shows that

$$\mathrm{Tr}(\sigma \,|\, \mathcal{R}_\infty)_{\mathrm{dis}} = \langle (1-e^{-2t})^{-1} \rangle \, .$$

Hence (5) can be rewritten as a sheaf theoretic Lefschetz trace formula

$$\sum_i (-1)^i \mathrm{Tr}(\phi^* \mid H^i(\text{“}\overline{\mathrm{Spec}\,\mathbb{Z}}\text{”}, \mathcal{R}))_{\mathrm{dis}} = \sum_{p \le \infty} \mathrm{Tr}(\phi^* \mid \mathcal{R}_p)_{\mathrm{dis}} \; . \tag{6}$$

For more on this see [D5], [DSch].

We now turn to Hasse–Weil zeta functions of algebraic schemes $\mathcal{X}/\mathbb{Z}$. A similar argument as for the Riemann zeta function suggests that

$$\zeta_{\mathcal{X}}(s) = \prod_{i=0}^{2d} \mathrm{det}_\infty \left( \frac{1}{2\pi}(s - \Theta) \mid H^i_c(\text{“}\mathcal{X}\text{”}, \mathcal{R}) \right)^{(-1)^{i+1}} \tag{7}$$

where $H^i_c(\text{“}\mathcal{X}\text{”}, \mathcal{R})$ is some real cohomology with compact supports associated to a dynamical system “$\mathcal{X}$” attached to $\mathcal{X}$ and $d = \dim \mathcal{X}$. Here $\Theta$ should be the infinitesimal generator of the induced flow on cohomology. In particular we would have

$$\mathrm{ord}_{s=\alpha}\, \zeta_{\mathcal{X}}(s) \;=\; \sum_{i=0}^{2d} (-1)^{i+1} \dim H^i_c(\text{“}\mathcal{X}\text{”}, \mathcal{C})^{\Theta \sim \alpha} \; .$$

For a regular connected $\mathcal{X}$ the Poincaré duality pairing

$$\cup : H^i_c(\text{“}\mathcal{X}\text{”}, \mathcal{R}) \times H^{2d-i}(\text{“}\mathcal{X}\text{”}, \mathcal{R}) \longrightarrow H^{2d}_c(\text{“}\mathcal{X}\text{”}, \mathcal{R}) \xrightarrow{\sim} \mathbb{R}(-d) \tag{8}$$

should identify

$$H^i_c(\text{“}\mathcal{X}\text{”}, \mathcal{C})^{\Theta \sim \alpha} \quad \text{with the dual of} \quad H^{2d-i}(\text{“}\mathcal{X}\text{”}, \mathcal{C})^{\Theta \sim d - \alpha} \; .$$

In particular we would get:

$$\mathrm{ord}_{s=d-n}\, \zeta_{\mathcal{X}}(s) \;=\; \sum_{i=0}^{2d} (-1)^{i+1} \dim H^i(\text{“}\mathcal{X}\text{”}, \mathcal{C}(n))^{\Theta \sim 0} \; ,$$

where $\mathcal{C}(\alpha)$ is the sheaf $\mathcal{C}$ on “$\mathcal{X}$” with action of the flow twisted by $e^{-\alpha t}$. Thus

$$H^i(\text{“}\mathcal{X}\text{”}, \mathcal{C}(n))^{\Theta \sim 0} = H^i(\text{“}\mathcal{X}\text{”}, \mathcal{C})^{\Theta \sim n} \; .$$

For a regular $\mathcal{X}$ we expect formal analogues of Tate's conjecture

$$H^i_{\mathcal{M}}(\mathcal{X}, \mathbb{C}(n)) := \mathrm{Gr}^{\,n}_{\gamma} K_{2n-i}(\mathcal{X}) \otimes \mathbb{C} \xrightarrow{\sim} H^i(\text{“}\mathcal{X}\text{”}, \mathcal{C}(n))^{\Theta \sim 0} \; , \tag{9}$$

and in particular that

$$H^i(\text{“}\mathcal{X}\text{”}, \mathcal{C}(n))^{\Theta \sim 0} = 0 \quad \text{for } i > 2n \; .$$

Note that the latter assertion says that the weights of $\Theta$ on $H^i(\text{“}\mathcal{X}\text{”}, \mathcal{C})$, i.e. twice the real parts of its eigenvalues, should be $\ge i$. This would imply Soulé's conjecture

(2.1).
Again the explicit formulas could be expressed in terms of cohomology in the form

$$\sum_i (-1)^i \mathrm{Tr}(\phi^* \mid H^i_c(\text{“}\mathcal{X}\text{”}, \mathcal{R}))_{\mathrm{dis}} = \sum_{x \in |\mathcal{X}|} \log N(x) \sum_{k=1}^{\infty} \delta_{k \log N(x)} \; . \tag{10}$$

In support of these ideas we have the following result.

THEOREM 3.3 *On the category of algebraic $\mathbb{F}_p$-schemes $\mathcal{X}$ there is a cohomology theory in $\mathbb{C}$-vector spaces with a linear flow such that* (7) *holds. For a regular connected $\mathcal{X}$ of dimension $d$ it satisfies Poincaré duality* (8). *Moreover* (9) *reduces to the Tate conjecture for $l$-adic cohomology.*

See [D3] § 4, [D4] § 2 for more precise statements and the simple construction based on $l$-adic cohomology. This approach cannot be generalized to non-equicharacteristic $\mathcal{X}/\mathbb{Z}$.

If there were a dynamical cohomology theory $H^i(\text{“}\overline{\mathcal{X}}\text{”}, \mathcal{R})$ attached to some Arakelov compactification $\overline{\mathcal{X}}$ of $\mathcal{X}$ such that

$$\hat{\zeta}_{\mathcal{X}}(s) = \prod_{i=0}^{2d} \det\nolimits_{\infty} \left( \frac{1}{2\pi}(s - \Theta) \mid H^i(\text{“}\overline{\mathcal{X}}\text{”}, \mathcal{R}) \right)^{(-1)^{i+1}} ,$$

then as above Poincaré duality for $H^i(\text{“}\overline{\mathcal{X}}\text{”}, \mathcal{R})$ would be in accordance with the expected functional equation for $\hat{\zeta}_{\mathcal{X}}(s)$. A Hodge $*$-operator

$$* : H^i(\text{“}\overline{\mathcal{X}}\text{”}, \mathcal{R}) \longrightarrow H^{2d-i}(\text{“}\overline{\mathcal{X}}\text{”}, \mathcal{R})$$

defining a scalar product via $\langle f, f' \rangle = \mathrm{tr}(f \cup (*f'))$ and for which

$$\phi^{t*} \circ * = (e^t)^{d-i} * \circ \phi^{t*} \; , \quad \text{i.e.} \quad \Theta \circ * = * \circ (d - i + \Theta) \; ,$$

holds, would imply that $\Theta - i/2$ is skew symmetric, hence the Riemann hypotheses for $\hat{\zeta}_{\mathcal{X}}(s)$. The last equation means that the flow changes the metric defining the $*$-operator by the conformal factor $e^t$.

As we mentioned above the zeta function $\zeta_{\mathcal{X}}(s)$ is up to finitely many Euler factors the alternating product of the $L$-functions of the motives $H^i(X)$. In [D1] we constructed cohomology $\mathbb{R}$-vector spaces $H^w_{\mathrm{ar}}$ with a linear flow on the category of varieties over $\mathbb{R}$ or $\mathbb{C}$ such that

$$\zeta_{\mathcal{X}_\infty}(s) = \prod_{i=0}^{2 \dim \mathcal{X}_\infty} \det\nolimits_{\infty} \left( \frac{1}{2\pi}(s - \Theta) \mid H^i_{\mathrm{ar}}(\mathcal{X}_\infty) \right)^{(-1)^{i+1}} .$$

Cup product and functoriality turn the spaces $H^i_{\mathrm{ar}}(\mathcal{X}_\infty)$ into modules under $H^0_{\mathrm{ar}}(\mathcal{X}_\infty) = H^0_{\mathrm{ar}}(\mathrm{Spec}\, \mathbb{R}) = \mathcal{R}_\infty$ of rank equal to $\dim H^i(\mathcal{X}_\infty, \mathbb{Q})$. Philosophically the scheme $\mathcal{X}$ should have bad semistable “reduction” at infinity. In accordance with this idea Consani [Cons] has refined the theory $H^i_{\mathrm{ar}}$ to a cohomology theory with a linear flow and a monodromy operator $N$ which contains $H^i_{\mathrm{ar}}$ as the kernel of $N$.

We now turn our attention to motivic $L$-series. The first task is to express the local Euler factors $L_p(M,s)$ in terms of regularized determinants on some spaces functorially attached to $M$.

Theorem 3.4 *For every $p \leq \infty$, there is a left exact additive functor $\mathcal{F}_p$ from motives over $\mathbb{Q}$ to the category of $\mathbb{C}$-vector spaces with a linear flow such that*

$$L_p(M,s) = \det_\infty\left(\frac{1}{2\pi}(s-\Theta)\,|\,\mathcal{F}_p(M)\right)^{-1} .$$

*The functor $\mathcal{F}_p$ commutes with Tate twists, and there are natural flow equivariant maps*

$$\mathcal{F}_p(M) \otimes \mathcal{F}_p(M') \longrightarrow \mathcal{F}_p(M \otimes M') \tag{11}$$

*turning $\mathcal{F}_p(M)$ into an $\mathcal{F}_p(\mathbb{Q}(0)) = \mathcal{C}_p$-module of rank equal to* $\dim M_l^{I_p}$ *for finite $p$ and equal to* rk$M$ *for $p = \infty$. On the category of motives integral at $p$ the functor $\mathcal{F}_p$ is exact. On motives with good reduction at $p$ the map* (11) *is an isomorphism and $\mathcal{F}_p$ commutes with duals. For $p = \infty$ it has a real structure $\mathcal{F}_\infty^{\mathbb{R}}$ and there is a natural perfect pairing:*

$$\mathcal{F}_\infty^{\mathbb{R}}(M)^{\Theta=0} \times \mathrm{Ext}^1_{MH_{\mathbb{R}}}(\mathbb{R}(0), M_B^*(1)) \longrightarrow \mathbb{R} ,$$

*where $MH_{\mathbb{R}}$ is the category of real mixed Hodge structures over $\mathbb{R}$. For varieties $X/\mathbb{R}$ we have*

$$H^w_{\mathrm{ar}}(X) = \mathcal{F}_\infty^{\mathbb{R}}(H^w(X)) .$$

The proofs – which are quite formal – can be found in [D3]. The functor $\mathcal{F}_\infty$ is constructed from $M_B$ by a construction á la Fontaine using a simple Barsotti–Tate ring. For finite $p$, the construction applies an elementary case of the Riemann–Hilbert correspondence to $M_l^{I_p} \otimes_{\mathbb{Q}_l} \mathbb{C}$ with the Frobenius action. It can also be viewed as an association of Fontaine's type.

By the theorem

$$\hat{L}(M,s) = \prod_{p \leq \infty} \det_\infty\left(\frac{1}{2\pi}(s-\Theta)\,|\,\mathcal{F}_p(M)\right)^{-1} ,$$

and this suggests that

$$\hat{L}(M,s) = \prod_{i=0}^{2} \det_\infty\left(\frac{1}{2\pi}(s-\Theta)\,|\,H^i(\text{“}\overline{\mathrm{Spec}\,\mathbb{Z}}\text{”}, \mathcal{F}(M))\right)^{(-1)^{i+1}} \tag{12}$$

for some sheaf with action of the flow $\mathcal{F}(M)$ on “$\overline{\mathrm{Spec}\,\mathbb{Z}}$” whose stalks “at the points $p$” should be isomorphic to $\mathcal{F}_p(M)$. It should be thought of as an analogue of the sheaf $\mathcal{F}(\mathcal{M}) = j_*\mathcal{M}$ for a $\mathbb{Q}_l$-sheaf $\mathcal{M}$ on the generic point $\eta$ of a curve $Y$ over a finite field, where $j : \eta \hookrightarrow Y$ is the inclusion.
Formula (12) would represent $\hat{L}(M,s)$ as a quotient of entire functions – at least

if the regularized determinants are of the Cartier–Voros type [CV].
This together with Poincaré duality for the sheaf cohomologies $H^i(\text{“}\overline{\operatorname{Spec}\mathbb{Z}}\text{”}, \mathcal{F}(M))$ would explain the first part of conjecture 2.1 c.f. [D3] 7.19.
The assertion about $\hat{L}_{02}(M,s)$ in the second part of 2.1 means that $H^0(\text{“}\overline{\operatorname{Spec}\mathbb{Z}}\text{”}, \mathcal{F}(M))$ and $H^2(\text{“}\overline{\operatorname{Spec}\mathbb{Z}}\text{”}, \mathcal{F}(M))$ should be finite dimensional with $\Theta$ having only integer eigenvalues.
The Riemann conjecture would follow from purity: For a pure motive $M$ of weight $w$ the eigenvalues of $\Theta$ on $H^i(\text{“}\overline{\operatorname{Spec}\mathbb{Z}}\text{”}, \mathcal{F}(M))$ should have real part $\frac{w+i}{2}$. As before there is a Hodge $*$-argument for this c.f. [D3] 7.11.
For $L(M,s)$ we expect the formula

$$L(M,s) = \prod_{i=0}^{2} \det{}_\infty \left( \frac{1}{2\pi}(s-\Theta) \,|\, H^i_c(\text{“}\operatorname{Spec}\mathbb{Z}\text{”}, \mathcal{F}(M)) \right)^{(-1)^{i+1}} \tag{13}$$

and by Poincaré duality

$$L(M,s) = \prod_{i=0}^{2} \det{}_\infty \left( \frac{1}{2\pi}(s+\Theta) \,|\, H^i(\text{“}\operatorname{Spec}\mathbb{Z}\text{”}, \mathcal{F}(M^*(1))) \right)^{(-1)^{i+1}} . \tag{14}$$

See [D3] (7.19.1). This implies that

$$\operatorname{ord}_{s=0} L(M,s) = \sum_{i=0}^{2} (-1)^{i+1} \dim H^i(\text{“}\operatorname{Spec}\mathbb{Z}\text{”}, \mathcal{F}(M^*(1)))^{\Theta\sim 0} .$$

On the category $\mathcal{M}_\mathbb{Z}$ all functors $\mathcal{F}_p$ are exact by the theorem. Hence $\mathcal{F}$ should be exact and therefore induce maps for all $N$ in $\mathcal{M}_\mathbb{Z}$

$$\mathcal{F} : \operatorname{Ext}^i_{\mathcal{M}_\mathbb{Z}}(\mathbb{Q}(0), N) \otimes \mathbb{C} \longrightarrow \operatorname{Ext}^i_{\text{“}\operatorname{Spec}\mathbb{Z}\text{”}}(\mathcal{C}(0), \mathcal{F}(N))^{\Theta\sim 0} = H^i(\text{“}\operatorname{Spec}\mathbb{Z}\text{”}, \mathcal{F}(N))^{\Theta\sim 0} . \tag{15}$$

If these are isomorphisms (2.1) part 5. follows. Note that because $\operatorname{Spec}\mathbb{Z}$ is an affine curve it is reasonable to expect $H^i(\text{“}\operatorname{Spec}\mathbb{Z}\text{”}, \mathcal{F}(N))$ to vanish for $i \geq 2$. Similarly (15) with $\mathbb{Z}$ replaced by $\mathbb{Z}_S$ ought to be an isomorphism. The eigenvalues of $\Theta$ on

$$H^0(\text{“}\operatorname{Spec}\mathbb{Z}\text{”}, \mathcal{F}(N)) = H^0(\text{“}\overline{\operatorname{Spec}\mathbb{Z}}\text{”}, \mathcal{F}(N)) \quad \text{(c.f. [D4] § 4)}$$

being integers, we have

$$H^0(\text{“}\operatorname{Spec}\mathbb{Z}\text{”}, \mathcal{F}(N)) = \bigoplus_{n\in\mathbb{Z}} \operatorname{Hom}(\mathbb{Q}(0), N(n)) \otimes \mathbb{C}$$

by (15) applied to all twists $N(n)$. Together with (14) we would get the Artin conjecture (2.1) part 3. Further conjectures on $L$-functions and extensions of motives by Deligne, Scholl and Selberg are related to the cohomological formalism in [D4] §§ 4, 9.

Let us now turn to certain consequences of the formalism that have been proved. As for the Riemann zeta function we must have

$$\xi(M,s) := \hat{L}(M,s)\prod_{\tau}\frac{1}{2\pi}(s-\tau) = \prod_{\rho}\frac{1}{2\pi}(s-\rho) \tag{16}$$

where $\rho$ runs over the zeroes of $\hat{L}(M,s)$ and $\tau$ over its finitely many poles. This follows from the theory of [JL] or [I] assuming standard conjectures on the analytic behaviour of $L$-series. For example for $L(E,s)$, where $E$ is a modular elliptic curve, formula (16) is a theorem.

As explained above there should be a trace isomorphism

$$\mathrm{tr} : H^2(\text{“}\overline{\mathrm{Spec}\,\mathbb{Z}}\text{”}, \mathcal{R}(1)) = H^2(\text{“}\overline{\mathrm{Spec}\,\mathbb{Z}}\text{”}, \mathcal{R}(1))^{\Theta=0} \xrightarrow{\sim} \mathbb{R} \, .$$

Comparing this with (15) we are led to search for a category of (mixed) motives $\mathcal{M}_{\overline{\mathbb{Z}}}$ over $\overline{\mathrm{Spec}\,\mathbb{Z}}$ equipped with a non-trivial map

$$\mathrm{Ext}^2_{\mathcal{M}_{\overline{\mathbb{Z}}}}(\mathbb{Q}(0), \mathbb{Q}(1)) \longrightarrow \mathbb{R} \, .$$

Integrality at a finite prime $p$ can be expressed in terms of the functor $\mathcal{F}_p$, c.f. [DN] appendix. For $\mathcal{F}_\infty$ this condition means that the real Hodge structure $M_B$ be split. Taking this as our definition of integrality at $p=\infty$ we define $\mathcal{M}_{\overline{\mathbb{Z}}}$ to be the subcategory of motives in $\mathcal{M}_{\mathbb{Q}}$ which are integral at all primes $p \le \infty$. Under the natural injection [Sch] 2.7

$$\mathbb{Q}^* \hookrightarrow \mathrm{Ext}^1_{\mathcal{M}_{\mathbb{Q}}}(\mathbb{Q}(0), \mathbb{Q}(1)) \, , \tag{17}$$

the motive corresponding to $\alpha$ is integral at $p \le \infty$ iff $|\alpha|_p = 1$. In [DN] it was shown that if (17) is an isomorphism rationally then $\mathrm{Ext}^2_{\mathcal{M}_{\overline{\mathbb{Z}}}}(\mathbb{Q}(0), \mathbb{Q}(1))$ is non-zero. If $\mathcal{M}_{\overline{\mathbb{Z}}}$ is replaced by the category (1-motives$\,/\mathrm{Spec}\,\overline{\mathbb{Z}}) \otimes \mathbb{Q}$ then $\mathrm{Ext}^2(\mathbb{Q}(0), \mathbb{Q}(1))$ is non-zero unconditionally, [J] Cor. 5.5. Furthermore it was shown that the motivic height pairing of [Sch] could be interpreted as a Yoneda pairing followed by the degree map

$$\mathrm{Ext}^1_{\mathcal{M}_{\overline{\mathbb{Z}}}}(\mathbb{Q}(0), M) \times \mathrm{Ext}^1_{\mathcal{M}_{\overline{\mathbb{Z}}}}(\mathbb{Q}(0), M^*(1)) \longrightarrow \mathrm{Ext}^2_{\mathcal{M}_{\overline{\mathbb{Z}}}}(\mathbb{Q}(0), \mathbb{Q}(1)) \longrightarrow \mathbb{R} \, .$$

This is in accordance with the idea that under a suitable extension of the isomorphism (15) to $\overline{\mathrm{Spec}\,\mathbb{Z}}$, (c.f. [D4] (2.4)), the motivic height pairing will correspond to Poincaré duality

$$H^1(\text{“}\overline{\mathrm{Spec}\,\mathbb{Z}}\text{”}, \mathcal{F}(M)) \times H^1(\text{“}\overline{\mathrm{Spec}\,\mathbb{Z}}\text{”}, \mathcal{F}(M^*(1))) \longrightarrow H^2(\text{“}\overline{\mathrm{Spec}\,\mathbb{Z}}\text{”}, \mathcal{C}) \overset{\mathrm{tr}}{\cong} \mathbb{C}$$

restricted to the $\Theta \sim 0$ parts.

Apart from local $L$-factors there are also local $\varepsilon$-factors attached to motives. In [D6] the functors $\mathcal{F}_p$ and a notion of regularized super-dimension were used among other things to give a comparatively uniform description of these factors at all places.

A motive $M$ of weight $w$ with coefficients in a number field $T$ is called orthogonal if there is a symmetric morphism $M \otimes M \to T(-w)$ which induces an isomorphism $M^* \cong M(w)$. For example the Artin motive attached to a representation $\rho : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_N(T)$ is orthogonal if and only if $\rho$ is orthogonal. Our formalism implies that for orthogonal $M$ the cup product induces a symplectic form on $H^1(\text{“}\overline{\mathrm{Spec}\,\mathbb{Z}}\text{”}, \mathcal{F}(M))^{\Theta \sim \frac{w+1}{2}}$ which must therefore be of even dimension. Hence the order of vanishing of $L(M,s)$ at the central point $\frac{w+1}{2}$ must be even and the sign in the functional equation therefore be $+1$ c.f. [D4] § 6. For Artin motives this is a theorem of Fröhlich and Queyrut which was extended to more general motives by T. Saito in [S] using crystalline methods.

We close this section with some remarks on trace formulas. If the $L$-functions of motives satisfy the expected analytic properties, one can easily extend the explicit formulas of analytic number theory for the $\hat{L}_S$-function to this context, see for example [DSch] or [JL]. In terms of our conjectural cohomology theory these can be reformulated – as for the Riemann zeta function – as the following equalities of distributions on $\mathbb{R}^+$:

$$\sum_i (-1)^i \mathrm{Tr}(\psi^* \mid H^i_c(\text{“}\overline{\mathrm{Spec}\,\mathbb{Z}} \setminus S\text{”}, \mathcal{F}(M)))_{\mathrm{dis}} = \sum_{p \notin S} \log p \sum_{k=1}^{\infty} \mathrm{Tr}(\mathrm{Fr}_p^k \mid M_l^{I_p}) \delta_{k \log p} + \alpha(S) \left\langle \frac{\mathrm{Tr}(e^{\bullet t} \mid \mathrm{Gr}^{\bullet}_{\mathcal{V}} M_B)}{1 - e^{-2t}} \right\rangle \tag{18}$$

and

$$\sum_i (-1)^i \mathrm{Tr}(\psi^* \mid H^i_c(\text{“}\overline{\mathrm{Spec}\,\mathbb{Z}} \setminus S\text{”}, \mathcal{F}(M)))_{\mathrm{dis}} = \sum_{p \le \infty, p \notin S} \mathrm{Tr}(\psi^* \mid \mathcal{F}_p)_{\mathrm{dis}} \, .$$

Here “$\overline{\mathrm{Spec}\,\mathbb{Z}} \setminus S$” is the dynamical system corresponding to $\overline{\mathrm{Spec}\,\mathbb{Z}} \setminus S$ and we have written $\psi^{t*}$ for the induced flow on cohomology with sheaf coefficients in accordance with notations in the next section. Moreover $e^{\bullet t}$ is the map $e^{nt}$ on $\mathrm{Gr}^n_{\mathcal{V}} M_B$ and $\alpha(S)$ is zero or one according to whether $S$ contains $p = \infty$ or not.

In the next section we consider trace formulas for dynamical systems on foliated spaces which bear striking formal similarities with (5) and its generalization (18).

## 4 Dynamical systems on foliated spaces

We begin by recalling a formula due to Guillemin and Sternberg [GS] VI § 2. Consider a smooth compact manifold $X$ with a flow $\phi^t$, i.e. a smooth action

$$\phi : X \times \mathbb{R} \to X \, , \quad \phi^t(x) = \phi(x,t) \, .$$

The compact orbits are assumed to be non-degenerate in the following sense. If $x$ is a fixed point of the flow, i.e. $\phi^t(x) = x$ for all $t$, then the tangent map $T_x\phi^t : T_x X \to T_x X$ should not have 1 as an eigenvalue for any $t > 0$. The vector field $Y_\phi$ generated by the flow is $\phi$-invariant in the sense that $T_x\phi^t(Y_{\phi,x}) = Y_{\phi,\phi^t(x)}$ for all points $x$ in $X$. Thus for any point $x$ on a periodic orbit $\gamma$ of length $l(\gamma)$

and any positive integer $k$ the endomorphism $T_x\phi^{k\cdot l(\gamma)}$ of $T_xX$ has $Y_{\phi,x}$ as an eigenvector of eigenvalue 1. Non-degeneracy of $\gamma$ means that the eigenvalue 1 does not occur on $T_xX/T_x^0$ where $T_x^0 = \mathbb{R}\cdot Y_{\phi,x}$.
Let $E$ be a smooth vector bundle on $X$ with an action opposite to $\phi$, i.e. a family of maps

$$\psi^t : \phi^{t*}E \longrightarrow E$$

satisfying an obvious cocycle condition. Note that for any $x \in \gamma$ we get maps

$$\psi_x^{kl(\gamma)} : E_{\phi^{kl(\gamma)}(x)} = E_x \longrightarrow E_x \ ,$$

and that the traces $\mathrm{Tr}(\psi_x^{kl(\gamma)} \,|\, E_x)$ are independent of the choice of $x$ on $\gamma$. For a fixed point $x$ the traces $\mathrm{Tr}(\psi_x^t \,|\, E_x)$ are defined for all $t$.

Consider the endomorphisms

$$\psi^{t*} : \Gamma(X,E) \xrightarrow{\phi^{t*}} \Gamma(X,\phi^{t*}E) \xrightarrow{\psi^t} \Gamma(X,E)$$

of the Fréchet space $\Gamma(X,E)$. In order to define a distributional trace

$$\mathrm{Tr}(\psi^* \,|\, \Gamma(X,E)) \quad \text{in} \quad \mathcal{D}'(\mathbb{R}^+) \ ,$$

Guillemin and Sternberg proceed as follows. Consider the restriction $\phi : X\times\mathbb{R}^+ \to X$, the diagonal map $\Delta : X\times\mathbb{R}^+ \to X\times X\times\mathbb{R}^+, \Delta(x,t) = (x,x,t)$ and the projections $p : X\times\mathbb{R}^+ \to X, \pi : X\times\mathbb{R}^+ \to \mathbb{R}^+$. View $\psi$ as a map $\psi : \phi^*E \to p^*E$ and let $K_{\psi^*}$ be the Schwartz kernel of the composite map:

$$\psi^* : \Gamma(X,E) \xrightarrow{\phi^*} \Gamma(X\times\mathbb{R}^+,\phi^*E) \xrightarrow{\psi} \Gamma(X\times\mathbb{R}^+,p^*E) \ .$$

Thus $K_{\psi^*}$ is a generalized density on $X\times X\times\mathbb{R}^+$. The non-degeneracy assumptions above are equivalent to the image of $\Delta$ and the graph of $\phi$ intersecting transversally. Thus by the theory of the wave front set one can pull back $K_{\psi^*}$ via $\Delta$ and define

$$\mathrm{Tr}(\psi^* \,|\, \Gamma(X,E)) = \pi_*\Delta^*K_{\psi^*} \quad \text{in } \mathcal{D}'(\mathbb{R}^+) \ .$$

Intuitively,

$$\mathrm{Tr}(\psi^* \,|\, \Gamma(X,E)) = \int_X K_{\psi^*}(x,x,t)\,dx$$

as a distribution in $t$.

With this definition of a trace the following result becomes almost a tautology:

Proposition 4.1 (Guillemin, Sternberg) *Under the assumptions above, the following formula holds in $\mathcal{D}'(\mathbb{R}^+)$:*

$$\mathrm{Tr}(\psi^* \,|\, \Gamma(X,E)) \quad = \quad \sum_\gamma l(\gamma)\sum_{k=1}^\infty \frac{\mathrm{Tr}(\psi_x^{kl(\gamma)} \,|\, E_x)}{|\det(1 - T_x\phi^{kl(\gamma)} \,|\, T_xX/T_x^0)|}\delta_{kl(\gamma)}$$

$$+\sum_x \left\langle \frac{\mathrm{Tr}(\psi_x^t \,|\, E_x)}{|\det(1-T_x\phi^t \,|\, T_xX)|} \right\rangle \ .$$

*Here $\gamma$ runs over the periodic orbits and in the first sum $x$ denotes any point on $\gamma$. In the second sum $x$ runs over the stationary points of the flow.*

In order to get a formula that is closer in appearance to (5) and (18) we now apply a basic idea which Guillemin [G] and independently Patterson [P] used in the context of Selberg and Ruelle zeta functions. It involves the theory of foliations for which we refer e.g. to [Go]. Assume that $X$ carries a smooth foliation of codimension one such that $\phi^t$ maps leaves to leaves. By a theorem of Frobenius this is equivalent to specifying an integrable codimension one subbundle $T_0 \subset TX$ with $T\phi^t(T_0) = T_0$ for all $t$, the bundle of tangents to the leaves. Let $U \subset X$ be the open $\phi^t$-invariant subset of points $x$ where the flow line through $x$ intersects the leaf through $x$ transversally, i.e. where

$$T_{0x} \oplus T_x^0 = T_x X \; .$$

We assume that $U$ contains all periodic orbits.
If $x$ is a fixed point of $\phi$ there exists some real constant $\kappa_x$ such that $T_x\phi^t$ acts on the one dimensional space $T_xX/T_{0x}$ by multiplication with $e^{\kappa_x t}$. We set

$$\varepsilon_\gamma(k) = \mathrm{sgn}\;\det(1 - T_x\phi^{kl(\gamma)} \,|\, T_{0x}) \quad \text{and} \quad \varepsilon_x = \mathrm{sgn}\;\det(1 - T_x\phi^t \,|\, T_xX)$$

the latter being independent of $t > 0$. From the proposition applied to $\Lambda^i T_0^* \otimes E$ we get the following formula in $\mathcal{D}'(\mathbb{R}^+)$:

$$\sum_i (-1)^i \mathrm{Tr}(\psi^* \,|\, \Gamma(X, \Lambda^i T_0^* \otimes E)) \tag{19}$$
$$= \sum_\gamma l(\gamma) \sum_{k=1}^\infty \varepsilon_\gamma(k) \mathrm{Tr}(\psi_x^{kl(\gamma)} \,|\, E_x) \delta_{kl(\gamma)} + \sum_x \varepsilon_x \left\langle \frac{\mathrm{Tr}(\psi_x^t \,|\, E_x)}{1 - e^{\kappa_x t}} \right\rangle .$$

Here an action $\psi^t : \phi^{t*}T_0^* \to T_0^*$ is given by $\psi_x^t = (T_x\phi^t)^* : T^*_{0\phi^t(x)} \to T^*_{0x}$. Together with the action on $E$ we get an action opposite to $\phi$ on every $\Lambda^i T_0^* \otimes E$. In order to proceed we next assume that $E$ carries a flat connection along the leaves

$$\delta_0 : \mathcal{E} \longrightarrow \mathcal{T}_0^* \otimes \mathcal{E} \; ,$$

where $\mathcal{E}$ and $\mathcal{T}_0$ are the sheaves of smooth sections of $E$ and $T_0$. It gives rise to a fine resolution

$$\mathcal{E} \xrightarrow{\delta_0} \mathcal{T}_0^* \otimes \mathcal{E} \xrightarrow{\delta_0} \Lambda^2 \mathcal{T}_0^* \otimes \mathcal{E} \longrightarrow \ldots$$

of the sheaf

$$\mathcal{F} = \mathrm{Ker}\,(\delta_0 : \mathcal{E} \longrightarrow \mathcal{T}_0^* \otimes \mathcal{E})$$

of smooth sections of $\mathcal{E}$ which are constant along the leaves of the foliation. For the trivial bundle $E = X \times \mathbb{R}$ with its canonical $T_0$-connection we obtain the sheaf

$\mathcal{R}$ of smooth real valued functions on $X$ which are constant along the leaves. Note that $\mathcal{F}$ carries a canonical action

$$\psi^t : (\phi^t)^{-1}\mathcal{F} \longrightarrow \mathcal{F}$$

opposite to $\phi^t$ which is used to define a map on cohomology by composition:

$$\psi^{t*} : H^i(X, \mathcal{F}) \overset{(\phi^t)^{-1}}{\longrightarrow} H^i(X, (\phi^t)^{-1}\mathcal{F}) \overset{\psi^t}{\longrightarrow} H^i(X, \mathcal{F}) \; .$$

Then the canonical isomorphism:

$$H^i(X, \mathcal{F}) = H^i((\Gamma(X, \Lambda^\bullet \mathcal{T}_0^* \otimes \mathcal{E}), \delta_0))$$

becomes equivariant under the induced action of the flow and one might hope to replace the alternating sum in (19) by an alternating sum over traces on cohomology.

On the other hand the differential $\delta_0$ will not have closed image in general, so that the cohomology spaces will not even be Hausdorff [H] 2.1. Let $\overline{H}^i(X, \mathcal{F})$ be the maximal Hausdorff quotient of $H^i(X, \mathcal{F})$, the *reduced* leafwise cohomology. It seems reasonable to expect a dynamical trace formula of the form

$$\sum_i (-1)^i \operatorname{Tr}(\psi^* \mid \overline{H}^i(X, \mathcal{F})) \tag{20}$$
$$= \sum_\gamma l(\gamma) \sum_{k=1}^\infty \varepsilon_\gamma(k) \operatorname{Tr}(\psi_x^{kl(\gamma)} \mid E_x) \delta_{kl(\gamma)} + \sum_x \varepsilon_x \left\langle \frac{\operatorname{Tr}(\psi_x^t \mid E_x)}{1 - e^{\kappa_x t}} \right\rangle \; .$$

Note that for the trivial bundle $E = X \times \mathbb{R}$ we would get

$$\sum_i (-1)^i \operatorname{Tr}(\psi^* \mid \overline{H}^i(X, \mathcal{R})) = \sum_\gamma l(\gamma) \sum_{k=1}^\infty \varepsilon_\gamma(k) \delta_{kl(\gamma)} + \sum_x \varepsilon_x \langle (1 - e^{\kappa_x t})^{-1} \rangle \; . \tag{21}$$

For the geodesic flow on the sphere bundle of cocompact quotients of rank one symmetric spaces and the stable foliation, analogous formulas are consistent with the Selberg trace formula, as has been shown by Guillemin [G], Patterson [P] and later workers, e.g. Juhl, Schubert, Bunke, Olbrich and Deitmar. Strictly speaking in these investigations $\overline{H}^i(X, \mathcal{F})$ is replaced by a sum of representations suggested by this cohomology.

If $X$ is the suspension of a diffeomorphism on a compact manifold $M$, the leafwise cohomologies turn out to be Hausdorff and hence Fréchet spaces, and (20) holds with the straightforward definition of a distributional trace given in (4). This consequence of the ordinary Lefschetz trace formula seems to be well known. A proof is written up in [D7] § 3.

Apart from these cases which do not involve stationary points the formula (20) does not seem to be established. One of the main problems is of course the definition of a good trace on the cohomology spaces $\overline{H}^i(X, \mathcal{F})$ these being infinite dimensional in general [AH]. Even if all of the $\overline{H}^i(X,\mathcal{F})$ are finite dimensional, (20) does not seem to be known. However it appears that at least for Riemannian foliations something can be done using the recent Hodge theorem of Alvarez-Gomez and Kordyukov for leafwise cohomology. In this case there is also a Hodge $*$-operator on cohomology which is induced by the metrics on the leaves.

Let $U$ be the dynamical system obtained by removing all the leaves containing stationary points. Then a trace formula of the form

$$\sum_i (-1)^i \mathrm{Tr}(\psi^* \mid \overline{H}^i_c(U,\mathcal{F})) = \sum_\gamma l(\gamma) \sum_{k=1}^\infty \varepsilon_\gamma(k) \mathrm{Tr}(\psi_x^{kl(\gamma)} \mid E_x) \delta_{kl(\gamma)}$$

is expected. For $E = U \times \mathbb{R}$ in particular we should have

$$\sum_i (-1)^i \mathrm{Tr}(\psi^* \mid \overline{H}^i_c(U,\mathcal{R})) = \sum_\gamma l(\gamma) \sum_{k=1}^\infty \varepsilon_\gamma(k) \delta_{kl(\gamma)} \ . \tag{22}$$

It seems to be quite a challenge to establish such dynamical trace formulas in any generality and also for more general foliations. This would also be a major contribution to the theory of periodic solutions of ordinary differential equations.

Given a closed orbit $\gamma$ and a point $x$ on $\gamma$ consider the isomorphism

$$\overline{\gamma} = (\gamma, x) : \mathbb{R}/l(\gamma)\mathbb{Z} \xrightarrow{\sim} \gamma \ , \ \bar{t} \longmapsto \phi^t(x) \ .$$

The functor

$$\mathcal{F} \mapsto \mathcal{F}_{\overline{\gamma}} = \Gamma(\mathbb{R}/l(\gamma)\mathbb{Z}, \overline{\gamma}^{\,-1}\mathcal{F})$$

from $\mathcal{R}$-modules to $C^\infty(\mathbb{R}/l(\gamma)\mathbb{Z})$-modules is exact [D7] 3.22. We view $\mathcal{F}_{\overline{\gamma}}$ as the stalk of $\mathcal{F}$ in the "geometric point" $x$ of $\gamma$. The Poisson summation formula implies that

$$\mathrm{Tr}(\psi^* \mid \mathcal{F}_{\overline{\gamma}})_{\mathrm{dis}} = l(\gamma) \sum_{k=1}^\infty \mathrm{Tr}(\psi_x^{kl(\gamma)} \mid E_x) \delta_{kl(\gamma)} \ .$$

For a stationary point $x$ a suitable interpretation of the trace on $\mathcal{F}_x$ gives:

$$\mathrm{Tr}(\psi^* \mid \mathcal{F}_x)_{\mathrm{dis}} = \left\langle \frac{\mathrm{Tr}(\psi_x^t \mid E_x)}{1 - e^{\kappa_x t}} \right\rangle \quad \text{c.f. [D8].}$$

Thus the right hand side of the trace formulas can be rewritten in more sheaf theoretical terms as the sum of distributional traces of the flow on the stalks of $\mathcal{F}$ in the compact orbits of the flow. Incidentially, note that our former ring $\mathcal{R}_p$ is just the dense subalgebra of finite Fourier series in $C^\infty(\mathbb{R}/(\log p)\mathbb{Z})$.

Formula (21) resp. (22) resembles the cohomological version of the explicit formulas for the Riemann zeta function (5) resp. for the Hasse Weil zeta function (10). However, as we will see, the setting of this section and in particular the assumption that we are dealing with compact manifolds is too restrictive for the goal of realizing (5) and (10) as special cases of (21) and (22). Nonetheless we are led to expect the following structures on the searched for dynamical systems ("$\overline{\operatorname{Spec}\mathbb{Z}}$", $\phi^t$) and ("$\mathcal{X}$", $\phi^t$) corresponding to $\overline{\operatorname{Spec}\mathbb{Z}}$ resp. the algebraic scheme $\mathcal{X}/\mathbb{Z}$. The space "$\overline{\operatorname{Spec}\mathbb{Z}}$", whatever its nature, infinite dimensional, a Grothendieck topology, ..., should have some compactness property. The closed orbits $\gamma$ should correspond to the prime numbers such that $l(\gamma) = \log p$ if $\gamma \mathrel{\widehat{=}} p$. More generally on "$\mathcal{X}$" they should correspond to the closed points of $\mathcal{X}$ with $l(\gamma) = \log N(x)$ if $\gamma \mathrel{\widehat{=}} x$. On "$\overline{\operatorname{Spec}\mathbb{Z}}$" there should be a stationary point $x_\infty$ corresponding to the infinite prime $p = \infty$. All these compact orbits must appear with positive sign in the dynamical trace formulas. Of course there could also be more periodic orbits and stationary points if their contributions in the trace formula vanish because of opposite signs.

There are to be one-codimensional foliations on "$\overline{\operatorname{Spec}\mathbb{Z}}$" and "$\mathcal{X}$" such that the open subset of points where the leaf is transversal to the flow contains all periodic orbits. Moreover $\kappa_{x_\infty} = -2$, i.e. $T_{x_\infty}\phi^t$ operates on $T_{x_\infty}/T_{0x_\infty}$ by multiplication with $e^{-2t}$.

The cohomologies conjectured in section two should be the dense spaces of smooth vectors in the corresponding reduced leafwise cohomologies. Here a vector is smooth if it is contained in the sum of generalized eigenspaces of the induced flow on cohomology.

The leaves on "$\overline{\operatorname{Spec}\mathbb{Z}}$" (resp. "$\mathcal{X}$") should be two (resp. $2\dim\mathcal{X}$) dimensional in a suitable sense since for foliated manifolds $H^i(X, \mathcal{R}) = 0$ for $i > d$ where $d$ is the dimension of the leaves, and $\overline{H}^d(X, \mathcal{R}) \neq 0$ if there exists a non trivial holonomy invariant current on $X$. Thus "$\overline{\operatorname{Spec}\mathbb{Z}}$" (resp. "$\mathcal{X}$") should have dimension three (resp. $2\dim\mathcal{X} + 1$) in that sense. These dimensions agree with the étale cohomological dimensions of $\operatorname{Spec}\mathbb{Z}$ (resp. $\mathcal{X}$).

As for the structure of "$\overline{\mathcal{X}}$" $\setminus$ "$\mathcal{X}$" possibly the set of stationary points of the flow on "$\overline{\mathcal{X}}$" is $\mathcal{X}_\infty(\mathbb{C})/\langle F_\infty\rangle$, where $F_\infty$ is complex conjugation. This would generalize what we expect for $\mathcal{X} = \operatorname{Spec}\mathbb{Z}$ and more generally for $\mathcal{X} = \operatorname{Spec}\mathfrak{o}_k$. Note also that the set of closed points of $\mathcal{X}$ over $p$ can be identified with the set $\mathcal{X}_p(\overline{\mathbb{F}}_p)/\langle \mathrm{Fr}_p\rangle$ of Frobenius orbits on $\mathcal{X}_p(\overline{\mathbb{F}}_p)$, where $\mathcal{X}_p = \mathcal{X} \otimes \mathbb{F}_p$.

We now discuss the basic theory of flows with an integrable invariant complement. This is relevant for us since they appear as subsystems in the above. Let us define an $F$-flow $\phi^t$ to consist of a (Banach-)manifold $U$ with a flow generated by a smooth vector field which exists for all positive but possibly not for all negative times. By definition an $F$-system is an $F$-flow with a one-codimensional foliation $T_0$ which is everywhere transversal to the flow. In particular there are no fixed points. These systems form a category in an obvious way. Their theory is essentially well known and recalled for example in [D7] § 3. The foliation corresponds uniquely to a closed flow-invariant one form $\omega_\phi$ with $\langle \omega_\phi, Y_\phi\rangle = 1$, via $\ker\omega_\phi = T_0$. The period group $\Lambda \subset \mathbb{R}$ is defined as the image of the length homomorphism

$$l : \pi_1^{\mathrm{ab}}(U) \longrightarrow \mathbb{R} \quad , \quad l(c) = \int_c \omega_\phi \; .$$

If there is a morphism $U \to U'$ of $F$-systems then $\Lambda_U \subset \Lambda_{U'}$. Periodic orbits $\gamma$ give well defined elements $[\gamma]$ of $\pi_1^{\mathrm{ab}}(U)$ and one has $l([\gamma]) = l(\gamma)$, the length of $\gamma$.

For a variety $V/\mathbb{F}_q$ there is an analogous map

$$l : \hat{\pi}_1^{\mathrm{ab}}(V) \longrightarrow \hat{\pi}_1(\operatorname{Spec}\mathbb{F}_q) = \hat{\mathbb{Z}}$$

induced by the projection $V \to \operatorname{Spec}\mathbb{F}_q$. Closed points $x$ of $V$ give Frobenius conjugacy classes and hence well defined elements $[F_x]$ of $\hat{\pi}_1^{\mathrm{ab}}(V)$. They satisfy the equation $l([F_x]) = \deg x = \log_q N(x)$.
On an $F$-system the following three categories are equivalent:

- vector bundles $E$ with a flat $T_0$-connection $\delta_0$ and a compatible action $\psi$ which is opposite to $\phi$;
- locally free $\mathcal{R}$-modules $\mathcal{F}$ with an action $\psi$ opposite to $\phi$;
- local systems $F$ of $\mathbb{R}$-vector spaces.

Here $\quad \mathcal{E} \leftrightarrow \mathcal{F} = \operatorname{Ker}(\delta_0 : \mathcal{E} \to \mathcal{T}_0^* \otimes \mathcal{E}) \leftrightarrow F = \operatorname{Ker}(\Theta_{\mathcal{F}} : \mathcal{F} \to \mathcal{F})$,
where $\Theta_{\mathcal{F}} : \mathcal{F} \to \mathcal{F}$ is the derivative of $\psi$ at $t = 0$. Let $\alpha$ be a real number. To the twist $\mathcal{F}(\alpha)$, defined as $\mathcal{F}$ with action $\psi^t_{\mathcal{F}(\alpha)} = e^{-t\alpha}\psi^t_{\mathcal{F}}$, there corresponds a local system $F(\alpha)$. For $\mathcal{F} = \mathcal{R}$ it is denoted $\underline{\mathbb{R}}(\alpha)$. Its monodromy representation is $\exp(\alpha l)$. Hence $\Lambda \subset \log \mathbb{Q}_+^*$ if and only if there is a local system $\underline{\mathbb{Q}}(1)$ of $\mathbb{Q}$-vector spaces such that $\underline{\mathbb{R}}(1) = \underline{\mathbb{Q}}(1) \otimes \mathbb{R}$.
If we complexify we get analogous equivalences of categories.
There is an exact sequence

$$0 \longrightarrow H^{i-1}(U, \mathcal{F})/\operatorname{Im}\Theta \longrightarrow H^i(U, F) \longrightarrow H^i(U, \mathcal{F})^{\Theta=0} \longrightarrow 0$$

where $\Theta = (\Theta_{\mathcal{F}})_*$. This is analogous to the exact sequence

$$0 \longrightarrow H^{i-1}(\overline{V}, \overline{F})_{\mathrm{Fr}_q} \longrightarrow H^i(V, F) \longrightarrow H^i(\overline{V}, \overline{F})^{\mathrm{Fr}_q} \longrightarrow 0$$

for a $\mathbb{Q}_l$-sheaf $F$ on $V$ where $\overline{V} = V \otimes \overline{\mathbb{F}}_q$. In the language of arithmetic geometry, $H^*(U, F)$ is the arithmetic cohomology and $H^*(U, \mathcal{F})$ with its action of the flow the geometric cohomology. As usual the latter commutes with twists but not the former.

There is a classification theorem: Every $F$-system is canonically contained as an open subsystem in a complete such system, i.e. one where the flow exists for all times in $\mathbb{R}$; c.f. [D8]. All complete connected $F$-systems are obtained as follows: Let $M$ be any leaf of $U$. Then $M$ is connected and $\Lambda = \{t \in \mathbb{R} \mid \phi^t(M) = M\}$ so that $\Lambda$ operates on $M$. The system $U$ is then isomorphic to the suspension $M \times_\Lambda \mathbb{R}$ where $\Lambda$ acts on $\mathbb{R}$ by translation and the foliation is by the images of $M \times \{t\}$ for $t$ in $\mathbb{R}$.

## 5 Further comparison

For "Spec $\mathbb{Z}$" the period group $\Lambda$ must contain the numbers $\log p$ as they should be lengths of closed orbits. Hence $\Lambda \supset \log \mathbb{Q}_+^*$. On the other hand $\underline{\mathbb{R}}(1)$ will have a rational structure (see below) and hence $\Lambda \subset \log \mathbb{Q}_+^*$, so that $\Lambda = \log \mathbb{Q}_+^*$. Writing

the flow multiplicatively we therefore expect "Spec $\mathbb{Z}$", if its flow is complete, to have the form $M \times_{\mathbb{Q}_+^*} \mathbb{R}_+^*$ for some "space" with $\mathbb{Q}_+^*$-action $M$ reminiscent of the idèlic picture. A similar argument for varieties $V/\mathbb{F}_q$ with a rational point suggests that "$V$" $\cong N \times_{q^{\mathbb{Z}}} \mathbb{R}_+^*$.
As mentioned above, the leafwise cohomology of "$V$" should be isomorphic to a theory constructed from the $\mathbb{Q}_l$-cohomology of $\overline{V}$ after the choice of an embedding $\mathbb{Q}_l \subset \mathbb{C}$. Comparing the kernels of $\Theta$, it follows on combining [D3] (2.4) and § 4 with [D7] (3.19) that the singular cohomology of $N$ with $\mathbb{C}$-coefficients, endowed with the automorphism $q^*$, must be isomorphic to $H^*(\overline{V}, \mathbb{Q}_l) \otimes \mathbb{C}$ with $\mathrm{Fr}_q^*$-action. It follows that $H^d(N, \mathbb{Z})$, where $d = \dim N$, must be a $\mathbb{Z}[q^{-1}]$-module, since $\mathrm{Fr}_q^*$ acts by multiplication with $q^d$ on $H^d(\overline{V}, \mathbb{Q}_l)$.
The natural way to obtain such $N$ is to take a compact manifold $\tilde{N}$ with a finite map $q : \tilde{N} \to \tilde{N}$ of degree $q$ and set $N = \varprojlim(\ldots \to \tilde{N} \xrightarrow{q} \tilde{N} \to \ldots)$. Note the continuity theorems for cohomology in this regard, c.f. [Br] II.14. The most naive way to obtain $(\tilde{N}, q)$ would be by lifting $(V, \mathrm{Fr}_q)$ to $\mathbb{C}$. For cellular varieties and ordinary abelian varieties over $\mathbb{F}_q$ this is possible but of course not in general.
It seems possible that in the above isomorphism "Spec $\mathbb{Z}$" $\cong M \times_{\mathbb{Q}_+^*} \mathbb{R}_+^*$ the leaf $M$ is obtained from a "space" $\tilde{M}$ with commuting operators for every prime number $p$, by an analogous projective limit. This puts $M \times_{\mathbb{Q}_+^*} \mathbb{R}_+^*$ even closer to the idèlic view point.

Allowing such more general spaces removes a difference between dynamical trace formulas and explicit formulas in cohomological form: Both can be extended to test functions on $\mathbb{R}^*$, but whereas for compact manifolds the former become symmetric under $t \leftrightarrow -t$, the latter exhibit a twisted symmetry. A closely related point is this: For a finite dimensional $F$-system the flow acts with weight zero on the top leafwise reduced cohomology with compact supports. This follows by looking at the invariant currents and noting that automorphisms act by $\pm 1$ on top compactly supported cohomology with $\mathbb{Z}$-coefficients. Since we want weights different from zero, e.g. equal to one for Spec $\mathbb{Z}$, we are forced to allow more general spaces than finite dimensional manifolds as leaves. For ordinary abelian varieties over $\mathbb{F}_p$ the theory of the zeta function can in fact be established dynamically using pro-manifolds but in general – at least in characteristic $p$ – even pro-manifolds as leaves are not the right kind of space.

If the association from schemes to foliated dynamical systems is functorial one has a natural construction of sheaves $\mathcal{F}(M)$ for any motive $M$. For a variety $\pi : X_0 \to \operatorname{Spec} \mathbb{Q}$ let $\pi =$ "$\pi$" $: X =$ "$X_0$" $\to$ "Spec $\mathbb{Q}$" be the associated morphism of foliated dynamical systems. The functors

$$X_0 \longmapsto R^i \pi_*(\mathcal{R}_X) \quad \text{and} \quad X_0 \longmapsto R^i \pi_*(\underline{\mathbb{R}}_X)$$

define cohomology theories which by universality factor over the category of motives. They are denoted $M \mapsto \mathcal{G}(M)$ and $M \mapsto G(M)$. The morphism $j_0 : \operatorname{Spec} \mathbb{Q} \to \operatorname{Spec} \mathbb{Z}$ will induce a morphism $j =$ "$j_0$" of dynamical systems and we get functors $\mathcal{F} = j_* \circ \mathcal{G}$ and $F = j_* \circ G$. The two constructions are related by $F = \operatorname{Ker}(\Theta : \mathcal{F} \to \mathcal{F})$. Moreover $F$ has a natural $\mathbb{Q}$-structure $F_{\mathbb{Q}}$ obtained by starting with rational coefficients. In fact over "Spec $\mathbb{Z}_S$", where $S$ is a finite or cofinite set of prime numbers, we should get a $\mathbb{Z}[l^{-1} \mid l \notin S]$-structure on $F$ by taking $\underline{\mathbb{Z}}_X$-coefficients[2] above. For $n \in \mathbb{Z}$, we get $\mathcal{F}(\mathbb{Q}(n)) = \mathcal{R}(n), F(\mathbb{Q}(n)) = \underline{\mathbb{R}}(n)$ and $F_{\mathbb{Q}}(\mathbb{Q}(1))$ provides the rational structure on $\underline{\mathbb{R}}(1)$ alluded to above.

Comparing formulas (18) and (20) over "Spec $\mathbb{Z}_S$", we see that the semisimplifications of $(M_l^{I_p}, \mathrm{Fr}_p)$ and $(E_x, \psi_x^{\log p})$ for $x \in \gamma \mathrel{\widehat{=}} p \notin S$ should be isomorphic. Since $E$ is a vector bundle the dimensions of $M_l^{I_p}$ must be constant, i.e. $M$ must have good reduction at the finite primes $p \notin S$. Note that via the equivalence of categories above, $(E_x, \psi_x^{\log p})$ is isomorphic to $F_x$ with its monodromy representation along $\gamma$. The rational structure $F_{\mathbb{Q},x}$ on $F_x$ thus implies that the characteristic polynomial of the monodromy representation has rational coefficients. The same must therefore hold for the Frobenius action on $M_l$ if our picture is correct. This is well known for many motives by the work of Deligne and conjectured in general.

We now reinterpret part of (2.1) 5. as a fully faithfulness assertion. For finite $S$ consider a motive $M$ over $\mathbb{Q}$ with good reduction outside of $S$. Using the expected isomorphism (15) over Spec $\mathbb{Z}_S$ we get a commutative diagram

$$\begin{array}{ccc} \mathrm{Hom}(\mathbb{Q}(0), M) \otimes \mathbb{R} & \xrightarrow[\sim]{\mathcal{F}} & H^0(\text{“Spec}\,\mathbb{Z}_S\text{”}, \mathcal{F}(M))^{\Theta\sim 0} \\ F \big\downarrow & & \big\uparrow \\ H^0(\text{“Spec}\,\mathbb{Z}_S\text{”}, F(M)) & \xrightarrow{\sim} & H^0(\text{“Spec}\,\mathbb{Z}_S\text{”}, \mathcal{F}(M))^{\Theta=0}, \end{array}$$

noting that $H^0$ is Hausdorff. Hence all arrows must be isomorphisms. Replacing $M$ by $M_1^* \otimes M_2$, it follows that the exact tensor functor $F_{\mathbb{Q}}$ from motives with good reduction on Spec $\mathbb{Z}_S$ to $\mathbb{Q}$-local systems on "Spec $\mathbb{Z}_S$", must be fully faithful. The map induced by Tannakian duality fits very nicely into a diagram comparing topological fundamental groups and Galois groups of number fields, see [D7] (42).

The constructions in the real manifold setting of section three, even if we allow infinite-dimensional or pro-manifolds, always lead to sheaves of real vector spaces $\mathcal{F}$. On the other hand the spaces $\mathcal{F}_p(M)$ are by construction ([D3] § 3) complex vector spaces with no evident real structure. For motives over $\mathbb{Q}$ this is not a contradiction, but the analogue for motives over finite fields is impossible. This is so because the functors $F_x$ would give exact faithful tensor functors into $\mathbb{R}$-vector spaces which are known not to exist. On the other hand on the subcategory of ordinary motives over finite fields the predictions of the dynamical formalism work out correctly by a result of Deligne, see [D7] 4.7.

## CONCLUSION

Apart from stating his famous conjectures on zeta functions, A. Weil also explained how they could be attacked given a cohomology theory for varieties in characteristic $p$ with properties similar to those of singular cohomology. For varieties over number fields the analogues of the Weil conjectures and further conjectures have by now been checked in numerous cases except for the Riemann conjecture 2.1 part 4 of course. In this article we have outlined a strategy to approach them. This program requires a cohomology theory for algebraic schemes over the integers

[2]This is not a misprint.

with properties similar to those of the reduced leafwise cohomology of a class of dynamical systems with one-codimensional foliations by pro-manifolds.

## References


[AH] J.A. Alvarez Lopez, S. Hurder, Pure-point spectrum for foliation geometric operators, preprint 1995

[B] M. Berry, Riemann's zeta function: a model of quantum chaos, Lecture Notes in Physics 263, Springer 1986

[Br] G. Bredon, Sheaf theory. Springer 1997

[BK] S. Bloch, K. Kato, $L$-functions and Tamagawa numbers of motives. (Grothendieck Festschrift I) 333–400, Birkhäuser 1991

[CV] P. Cartier, A. Voros, Une nouvelle interprétation de la formule des traces de Selberg, (Grothendieck Festschrift II) 1–67, Birkhäuser 1991

[C] A. Connes, Formule de trace en géometrie non-commutative et hypothèse de Riemann. C.R. Acad. Sci. 323 (1996), 1231–1236

[Cons] K. Consani, Double complexes and Euler $L$-factors, Compos. Math. 111 (1998), 323–358

[De] P. Deligne, La conjecture de Weil I. Inst. Hautes Études Sci. Publ. Math. 43 (1974), 273–307

[D1] C. Deninger, On the $\Gamma$-factors attached to motives. Invent. Math. 104 (1991), 245–261

[D2] C. Deninger, Local $L$-factors of motives and regularized determinants. Invent. Math. 107 (1992), 135–150

[D3] C. Deninger, Motivic $L$–functions and regularized determinants. Proc. Symp. Pure Math. 55, 1 (1994), 707–743

[D4] C. Deninger, Motivic $L$–functions and regularized determinants II. In: F. Catanese (ed.), Arithmetic Geometry, Cortona, 1994, Symp. Math. 37, 138–156, Cambridge Univ. Press 1997

[D5] C. Deninger, Lefschetz trace formulas and explicit formulas in analytic number theory. J. Reine Angew. Math. 441 (1993), 1–15

[D6] C. Deninger, Motivic $\varepsilon$–factors at infinity and regularized dimensions. Indag. Math., N.S. 5 (4), (1994), 403–409

[D7] C. Deninger, On Dynamical systems and their Possible Significance for Arithmetic Geometry. To appear in a conference proceedings on regulators

[D8] C. Deninger, On Dynamical systems and their Possible Significance for Arithmetic Geometry II. In preparation

[DN] C. Deninger, E. Nart, On $\mathrm{Ext}^2$ of motives over arithmetic curves. Amer. J. of Math. 117 (1995), 601–625

[DSch] C. Deninger, M. Schröter, A distribution theoretic interpretation of Guinand's functional equation for Cramér's $V$–function and generalizations. J. London Math. Soc. (2) 52 (1995) 48–60

[F-PR] J-M. Fontaine, B. Perrin-Riou, Autour des conjectures de Bloch et Kato, cohomologie galoisienne et valeurs de fonctions $L$, Proc. Symp. Pure Math. 55, 1 (1994), 599–706

[Go] C. Godbillon, Feuilletages, Etudes géométriques. Progress in Math. 98, Birkhäuser 1991

[G] V. Guillemin, Lectures on spectral theory of elliptic operators. Duke Math. J. 44 (1977), 485–517

[GS] V. Guillemin, S. Sternberg, Geometric asymptotics, Math. Surveys 14, Amer. Math. Soc., Providence, R.I. 1977

[H] A. Haefliger, Some remarks on foliations with minimal leaves. J. Diff. Geo. 15 (1980), 269–284

[HW] A. Huber, J. Wildeshaus, Classical motivic polylogarithm according to Beilinson and Deligne, Doc. Math. J. DMV 3 (1998) 27–133

[I] G. Illies, Regularisierte Produkte, Spurformeln und Cramér-Funktionen, Schriftenreihe des Mathematischen Instituts der Univ. Münster, 3. Serie, Heft 22, 1998

[J] U. Jannsen, Mixed motives, motivic cohomology, and Ext-groups. Proc. ICM Zürich 1994, pp. 667–679, Birkhäuser 1995

[JL] J. Jorgenson, S. Lang, Explicit formulas and regularized products, Lecture Notes in Mathematics 1593, Springer 1994

[K] N. Kurokawa, Parabolic components of zeta function. Proc. Japan Acad. Ser. A64 (1988) 21–24

[P] S.J. Patterson, On Ruelle's zeta function. Israel Math. Conf. Proc. 3 (1990), 163–184

[S] T. Saito, The sign of the functional equation of the $L$-function of an orthogonal motive. Invent. Math. 120 (1995), 119–142

[Sa] P. Sarnak, Quantum Chaos, Symmetry and Zeta Functions, Lecture I: Quantum Chaos, Lecture II: Zeta Functions, Curr. Dev. Math. 84–115, International Press 1997

[Sch] A.J. Scholl, Height pairings and special values of $L$-functions. Proc. Symp. Pure Math. 55, 1 (1994), 571–598

[SchS] M. Schröter, C. Soulé, On a result of Deninger concerning Riemann's zeta function, Proc. Sympos. Pure Math. 55 (1994), 745–747

[Se] J-P. Serre, Facteurs locaux des fonctions zêta des variétés algebriques (définitions et conjectures), Sém. Delange-Pisot-Poitou, exp. 19 (1969/70)

[So] C. Soulé, $K$-théorie et zéros aux points entiers de fonctions zêta, Proc. ICM, Warsaw, 1983, pp. 437–445

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