Solvable Lattice Models and Representation Theory of Quantum Affine Algebras

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Abstract. A review on some recent developments in solvable lattice models in connection with the representation theory of the quantum affine algebras is given.

Keywords and Phrases: the XXZ model, the six-vertex model, the quantum affine algebras, the qKZ equation, the corner transfer matrix, the intertwiner

1 The XXZ model: a solvable system of infinite degrees of freedom

The aim of this talk is to review some recent (in 90's) progress in solvable lattice models. I will, in particular, stress the connection with the representation theory of the quantum affine algebras. In this section, I introduce the XXZ model and the six-vertex model, state the problems we wish to solve and give the clue to the solvability of these models. I also give some results in prehistoric ages (i.e., before '85, the birth of Quantum Groups) which led us to this connection.

1.1 The XXZ Hamiltonian

Consider the one-dimensional quantum Hamiltonian with a real parameter Δ,

$$H \quad = \quad -\frac{1}{2}\sum_k(\sigma_k^x\sigma_{k+1}^x+\sigma_k^y\sigma_{k+1}^y+\Delta\sigma_k^z\sigma_{k+1}^z). \tag{1}$$

Here $\sigma^x,\sigma^y,\sigma^z$ are the Pauli matrices, and the index k signifies the k-th component of the tensor product $\otimes_k V_k$ of the two-dimensional spaces $V_k\simeq V=\mathbf{C}v_0\oplus\mathbf{C}v_1$. The Hamiltonian (1) is called the XXZ Hamiltonian.

Here we have not specified the range of the index k. If the range is finite, e.g., an interval $0\le k\le N-1$ or a periodic chain $k\in\mathbf{Z}/N\mathbf{Z}$, both the space $\otimes_k V_k$ and the operator H are well-defined. However, in physics, we are interested in the large volume limit, i.e., $N=\infty$, where the number of degrees of freedom of the system becomes infinite. There is no apriori meaning of these expressions in this limit. In fact, some physical quantities are divergent (e.g., the trace of $e^{-H/kT}$).

We say a model is solved if we can extract finite quantities and give them closed expressions.

The problems we are interested in, in general, are

(A) the diagonalization of the Hamiltonian; and

(B) the computation of the matrix elements of the local operators; a particular case of (B) is

(C) the computation of the correlation functions.

1.2 Vacuum states as infinite linear combinations of paths

Our consideration is restricted to the $T=0$ case. In this case, we are interested in the lowest eigenvalue of the Hamiltonian and the corresponding eigenvectors (the vacuum states). We are also interested in those eigenvectors whose eigenvalues have finite differences to the lowest one in the large volume limit (the excited states).

If $\Delta \to \pm\infty$, the Hamiltonian effectively approaches a diagonal one $H \sim \mp\frac{1}{2}\sum_k \sigma_k^z\sigma_{k+1}^z$. If $\Delta=\infty$, there are two vacuums ($i=0,1$),

$$|\bar{p}^{(i)}\rangle = \otimes_k v_{\bar{p}^{(i)}(k)} \quad \text{where } \bar{p}^{(i)}(k)=\tfrac{1}{2}(1-(-1)^i). \tag{2}$$

All the spins are equal in the vacuum states. The corresponding eigenvalue is $-\sharp\{k\}$, and therefore divergent in the large volume limit. However, we renormalize the Hamiltonian by replacing $\sigma_k^z\sigma_{k+1}^z$ by $\sigma_k^z\sigma_{k+1}^z-1$ so that its lowest eigenvalue is 0. On the other hand, if $\Delta=-\infty$, the vacuums ($i=0,1$) are

$$|p^{(i)}\rangle = \otimes_k v_{p^{(i)}(k)} \quad \text{where } p^{(i)}(k)=\tfrac{1}{2}(1-(-1)^{k+i}). \tag{3}$$

The spins are alternating in the vacuum states. The renormalization of the Hamiltonian is such that $\sigma_n^z\sigma_{n+1}^z+1$.

If Δ is finite we must take account of the interaction terms $\sigma_k^x\sigma_{k+1}^x+\sigma_k^y\sigma_{k+1}^y$. These terms are non-diagonal and mix the vectors of the form $|p\rangle=\otimes_k v_{p(k)}$. However, they preserve the total spin of the vectors, i.e., $\frac{1}{2}\sum_k(1-2p(k))$. Therefore, if $|\Delta|$ is sufficiently large, it is natural to expect that the vacuum states are contained in the same subspace of total spin as $|\bar{p}^{(i)}\rangle$ or $|p^{(i)}\rangle$. In fact, this is true. For $\Delta\sim\infty$, this implies that $|\bar{p}^{(i)}\rangle$ remains as the vacuum.

The case $\Delta\sim-\infty$ is more interesting because the vacuum states are linear combinations of (3) and other vectors of total spin 0 (we assume N is even). If N is infinite, infinitely many terms appear in the linear combination. Mathematically, this is a serious problem because it is not clear if we can introduce a suitable topology in order to deal with this infinite sum.

One can make a perturbation expansion of the vacuum state in the form

$$|vac\rangle_i = \sum_p c(p)|p\rangle \quad \text{where } |p\rangle = \otimes_{k\in\mathbf{Z}} v_{p(k)}. \tag{4}$$

Note that $N=\infty$ in this formula. We set $c(p^{(i)})=1$ and the other coefficients are of the form $c(p)=\sum_{j\geq 1} c_j(p)\varepsilon^j$ with $\varepsilon=\Delta^{-1}$. In principle, one can determine each coefficient $c_j(p)$ recursively by solving the equation $H_{re}|vac\rangle_i = 0$. Here the renormalized Hamiltonian H_{re} is given in the form

$$H_{re} = -\frac{1}{2}\sum_k \Big(\sigma_k^x\sigma_{k+1}^x + \sigma_k^y\sigma_{k+1}^y + \Delta(\sigma_k^z\sigma_{k+1}^z + 1 + \sum_{j\geq 1} r_j\varepsilon^j)\Big), \tag{5}$$

in which the coefficients r_j are determined in each step of the recursion to remove the divergence and to make the eigenvalue 0.

An important feature of this expansion is that $c(p)$ is zero unless $p(k) = p^{(i)}(k)$ for all but finite k. We call such p *a path* belonging to the i-th ground-state.

1.3 Phases of the XXZ model

If we vary Δ from $-\infty$ to ∞, the eigenvalues cross each other. The vacuum states change from one region to another when the eigenvalues cross. In the infinite volume limit, it is known that there are three different phases (see, e.g., [1]),

$$\text{(i) } \Delta < -1, \quad \text{(ii) } -1 \leq \Delta \leq 1, \quad \text{(iii) } \Delta > 1. \tag{6}$$

We have already mentioned (i) and (iii). The phase (ii) is such that the vacuum state belongs to the subspace of the total spin 0. In this phase, there is a unique vacuum state, which belongs to the space of total spin 0. Nothing like the path expansion (4) is available because there is no special limit where the Hamiltonian is diagonal. I will discuss that this difference between phase (i) and (ii) causes an essential difference in our treatment of the model in the representation theory. As for the phase (iii), where the vacuum states are trivial, there is nothing to say about from the representation theory, and I will not discuss this phase any further.

1.4 Excited states and particles

The method invented by Bethe when he solved the XXX model is called the Bethe Ansatz. It starts with finite periodic N, and consider the infinite volume limit in the second step. The key idea in this method is to introduce the notion of quasi-particles borrowed from the quantum field theory.

For finite N, there exist only finitely many eigenvectors of the Hamiltonian. It has only discrete eigenvalues. However, in the infinite volume limit, continuous spectra appear. To parametrize the eigenvectors belonging to the continuous spectra we need continuous parameters. The Bethe Ansatz uses a set of continuous parameters $\beta_1, \ldots, \beta_n$, called the rapidity variables, to parametrize the eigenvectors in the finite volume. An eigenstate parametrized by n continuous parameters is called an n quasi-particle state. Since there are only finitely many eigenvectors, only some discrete values of the quasi-momenta are allowed to give actual eigenvectors.

The vector $|p^{(0)}\rangle = \otimes_k v_0$ is the 0 quasi-particle state. One quasi-particle state is a linear combination of $|p\rangle$ such that $p(k) = 1$ for one and only one k, and so on for two and more quasi-particle states. This picture is not appropriate in the phases (i) and (ii), and, in particular, in the large volume limit, because the vacuum states in this terminology are $\frac{N}{2}$ particle states. In these phases, $n(>0)$ quasi-particle states may have lower 'energies' (=eigenvalues of the Hamiltonian) than the 0 quasi-particle state. There is a trick to reparametrize the vacuum and the excited states in such a way that the vacuum states are the 0 particle states and the excited states are the $n(>0)$ particle states. This is possible only in the infinite volume limit. I stress this point because in many cases something good happens only in the infinite volume limit. The remarkable thing in this parametrization is that the renormalized energy of an n-particle state with the rapidities $\beta_j(1 \le j \le n)$ is given by an additive formula $\sum_j \varepsilon(\beta_j)$. The function $\varepsilon(\beta)$ is a simple function, e.g., if $\Delta = -1$, we have $\varepsilon(\beta) = \frac{\pi}{\operatorname{ch}\beta}$. Each particle with the rapidity β_j carries the energy $\varepsilon(\beta_j)$. This is the reason why these states are called the n-particle states.

Note that, if $\Delta = -1$ the above formula tells that there is no energy gap between the vacuum and the excited states: The energy difference $\varepsilon(\beta)$ approches 0 if $|\beta| \to \infty$. This property is called 'massless' by using the language of quantum field theory. In statistical mechanics, this is called 'critical'. In the phase (ii) the particles are massless, while in the phase (i) they are massive.

A further remarkable fact about the particle structure, valid both in the massive and the massless phases, is the degeneracy of the n-particle states ([15, 6]). A clear view of this fact was given in [6] for $\Delta = -1$. I write their formula in the form adapted to our notation. Denote the space of the eigenvectors of the Hamiltonian by $\mathcal{F}$. We call it the physical space. We have the decomposition

$$\mathcal{F} \quad = \quad \oplus_{n\ge 0,\text{even}} \prod_{j=1}^{n} \int_{-\infty}^{\infty} \frac{d\beta_j}{2\pi} \left[\otimes_{j=1}^{n} (\mathbf{C}^2)_{\beta_j}\right]_{\text{sym}} \tag{7}$$

It means that the n-particle states with a fixed set of rapidities $(\beta_1, \dots, \beta_n)$ have 2^n-fold degeneracy. This degeneracy is identified with the tensor product $\otimes^n \mathbf{C}^2$.

Here is a key to the connection with the representation theory. The Hamiltonian (1) with $\Delta = -1$ has a global sl_2 symmetry, i.e., there exists an sl_2 action on $\otimes_k V_k$ which commutes with the Hamiltonian. The formula (7) claims that the vector space of the n-particle states with rapidities $(\beta_1, \dots, \beta_n)$ is isomorphic to

$$\otimes^n \mathbf{C}^2 \quad = \quad \oplus_{\varepsilon_1,\dots,\varepsilon_n=0,1} \mathbf{C} v_{\varepsilon_1} \otimes \cdots \otimes \mathbf{C} v_{\varepsilon_n}. \tag{8}$$

as sl_2-module. In other words, we have a complete parameterization of the excited states by the rapidities $(\beta_1, \dots, \beta_n)$ and the isospins $(\varepsilon_1, \dots, \varepsilon_n)$. Let us denote this state by $|\beta_n, \dots, \beta_1\rangle_{\varepsilon_n,\dots,\varepsilon_1}$.

There is a further symmetry of the n-particle states that is indicated by the symbol $\Big[\quad\Big]_{\text{sym}}$ in (7): There exists a matrix $S(\beta)$ depending on the rapidity variable β, which acts on $\mathbf{C}^2 \otimes \mathbf{C}^2$. This is called the S-matrix. The S-matrix exchanges the rapidities of n-particle states.

$$\begin{aligned} & |\beta_n, \dots, \beta_j, \beta_{j+1}, \dots, \beta_1\rangle_{\varepsilon_n,\dots,\varepsilon_j,\varepsilon_{j+1},\dots,\varepsilon_1} \\ = & \sum_{\varepsilon'_j,\varepsilon'_{j+1}} S(\beta_j - \beta_{j+1})_{\varepsilon_j,\varepsilon_{j+1}}^{\varepsilon'_j,\varepsilon'_{j+1}} |\beta_n, \dots, \beta_{j+1}, \beta_j, \dots, \beta_1\rangle_{\varepsilon_n,\dots,\varepsilon'_{j+1},\varepsilon'_j,\dots,\varepsilon_1}. \end{aligned} \tag{9}$$

I will give the explicit formula of the S-matrix only later when I discuss the sine-Gordon theory.

The meaning of the rapidity variables in the representation theory is unclear at this stage because a larger symmetry is still hidden behind. In Section 2, I will show that the hidden symmetry distinguishes the rapidities. The particles with different rapidities correspond to different (i.e., non-isomorphic) representations.

1.5 Correlation functions

Now, I will explain what (B) and (C) mean. We call a linear operator acting on $\otimes_{k\in\mathbf{Z}}V_k$ *local* if its action is restricted to a finite interval of the one-dimensional lattice $\mathbf{Z}$ where the index k runs. The Hamiltonian is not local though each summand in (1) is local.

The correlation functions are the vacuum-to-vacuum matrix element of local operators. If we take a local operator acting on n sites of the lattice, its correlation function is called an n-point function. Quantities of physical interest are often given in terms of the correlation functions. For example, the one point function

$$P^{(i)}(k) = \frac{{}_i\langle vac|\sigma^z_k|vac\rangle_i}{{}_i\langle vac|vac\rangle_i} \tag{10}$$

gives the magnetization.

Introduce a new parameter q by $\Delta = \frac{1}{2}(q+q^{-1})$. The massive phase is $-1 < q < 0$ and, the massless phase is $|q| = 1$. Here we are considering the one-point function in the massive phase.

By obvious reasons, the one-point function satisfies $P^{(1-i)}(k+1) = P^{(i)}(k)$ and $P^{(0)}(0) + P^{(1)}(0) = 0$. The function $P^{(0)}(0)$ was computed by Baxter ([2]):

$$P^{(0)}(0) = \prod_{k=1}^{\infty}\left(\frac{1-q^{2k}}{1+q^{2k}}\right)^2. \tag{11}$$

The above q is identified with the q in the affine quantum algebra $U_q(\widehat{sl}_2)$. The representation theory of $U_q(\widehat{sl}_2)$ provides us with the scheme for computing the general correlation functions, and the general matrix elements of local operators with respect to the excited states. I will explain this in Section 4.

2 Quantum affine algebras: the structure underlying the solvability

An operator which commutes with the Hamiltonian is called its symmetry. In this section I discuss the symmetries of the XXZ Hamiltonian. There are two kinds of symmetries, abelian and non-abelian. The latter is the symmetry of the quantum affine algebra $U_q(\widehat{sl}_2)$. This algebra underlies the solvability of the XXZ Hamiltonian.

2.1 Integrability and the transfer matrix

What I have described in the previous section is heavily dependent on the special choice of the Hamiltonian (1). In the infinite volume limit, in general, Hamiltonians have infinitely degenerate eigenvalues. This is an obstacle for the diagonalization. In the XXZ case, this infinite degeneracy is decomposed into finite degenaracy in the particle structure: If the number of particles and their rapidities are fixed, the degeneracy reduces to finite. The decomposition is explained as follows.

The XXZ Hamiltonian on the finite N-periodic lattice has an abelian (i.e., mutually commuting) family of symmetries. The simultaneous eigenspaces of this commuting family of operators give rise to the decomposition into the particles in the infinite volume limit.

Let us discuss the commuting family. There exists a family of operators $T(\zeta)$ parametrized by a complex parameter ζ

$$T(\zeta)(\otimes_k v_{\varepsilon_k}) = \sum_{\{\varepsilon'_k\}_{k\in\mathbf{Z}/N\mathbf{Z}}} T(\zeta)^{\{\varepsilon_k\}}_{\{\varepsilon'_k\}}(\otimes_k v_{\varepsilon'_k}). \tag{12}$$

We have

$$[T(\zeta_1),T(\zeta_2)]=0, \tag{13}$$

$$T(1) \text{ is the shift operator, i.e., } T(1)^{\{\varepsilon_k\}}_{\{\varepsilon'_k\}}=\prod_k \delta_{\varepsilon_{k+1},\varepsilon'_k}, \tag{14}$$

$$T(1)^{-1}T(\zeta)=1+(c_1H+c_2)(\zeta-1)+O((\zeta-1)^2). \tag{15}$$

This operator naturally appears in the study of a statistical mechanical model of a different kind, which I will explain in the next section.

2.2 The six-vertex model

The operator $T(\zeta)$ appears in the six-vertex model, a model in classical statistical mechanics on the two dimensional lattice. Consider a 'lattice' consisting of lines in the two dimensional plane. The lines are either horizontal or vertical. We call an intersection of two lines *a vertex*. We associate a local variable ε_k to each edge k, which is a line segment between two neighboring vertices. The variable ε_k takes values 0 or 1.

A configuration $\mathcal{C}$ is an assignment of values 0 or 1 to all the local variables. Consider a vertex v, and a local configuration around the vertex, say ε'_1 and ε_1 for the upper and the lower edges on the vertical line, and ε'_2 and ε_2 for the right and the left edges on the horizontal line. We associate a local weight, $R^{\varepsilon_1,\varepsilon_2}_{\varepsilon'_1,\varepsilon'_2}$, called the Boltzmann weight, to each local configuration. We consider these weights as the matrix elements of an matirx R acting on $V\otimes V$:

$$R(v_{\varepsilon_1}\otimes v_{\varepsilon_2}) = \sum_{\varepsilon'_1,\varepsilon'_2} R^{\varepsilon_1,\varepsilon_2}_{\varepsilon'_1,\varepsilon'_2} v_{\varepsilon'_1}\otimes v_{\varepsilon'_2}. \tag{16}$$

The most basic quantity in classical statistical mechanics is the partition function Z. This is the sum of the product of the local Boltzmann weights; the sum is taken over all the configurations and the product is taken over all the vertices.

$$Z = \sum_{\mathcal{C}} \prod_{v} R^{\varepsilon_1(v,\mathcal{C}),\varepsilon_2(v,\mathcal{C})}_{\varepsilon'_1(v,\mathcal{C}),\varepsilon'_2(v,\mathcal{C})}. \tag{17}$$

Sometimes it is necessary to consider similar configuration sums for a different arrangement of lines, e.g., by introducing lines with different angles.

Now, consider a vertical slice of the whole lattice, i.e., a vertical line and the two sets of horizontal edges in the right and left sides of the vertical line. Let us denote the local variables on the right edges by $\{\varepsilon'_k\}$ and those on the left by $\{\varepsilon_k\}$. One can associate a matrix T acting on $\otimes_k V_k$. This is called the transfer matrix:

$$T^{\{\varepsilon_k\}}_{\{\varepsilon'_k\}} = \sum_{\mathcal{C}_s} \prod_{v_s} R^{\varepsilon_1(v_s,\mathcal{C}_s),\varepsilon_2(v_s,\mathcal{C}_s)}_{\varepsilon'_1(v_s,\mathcal{C}_s),\varepsilon'_2(v_s,\mathcal{C}_s)}. \tag{18}$$

Here the subscript s is put to indicate the restriction to the slice. The configuration $\mathcal{C}_s$ is fixed to $\{\varepsilon'_k\}$ and $\{\varepsilon_k\}$ on the horizontal edges.

The transfer matrix is convenient in the calculation of the partition function. For a finite lattice on the torus $Z = \operatorname{tr} T^N$ where N is the number of the vertical lines on the torus.

So far, I have discussed general setting for a type of models called *vertex models*. Now, I introduce the six-vertex model whose transfer matrix gives the commuting family of operators satisfying (13-15).

We associate a rapidity variable β_j to each line j in the lattice. We set $\zeta_j = e^{\frac{\pi\beta_j}{\xi}}$, where ξ and q are related by $q = -e^{-\frac{\pi^2 i}{\xi}}$. In the massive phase, ξ is purely imaginary ($\operatorname{Im}\xi < 0$), and in the massless phase, $\xi > 0$.

Consider the following $\bar{R}$ depending on the parameters q and ζ.

$$\bar{R}^{\varepsilon,\varepsilon}_{\varepsilon,\varepsilon} = 1, \bar{R}^{\varepsilon,1-\varepsilon}_{\varepsilon,1-\varepsilon} = \frac{q(1-\zeta^2)}{1-q^2\zeta^2}, \bar{R}^{\varepsilon,1-\varepsilon}_{1-\varepsilon,\varepsilon} = \frac{\zeta(1-q^2)}{1-q^2\zeta^2} \quad (\varepsilon = 0, 1), \tag{19}$$

all the other weights are zero.

The vertex model given by this R-matrix is called the six-vertex model. Note that only 6 out of 16 local configurations have a non-zero weight.

In general, we choose the Boltzmann weights at a vertex v to be $\bar{R}(\zeta_1/\zeta_2)$ if the vertical line passing thorough v carries the parameter β_1 and the horizontal line β_2. With this special choice of the Boltzmann weights, the partition function has a large symmetry, i.e., it is invariant under deformation of the arrangement of the lines. This is called the Z-invariance. General Z-invariance is a straightforward consequesnce of the simplest case where only three lines are involved. The equation of the Z-invariance in this case is called the Yang-Baxter equation.

Suppose we define the transfer matrix T by choosing the parameter ζ for the vertical line, and 1 commonly for the horizontal lines. With this choice the transfer matrix $T(\zeta)$ satisfies (13-15). Note, in particular, that (13) follows from the Z-invariance.

The origin of the Z-invariance, or the Yang-Baxter equation, is clarified in the theory of quantum groups. I will explain this in the particular context of the six-vertex model.

2.3 R-MATRICES AS INTERTWINERS [5, 9]

The quantum affine algebra $U_q(\widehat{sl}_2)$ is a q-deformation of the universal enveloping algebra $U(\widehat{sl}_2)$ of the affine Lie algebra $\widehat{sl}_2$. The structure and the representation theory of the former for a generic value of q is not very far from those of the latter which I will recall partly.

The Lie algebra $\widehat{sl}_2$ is a central extension of the infinite dimensional Lie algebra $sl_2 \otimes \mathbf{C}[t, t^{-1}]$. The last one contains two subalgebras that are isomorphic to sl_2: $(sl_2)_i = \mathbf{C}e_i \oplus \mathbf{C}f_i \oplus \mathbf{C}h_i$ $(i = 0, 1)$ where

$$e_0 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \otimes t, f_0 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \otimes t^{-1}, h_0 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \otimes 1 + c, \quad (20)$$
$$e_1 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \otimes 1, f_1 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \otimes 1, h_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \otimes 1.$$

Here, c is the central element.

There are two important categories of representations of $\widehat{sl}_2$:

the affinization of finite dimensional representations; (21)

and

the integrable highest weight representations (IHWR). (22)

There exists one-parameter family of automorphisms $A_\zeta : U(\widehat{sl}_2) \to U(\widehat{sl}_2)$ sending the generators e_i, f_i, h_i to $\zeta e_i, \zeta^{-1} f_i, h_i$. Given a finite dimensional repere-sentation M, i.e., an algebra map $\rho : U(\widehat{sl}_2) \to \mathrm{End}\,(M)$, one can define a new representation by $\rho \circ A_\zeta$. This representation is called the affinization of M. For example, there is a natural action of $\widehat{sl}_2$ on $V \simeq \mathbf{C}^2$ given by the matrix part of (20). The affinization of V is denoted by V_ζ.

The value of c is called the level of representation. The level of V_ζ, as well as the affinizations of all the finite dimensional representations, is zero.

I will say a few words on IHWR. A representation of $\widehat{sl}_2$ is called integrable if M is decomposed into a direct sum of finite dimensional modules by the action of each subalgebra $(sl_2)_i$. Let $\lambda \in (\mathbf{C}h_0 \oplus \mathbf{C}h_1)^*$ be an $\widehat{sl}_2$-weight. A vector u_λ is called a highest weight vector with the highest weight λ if $e_i u_\lambda = 0$, $h_i u_\lambda = \lambda(h_i) u_\lambda$ $(i = 0, 1)$. A representation M is called a highest weight representation if it is generated by a highest weight vector: $M = U(\widehat{sl}_2) u_\lambda$. There exists (and, in fact, uniquely exists) an integrable highest weight representation with the highest weight λ if and only if $\lambda_i = \lambda(h_i)$ is non-negative integer for each i. We denote it by $V(\lambda)$. The level of this representation is equal to $l = \lambda_0 + \lambda_1$.

The above story of the representation theory of $\widehat{sl}_2$ is 'deformed' to that of $U_q(\widehat{sl}_2)$. There is, however, one significant difference in the two theories. The tensor product of two representations is defined in both theories. The action is given by the canonical algebra map $\Delta : U \to U \otimes U$ $(U = U(\widehat{sl}_2)$ or $U = U_q(\widehat{sl}_2))$. This map (unfortunately, there is a conflict in the notation 'Δ') is called the coproduct. For $U = U(\widehat{sl}_2)$ the coproduct is given by $\Delta(X) = X \otimes 1 + 1 \otimes X$ for

$X \in sl_2$. It is invariant with respect to the transposition $\sigma : U \otimes U \to U \otimes U$, $\sigma(x \otimes y) = y \otimes x$. Namely, we have $\sigma \circ \Delta = \Delta$. This is no longer true after the deformation: Δ and $\Delta' = \sigma \circ \Delta$ are differennt.

A question arises. Are the two actions on the tensor product, one given by Δ and the other given by Δ', isomorphic? The answer is 'no' in general. However, it is 'yes' in certain situation including the tensor product of two representations from the union of the categories (21) and (22).

I recall the notion of intertwiner, which plays the central role in the following story. Consider two actions of an algebra A, M_i with the action given by ρ_i $(i = 1, 2)$. A map $F : M_1 \to M_2$ is called an intertwiner if the following diagram commutes:

$$\begin{array}{ccc} M_1 & \xrightarrow{F} & M_2 \\ \rho_1(a) \downarrow & & \rho_2(a) \downarrow \\ M_1 & \xrightarrow{F} & M_2 \end{array} \quad (x \in A). \tag{23}$$

Consider the tensor product of two affinizations $V_{\zeta_i}(i = 1, 2)$ of the two dimensional representation V of $U_q(\widehat{sl}_2)$. The R-matrix $\bar{R}(\zeta_1/\zeta_2) \in \mathrm{End}\,(V_{\zeta_1} \otimes V_{\zeta_2})$, which gives the Boltzmann weights of the six-vertex model, is the intertwiner of the two representations, one given by Δ and the other given by Δ'. Namely, we have an equality $\bar{R}(\zeta_1/\zeta_2)\Delta(x) = \Delta'(x)\bar{R}(\zeta_1/\zeta_2)$ for all $x \in U_q(\widehat{sl}_2)$.

2.4 $U_q(\widehat{sl}_2)$ symmetry of the XXZ model

After these preparation from the representation thoery, it is high time that I told the main idea of this talk: the $U_q(\widehat{sl}_2)$ symmetry of the XXZ Hamiltonian and the transfer matrix of the six-vertex model. It exists only for the massive phase and only in the infinite volume limit. This limitation makes a clear distinction of this symmetry from the abelian symmetry given by the transfer matrix itself.

Formally speaking, the space on which these operators act is the infinite tensor product $\otimes_{k\in\mathbf{Z}} V_k$ of the two dimensional spaces $V_k \simeq \mathbf{C}^2$. We consider these spaces as the two dimensional $U_q(\widehat{sl}_2)$ module with the following actions of the generators.

$$e_0 = f_1 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, e_1 = f_0 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, t_0^{-1} = t_1 = \begin{pmatrix} q & 0 \\ 0 & q^{-1} \end{pmatrix}. \tag{24}$$

Formally speaking again, an action ρ_∞ on $\otimes_{k\in\mathbf{Z}} V_k$ is given by the coproduct, $\Delta(e_i) = e_i \otimes 1 + t_i \otimes e_i$, $\Delta(f_i) = f_i \otimes t_i^{-1} + 1 \otimes f_i$, $\Delta(t_i) = t_i \otimes t_i$. Namely, we have,

$$\Delta_\infty(e_0) = \sum_k \cdots \otimes \begin{pmatrix} q^{-1} & 0 \\ 0 & q \end{pmatrix} \otimes \overset{k\text{-th}}{\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \otimes \cdots, \tag{25}$$

$$\Delta_\infty(f_0) = \sum_k \cdots \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \otimes \overset{k\text{-th}}{\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}} \otimes \begin{pmatrix} q & 0 \\ 0 & q^{-1} \end{pmatrix} \otimes \cdots,$$

$$\Delta_\infty(e_1) = \sum_k \cdots \otimes \begin{pmatrix} q & 0 \\ 0 & q^{-1} \end{pmatrix} \otimes \overset{k\text{-th}}{\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \otimes \cdots,$$

$$\Delta_\infty(f_1) = \sum_k \cdots \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \otimes \overset{k-\mathrm{th}}{\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}} \otimes \begin{pmatrix} q^{-1} & 0 \\ 0 & q \end{pmatrix} \otimes \cdots,$$

$$\Delta_\infty(t_0^{-1}) = \Delta_\infty(t_1) = \cdots \otimes \begin{pmatrix} q & 0 \\ 0 & q^{-1} \end{pmatrix} \otimes \begin{pmatrix} q & 0 \\ 0 & q^{-1} \end{pmatrix} \otimes \cdots.$$

This action is obviously not well-defined on arbitrary vectors of the form $\otimes_{k\in\mathbf{Z}} v_{\varepsilon_k}$ because $\Delta_\infty(t_1)$ counts the total spin $q^{\sum_{k\in\mathbf{Z}}\frac{1}{2}(1+(-1)^{\varepsilon_k})}$. The total spin is finite if we restrict to the vectors $|p\rangle$ considered in Section 1. Hopefully, if $-1 < q < 0$, the formal expressions (25) define actions on certain vectors of the form (4). One can check this idea in the small q expansion. For example, one can seek for a singlet, i.e., a vector annihilated by all $\rho_\infty(e_i)$ and $\rho_\infty(f_i)$ $(i = 0, 1)$ starting from the ground state vector $|p^{(0)}\rangle$ of (3). The result is remarkable. We get the same expansion as the vector $|vac\rangle_0$.

Denote the physical space corresponding to the i-th ground state by $\mathcal{F}_i$. We postulate that there is an action $\rho^{(i)}$ of $U_q(\widehat{sl}_2)$ on $\mathcal{F}_i$, and that the transfer matrix $T(\zeta)$ intertwines $\rho^{(0)}$ with $\rho^{(1)}$. In other words, the transfer matrix, and in particular, the XXZ hamiltonian, has the $U_q(\widehat{sl}_2)$ symmetry. If this is true, the $U_q(\widehat{sl}_2)$ module $\mathcal{F}_i$ must be highly reducible because the space of the intertwiners, containing all $T(\zeta)$, is infinite dimensional. Recall the decomposition (7) for $\Delta = -1$. This result suggests how the space $\mathcal{F}_i$ for $\Delta < -1$ decomposes with respect to the $U_q(\widehat{sl}_2)$ action. The rapidity variables β_j in (7) should be the parameters of the affinization $\zeta_j = e^{\frac{\pi\beta_j}{\xi}}$.

This ia a nice picture. However, its mathematical content is still unclear because we have no means to make a rigorous meaning of the infinite tensor product. In the following sections, I will give a different picture to the space $\mathcal{F}_i$ which enables us to formulate everything in the representation theory without using the infinite tensor product.

3 CFT and the SG model:integrable quantum field theories

Quantum field theory and statistical mechanics are twins. They share similar ideas in many aspects. Integrable QFT and solvable lattice models, in particular, have a common algebraic structure. In this section, I review a few results of the former, from which we learn how to solve the models by using the symmetry algebras.

3.1 Lattice theory and continuum limit

In Section 1, I have described the structure of the eigenvectors of the XXZ model by using the language of QFT. This is possible because of the similarlity between QFT and statistical mechanics. In fact, the connection between these two theories is more than a mere analogy because in the continuum limit, lattice theories are described by QFT. The correlation functions of local variables in the former are scaled to those of local fields in the latter. For example, take the two dimensional Ising model. This is a model in classical statistical mechanics on the two dimensional lattice. We have the scaling identity (see [21, 18])

$$\langle\varphi(0)\varphi(x)\rangle \quad = \lim_{\substack{\varepsilon\to 0,\, m,n\to\infty \\ x=(m\varepsilon,n\varepsilon)}} \langle\sigma_{0,0}\sigma_{m,n}\rangle. \tag{26}$$

Here, ε is a parameter in the Ising model such that the system becomes massless at $\varepsilon = 0$. This identity along with the n point generalization defines a massive QFT with the local field $\varphi(x)$.

In general, it is rather difficult to carry out the computation in the right hand side. Instead one can study the left hand side by using some other principle, and then identify it with the continuum limit of some statistical mechanical system. This idea was fully developed and extremely successful in the two dimensional conformal field theory, which deals with the short distance behavior of massive QFT.

The success of CFT came from the principle of conformal invariance. The conformal invariance forces the theory to be massless. Therefore, it has no power to say something about the scaling limit of off-critical (massive) models except in the short distance limit. My interest in CFT in this talk lies not in taking the scaling limit like (26) for critical models but in seeking for an algebraic machinery applicable to off-critical models.

3.2 Primary fields and vertex operators [3, 16, 20]

The local fields in CFT have the conformal invariance. This is a symmetry of the Virasoro algebra, which is a central extension of the Lie algebra of vector fields on the unit circle. (I restrict the discussion to the so-called chiral CFT.) This symmetry is a little bit different from the symmetry of the XXZ Hamitonian discussed in Section 2. The action of $U_q(\widehat{sl}_2)$ commutes with the XXZ Hamiltonian. The action of Virasoro algebra does not commute with the loacl fields. However, it induces an adjoint action on the set of local fields, and this action is identified with a highest weight representation.

The operators serving as a highest weight vector in this representation are called the primary fields. It is important to know the primary fields as an operator acting on the physical space of the conformal field theory. The operators in this context is called the vertex operators.

Let us consider the conformal field theory with the symmetry of the affine Lie algebra $\widehat{sl}_2$. We fix a positive integer l. The physical space of this theory is the direct sum of the level l integrable highest weight representations: $\mathcal{F}_{\mathrm{CFT},l} = \oplus_\lambda V(\lambda)$.

Let $V_\zeta^{(j)}$ be the affinization of the $2j+1$ dimensional representation of $\widehat{sl}_2$. We considered a special case, $j = \frac{1}{2}$ in Section 2. The intertwiner of the form

$$\phi^{(j)}(\zeta) : \mathcal{F}_{\mathrm{CFT},l} \to \mathcal{F}_{\mathrm{CFT},l} \otimes V_\zeta^{(j)} \tag{27}$$

exists if and only if $0 \le j \le \frac{l}{2}$. It is called the vertex operator of level l and spin j. This is identified with the primary field which generates the highest weight module with the highest weight λ such that $\lambda(h_1) = 2j$.

3.3 The KZ equation

The two-point scaling funtion of the Ising model (26) is expressed in a closed form by using a solution of the non-linear ordinary differential equation called the Painleve equation. No such result is known for other solvable lattice models that are essentially different from the Ising model. In CFT, the correlation functions satisfy a system of linear partial equations which is a generalization of the hypergeometric differential equation.

Let us consider a particular example, the operator $\phi^{(\frac{1}{2})}(\zeta)$ in (27). For simplicity we denote it by $\phi(\zeta)$. This operator has two components $\phi_0(\zeta), \phi_1(\zeta)$ corresponding to $v_0, v_1 \in V_\zeta$, each of which acts on $\mathcal{F}_{\mathrm{CFT},l}$. Denote by $|0\rangle$ the highest weight vector in the spin 0 highest weight module. Set

$$f(\zeta_1,\ldots,\zeta_n) = \sum_{\varepsilon_1,\ldots,\varepsilon_n} f_{\varepsilon_1,\ldots,\varepsilon_n}(\zeta_1,\ldots,\zeta_n) v_{\varepsilon_1}\otimes\cdots\otimes v_{\varepsilon_n} \in V_{\zeta_1}\otimes\cdots\otimes V_{\zeta_n} \quad (28)$$

where

$$f_{\varepsilon_1,\ldots,\varepsilon_n}(\zeta_1,\ldots,\zeta_n) \quad = \quad \langle 0|\phi_{\varepsilon_1}(\zeta_1)\cdots\phi_{\varepsilon_n}(\zeta_n)|0\rangle. \quad (29)$$

Let P_{jk} be the transposition of the j-th and the k-th components in the tensor product $V_{\zeta_1}\otimes\cdots\otimes V_{\zeta_n}$. After some trivial modification the function f satisfy the following system of linear partial differential equations called the Knizhnik-Zamolodchikov equation.

$$\frac{\partial}{\partial\zeta_j} f(\zeta_1,\ldots,\zeta_n) = \frac{1}{l+2}\sum_{k\neq j}\frac{P_{jk}}{\zeta_j-\zeta_k} f(\zeta_1,\ldots,\zeta_n). \quad (30)$$

3.4 Form factors of the SG model [19]

The two-point functions in CFT are simple power functions. This is clearly seen from the equation (30). The quantum field theory in the scaling limit of the Ising model is not conformally invariant. The two-point function is already highly non-trivial. There are a variety of quantum field theories obtained as the scaling limit of the off-critical solvable lattice models. These are massive field theories. Their correlation functions are, in general, not known. However, these theories have the integrability inherited from the lattice models. They have the factorized S-matrix and their form factors satisfy the q-deformation of the KZ equation. I will explain these points in the sine-Gordon model which are the scaling limit of the eight-vertex model (a generalization of the six-vertex model).

One way to compute the two-point function is to put a complete set of intermidiate states.

$$\langle vac|\phi(0)\phi(x)|vac\rangle = \sum_{n\geq 0,\mathrm{even}}\prod_{j=1}^{n}\int_{-\infty}^{\infty}\frac{d\beta_j}{2\pi}\frac{1}{n!}\sum_{\varepsilon_1,\ldots,\varepsilon_n} \quad (31)$$
$$\times \quad \langle vac|\phi(0)|\beta_n,\ldots,\beta_1\rangle_{\varepsilon_n,\ldots,\varepsilon_1\ \varepsilon_1,\ldots,\varepsilon_n}\langle\beta_1,\ldots,\beta_n|\phi(x)|vac\rangle.$$

The matrix elements $\langle vac|\phi(0)|\beta_n,\ldots,\beta_1\rangle_{\varepsilon_n,\ldots,\varepsilon_1}$ are called the form factors. For the Ising model, the form factors are given by the Pfaffian of the two-particle one, which is $\tanh\frac{\beta_1-\beta_2}{2}$.

There is a redundancy of the vectors $\langle vac|\phi(0)|\beta_n,\ldots,\beta_1\rangle_{\varepsilon_n,\ldots,\varepsilon_1}$. Only those with the restriction $\beta_1<\cdots<\beta_n$ are independent. The assumption of the factorized S-matrix is such that the linear relations among the vectors are given by the two-particle S-matrix in the form (9). For example, the two-particle S-matrix of the Ising model is -1.

The S-matrix of the sine-Gordon theory is given by (19) with a real parameter ξ, as $S=S_0\bar{R}$. The scalar factor S_0 is given by

$$S_0 = -e^{-i\int\frac{\sin\kappa\beta\,\mathrm{sh}\,\frac{\pi-\xi}{2}\kappa}{\mathrm{ch}\,\frac{\pi}{2}\kappa\,\mathrm{sh}\,\frac{\xi}{2}\kappa}\frac{d\kappa}{\kappa}}. \tag{32}$$

This function is expressed by means of the double gamma functions ([22, 13]). Note that S_0 depends on β,ξ in such a way that it is not single-valued in ζ,q as opposed to the matrix part $\bar{R}$.

In the limit $\xi\to\infty$, the double gamma function reduces to the usual gamma function, and the S_0 is given by

$$S_0(\beta) = \frac{\Gamma(\frac{1}{2}+\frac{\beta}{2\pi i})\Gamma(-\frac{\beta}{2\pi i})}{\Gamma(\frac{1}{2}-\frac{\beta}{2\pi i})\Gamma(\frac{\beta}{2\pi i})}. \tag{33}$$

The $\bar{R}$ reduces to $\frac{\beta-\pi iP}{\beta-\pi i}$ where P is the transposition.

The S-matrix of the SG theory is identical with the S-matrix of the six-vertex model in the massless phase. This is because the SG theory is the continuum limit of the eight-vertex model as I have already mentioned. The continuum limit is taken at the critical region of the eight-vertex model. This is nothing but the six-vertex model in the massless phase. The case discussed in Section 1 is a special case of this story where $\xi=\infty$.

Set

$$F_{\varepsilon_1,\ldots,\varepsilon_n}(\beta_1,\ldots,\beta_n) = \langle vac|\phi(0)|\beta_n,\ldots,\beta_1\rangle_{\varepsilon_n,\ldots,\varepsilon_1}. \tag{34}$$

Because of (9) it satisfies

$$F_{\varepsilon_1,\ldots,\varepsilon_{j+1},\varepsilon_j,\ldots,\varepsilon_n}(\beta_1,\ldots,\beta_{j+1},\beta_j,\ldots,\beta_n) = \sum_{\varepsilon'_j,\varepsilon'_{j+1}} S(\beta_j-\beta_{j+1})_{\varepsilon_j,\varepsilon_{j+1}}^{\varepsilon'_j,\varepsilon'_{j+1}} F_{\varepsilon_1,\ldots,\varepsilon'_j,\varepsilon'_{j+1},\ldots,\varepsilon_n}(\beta_1,\ldots,\beta_j,\beta_{j+1},\ldots,\beta_n). \tag{35}$$

There is another equation for the form factor. It gives the analytic continuation of the form factor in the last variable β_n:

$$F_{\varepsilon_1,\ldots,\varepsilon_n}(\beta_1,\ldots,\beta_n+2\pi i) = F_{\varepsilon_n,\varepsilon_1,\ldots,\varepsilon_{n-1}}(\beta_n,\beta_1,\ldots,\beta_{n-1}). \tag{36}$$

I will not explain why this is valid. In Section 4, however, its origin in the representation theory is given in the case of the XXZ model with $\Delta<-1$.

3.5 The quantum KZ equation [8]

Set

$$F(\beta_1,\ldots,\beta_n)=\sum_{\varepsilon_1,\ldots,\varepsilon_n}F_{\varepsilon_1,\ldots,\varepsilon_n}(\beta_1,\ldots,\beta_n)v_{\varepsilon_1}\otimes\cdots\otimes v_{\varepsilon_n}. \tag{37}$$

Combination of (9) and (36) gives the following difference equation for the form factor.

$$\begin{aligned}&F(\beta_1,\ldots,\beta_j+2\pi i,\ldots,\beta_n)=S_{j+1,j}(\beta_{j+1}-\beta_j-2\pi i)\cdots S_{n,j}(\beta_n-\beta_j-2\pi i)\\&\times S_{1,j}(\beta_1-\beta_j)\cdots S_{j-1,j}(\beta_{j-1}-\beta_j)F(\beta_1,\ldots,\beta_j,\ldots,\beta_n).\end{aligned} \tag{38}$$

Here, I denote by $S_{j,k}$ the action of S on the j-th and k-th components. In the limit where $\xi,\beta_1,\ldots,\beta_n\to\infty$, this equation scales to the differential equation (30) with the level l equal to 0.

One can repeat the story in 3.2 and 3.3 for $U_q(\widehat{sl}_2)$. Vertex operators are defined as the intertwiners between the highest weight representations with and without the tensor product by the affinization of a finite dimensional representation. The matrix elements of the product of vertex operators between the highest weight vectors satisfy a system of difference equation. This is called the quantum KZ equation. The above equation is a special case with level 0.

A question arises: Are these matrix elements representing the correlation functions of some integrable models? The answer is NO BUT. I will come back to this question later.

4 CTM and HTM: the key words in the dictionary

I present the algebraic structure of the XXZ and the six-vertex models in the language of representation theory. Two kinds of transfer matrices, that are acting on the half-infinite tensor product, play the central roles in the symmetry of $U_q(\widehat{sl}_2)$. I will explain how to identify these operators in the representation theory. This identification brings us the solutions to the problems mentioned before: the diagonalization of the XXZ Hamiltonian, and the computation of the form factors and the correlation functions.

4.1 CTM [1]

Our goal is to understand the infinite tensor product $\otimes_{k\in\mathbf{Z}}V_k$ as a $U_q(\widehat{sl}_2)$ module. It is rather a big representation, of course not irreducible. The half infinite tensor product is also a representation space of $U_q(\widehat{sl}_2)$. It is much smaller than the infinite tensor product in both directions. The idea is to study the content of this representation first, There are two operators which naturally act on this space. They are the corner transfer matrix (CTM) and the half transfer matrix (HTM).

I start from the CTM. Recall the Boltzmann weights given by (19). There are three different ones. Let us call them the a, b and c weights, respectively, from the left to the right. We restrict to the region

$$-1<q<0,\quad 1<\zeta<-q^{-1}. \tag{39}$$

In this region, the c weight dominates the others.

The XXZ Hamiltonian and the transfer matrix of the six-vertex model act formaly on the vectors parametrized by the paths, which satisfy certain boundary conditions. We consider similar boundary conditions for the configurations on the two-dimensional lattice. A configuration is called the ground state if it consists of the c weight only. There are two such configurations. The local variables take constant values 0 or 1 along the NE-to-SW diagonal lines, and these values alternates over the diagonal lines. Choose two vertical lines, and consider the set of horizontal edges between these two lines. We number these edges by $\mathbf{Z}$ (increasingly from S to N). The configuration of a ground state on these edges is equal to $p^{(0)}$ or $p^{(1)}$. Accordingly, we call it the i-th ground state.

Consider the half infinite tensor product $\otimes_{k=1}^{\infty} V_k$. We denote by $\mathcal{H}_i$ the space spanned by the vector of the form $\otimes_{k=1}^{\infty} v_{p(k)}$ where the half infinite path p satisfies $p(k) = \frac{1}{2}(1-(-1)^{k+i})$ for sufficiently large k. The corner transfer matrix $A(\zeta)$ formally acts on the space $\mathcal{H}_i$. Its matrix element is given as follows.

Consider the center of the plaquet in the lattice between the edges 0 and 1. Divide the whole lattice into four quadrants at this point making cuts in the N,E,W,S directions. Take the NW quadrant. Fix the local variables of the edges on the N-cut to $\{p'(k)\}_{k\in\mathbf{Z}_{\geq 1}}$, and those on the W-cut to $\{p(k)\}_{k\in\mathbf{Z}_{\geq 1}}$. Consider the configuration sum for this quadrant with this restriction on the N and W boundaries. We also restrict the sum to those configurations which belong to the i-th ground state, i.e., different from the i-th ground state at finitely many places. We define the matrix element $A(\zeta)^{\{p'(k)\}}_{\{p(k)\}}$ to be the configuration sum under these restrictions.

This is only a formal definition, and it is divergent. In the region (39), the CTM can be renormalized to a 'finite' operator with discrete (and, in fact, equally spaced) eigenvalues, while if $|q|=1$, the renormalized operator has a continuum spectrum. This difference comes from the difference in the analytic structure of the free energy.

Consider a finite lattice with N sites (i.e., $N = \sharp\{\text{vertex}\}$). The limit $\kappa = \lim_{N\to\infty} Z^{\frac{1}{N}}$ is called the partition function per site. (The free energy is given by its logarithm.) In the massive region, it is given by

$$\kappa = \zeta \frac{(q^4\zeta^2;q^4)_\infty (q^2\zeta^{-2};q^4)_\infty}{(q^4\zeta^{-2};q^4)_\infty (q^2\zeta^2;q^4)_\infty} \tag{40}$$

where $(z;p)_\infty = \prod_{n=0}^{\infty}(1-p^n z)$.

The above κ is a single-valued meromorphic function in ζ. It has a natural boundary at $|q|=1$. If $|q|=1$, the partition function per site has an different expression: it is given by $-S_0^{-1}$ (see (32)) with a real value of ξ and an imaginary value of β. (Note that in the sine-Gordon theory, β is real.) This is not single-valued in ζ, nor in q.

Physical intuition tells that the renormalization of CTM and HTM is done by choosing the overall factor of the Boltzmann weight in such a way that the partition function per site is 1. Therefore, the structure of the physical space and the renormalized operators acting on it differs in the massive and the massless phases.

In the region (39), we have

$$A_{\rm re}(\zeta)=\zeta^{-D}. \tag{41}$$

The operator D is independent of ζ and has the spectrum $\{0,1,2,\ldots\}$. This remarkable (however, no rigorous proof is available) property is a consequence of the single-valuedness of κ.

Let Λ_i be the affine sl_2 weight such that $\langle\Lambda_i,h_j\rangle=\delta_{ij}$ $(i,j=0,1)$. I state the main postulate:

the space of the eigenvectors of the CTM in the i-th ground state is isomorphic to the integrable and irreducible highest weight representation $V(\Lambda_i)$ of $U_q(\widehat{sl}_2)$.

Namely, the half infinite tensor product $\mathcal{H}_i$ is interpreted as the highest weight module ([7])

$$\mathcal{H}_i\simeq V(\Lambda_i). \tag{42}$$

I give the evidence for this statement: The character of the space $\mathcal{H}_i$ and that of $V(\Lambda_i)$ are equal. The former can be computed in the the crystal limit $q\to 0$ because we have

$$D \quad = \quad -\frac{1}{2}\sum_{k=1}^{\infty}k(\sigma^x_k\sigma^x_{k+1}+\sigma^y_k\sigma^y_{k+1}+\Delta\sigma^z_k\sigma^z_{k+1}), \tag{43}$$

and this is diagonal in the limit. The equality of the characters is equivalent to the combinatorial identity

$$\sum_{p\in\mathcal{H}_{i,m}} q^{\sum_{k=1}^{\infty}\left((-1)^{p(k)+p(k+1)}-(-1)^{p^{(i)}(k)+p^{(i)}(k+1)}\right)} \quad = \quad \frac{q^{(m-i)(m-i+1)}}{(q^2;q^2)_\infty}, \tag{44}$$

where $\mathcal{H}_{i,m}=\{p\in\mathcal{H}_i;2\sum_{k=1}^{\infty}(p(k)-p^{(i)}(k))=m\}$.

4.2 HTM [4, 14, 13]

The matrix element of the transfer matrix is formally given by the configuration sum (18) for a slice of the lattice consisting of one vertical line and horizontal lines indexed by $k\in\mathbf{Z}$ which intersect the vertical one. Cut the vertical edge between the $k=0,1$ horizontal lines. The matrix element of the half transfer matrix $\Phi^{(i)}_\varepsilon(\zeta)$ $(\varepsilon=0,1)$ is given by the configuration sum for the upper half of the slice where the local variables on the right and left edges are fixed to $\{\varepsilon'_k\}$ and $\{\varepsilon_k\}$, and the one on the cut edge is fixed to ε. The superscript i indicates the restriction of the sum to those configurations which belong to the i-th ground state.

The half transfer matrix acts as

$$\cdots\otimes V\otimes V\otimes V\to(\cdots\otimes V\otimes V)\otimes V_\zeta. \tag{45}$$

The components described by V corresponds to the horizontal lines and the one denoted by V_ζ corresponds to the vertical line.

In the dictionary, the half transfer matrix (45) is translated into the unique (up to the normalization) intertwiner

$$\Phi^{(i)}(\zeta)=\sum_{\varepsilon=0,1}\Phi^{(i)}_\varepsilon(\zeta)\otimes v_\varepsilon : V(\Lambda_i)\to V(\Lambda_{1-i})\otimes V_\zeta. \tag{46}$$

If $\zeta=1$ in (45), the mapping is nothing but the identity operator. However, its translation (46) is a highly non-trivial operator even if $\zeta=1$.

I list some properties of the intertwiners.

$$\xi^D\Phi^{(i)}_\varepsilon(\zeta)=\Phi^{(i)}_\varepsilon(\xi\zeta)\xi^D, \tag{47}$$

$$\Phi^{(1-i)}_{\varepsilon_2}(\zeta_2)\Phi^{(i)}_{\varepsilon_1}(\zeta_1)=\sum_{\varepsilon'_1,\varepsilon'_2=0,1}R^{\varepsilon'_1\varepsilon'_2}_{\varepsilon_1\varepsilon_2}(\zeta_1/\zeta_2)\Phi^{(1-i)}_{\varepsilon'_1}(\zeta_1)\Phi^{(i)}_{\varepsilon'_2}(\zeta_2), \tag{48}$$

$$\sum_\varepsilon\Phi^{(1-i)}_{1-\varepsilon}(-q^{-1}\zeta)\Phi^{(i)}_\varepsilon(\zeta)=\mathrm{id}_{\mathcal{H}_i} \tag{49}$$

The R-matrix in (48) is normalized as $R(\zeta)=\frac{1}{\kappa(\zeta)}\bar{R}(\zeta)$ (see (40)).

4.3 Space of the physical states

The identification of the physical space follows from (42) by a simple functorial argument.

Consider the inner product of V, $\langle v_i,v_j\rangle=\delta_{i+j,1}$. The $U_q(\widehat{sl}_2)$ action (24) on V satisfies $\langle xv,v'\rangle=\langle v,b(x)v'\rangle$ where b is the anti-automorphism of $U_q(\widehat{sl}_2)$ given by $b(e_i)=qt_ie_i$, $b(f_i)=qt_i^{-1}f_i$, $b(t_i)=t_i^{-1}$. Since the left half $\cdots\otimes V\otimes V\otimes V$ is equal to $\oplus_{i=0,1}V(\Lambda_i)$, the right half $V\otimes V\otimes V\otimes\cdots$ is equal to the dual space $\oplus_{i=0,1}V(\Lambda_i)^*$. The action on the dual space is given by the transposed action $b(x)^t$. The infinite tensor product $\mathcal{F}$ is identified with $\mathrm{End}(\mathcal{H})=\mathcal{H}\otimes\mathcal{H}^*$. The action on $\mathcal{F}$ is given by the adjont action.

$$\mathcal{F}=\mathrm{End}(\mathcal{H})=\oplus_{i,j=0,1}\mathrm{Hom}(V(\Lambda_i),V(\Lambda_j)), \tag{50}$$

$$x.f=\sum x_{(1)}\circ f\circ b(x_{(2)})\quad\text{for } x\in U_q(\widehat{sl}_2), f\in\mathrm{End}(\mathcal{H}). \tag{51}$$

Here $\Delta(x)=\sum x_{(1)}\otimes x_{(2)}$ is the coproduct of x.

The inner product on $\mathcal{F}$ is given by

$$\langle f,g\rangle=\mathrm{trace}_{\mathcal{H}}f\circ g\quad\text{for } f,g\in\mathrm{End}(\mathcal{H}). \tag{52}$$

The transfer matrix in the dictionary reads as

$$T(\zeta)f\ =\ \sum_\varepsilon\Phi_\varepsilon(\zeta)\circ f\circ\Phi_{1-\varepsilon}(\zeta). \tag{53}$$

Now, I will diagonalize this operator.

4.4 Vacuum and excited states

The vacuum state (4) is given by the iteration of the transfer matrix, because it is the largest eigenvector. Namely, the coefficient $c(p)$ in (4) is (up to a divergent scalar) written as $c(p) \sim \lim_{N\to\infty}\langle p|T(\zeta)^N|p^{(N+i)}\rangle$ where $p^{(i)} = p^{(i+2)}$ is the ground state path.

Thr right hand side is nothing but the partition function for the one half of the lattice, or equivalently, is equal to the matrix element $(A(\zeta)B(\zeta))^{\{1-p(1-k)\}_{k\ge1}}_{\{p(k)\}_{k\ge1}}$ of the product of the CTMs corresponding to the NW and the SW quadrants. Using the symmetry property of $R(\zeta)$, $R^{\varepsilon_2'\varepsilon_1}_{\varepsilon_2\varepsilon_1'}(\zeta_2/\zeta_1) = R^{1-\varepsilon_1',\varepsilon_2'}_{1-\varepsilon_1,\varepsilon_2}(-q^{-1}\zeta_1/\zeta_2)$, we obtain $B_{\rm re}(\zeta) = A_{\rm re}(-q^{-1}\zeta^{-1})$, and therefore $A_{\rm re}(\zeta)B_{\rm re}(\zeta) = (-q)^D$.

We reached the conclusion.

$$|vac\rangle_i = \chi^{-\frac{1}{2}}(-q)^D \in {\rm End}\,(\mathcal{H}_i). \tag{54}$$

Here $\chi = {\rm trace}_{\mathcal{H}_i} q^{2D} = \prod_{n=1}^{\infty}\frac{1}{1-q^{2n}}$ is the normailzation factor such that ${}_i\langle vac|vac\rangle_i = 1$. One can easily check $T(\zeta)|vac\rangle_i = |vac\rangle_{1-i}$ by using (48) and (49).

To find particles in $\mathcal{F}$ is equivalent to find submodules isomorphic to $V_{\xi_n}\otimes\cdots\otimes V_{\xi_1}$ in ${\rm End}\,(\mathcal{H})$. This problem is also solved by using intertwiners, but of a different kind:

$$\Psi^{*(i)}(\xi) : V_\xi \otimes V(\Lambda_i) \to V(\Lambda_{1-i}). \tag{55}$$

The essential difference of this intertwiner from $\Phi^{(i)}(\zeta)$ is that V_ξ is placed in the left of $V(\Lambda_i)$. In the CFT case, there are no such difference because the coproduct is symmetric.

We have

$$\xi^D\Psi^{*(i)}_\varepsilon(\zeta) = \Psi^{*(i)}_\varepsilon(\xi\zeta)\xi^D, \tag{56}$$

$$\Psi^{*(1-i)}_{\varepsilon_1}(\xi_1)\Psi^{*(i)}_{\varepsilon_2}(\xi_2) = -\sum_{\varepsilon_1',\varepsilon_2'} R^{\varepsilon_1,\varepsilon_2}_{\varepsilon_1',\varepsilon_2'}(\xi_1/\xi_2)\Psi^{*(1-i)}_{\varepsilon_2'}(\xi_2)\Psi^{*(i)}_{\varepsilon_1'}(\xi_1), \tag{57}$$

$$\Phi^{(1-i)}_{\varepsilon_1}(\zeta)\Psi^{*(i)}_{\varepsilon_2}(\xi) = \tau(\zeta/\xi)\Psi^{*(1-i)}_{\varepsilon_2}(\xi)\Phi^{(i)}_{\varepsilon_1}(\zeta). \tag{58}$$

Here, we set

$$\tau(\zeta) = \zeta^{-1}\frac{(q\zeta^2;q^4)_\infty(q^3\zeta^{-2};q^4)_\infty}{(q\zeta^{-2};q^4)_\infty(q^3\zeta^2;q^4)_\infty}. \tag{59}$$

Using these relations, one can show that the n-particle states is given by

$$|\xi_n,\ldots,\xi_1\rangle_{\varepsilon_n,\ldots,\varepsilon_1,i} = \Psi^{*(n-1+i)}_{\varepsilon_n}(\xi_n)\cdots\Psi^{*(i)}_{\varepsilon_1}(\xi_1)(-q)^D. \tag{60}$$

The eigenvalue of the transfer matrix on this states is given by $\prod_{j=1}^n\tau(\zeta/\xi_j)$.

4.5 Correlation functions and form factors [11]

The correlation functions of the XXZ model are by definition ${}_i\langle vac|\mathcal{O}|vac\rangle_i$ where $\mathcal{O}$ is some local operators. This expression is immediately written as the trace $\chi^{-1}\mathrm{trace}_{\mathcal{H}_i} q^{2D}\mathcal{O}$.

For example, let us consider the simplest case $\sigma_1^z \in \mathrm{End}\,(\mathcal{F}) = \mathrm{End}\,(\mathcal{H}\otimes\mathcal{H}^*)$. This operator, in fact, acts only on $\mathcal{H}$. Recall the half transfer matrix (we now abbreviate the notation by dropping the superscript i)

$$\Phi(1) = \Phi_0(1)\otimes v_0 + \Phi_1(1)\otimes v_1 : \cdots\otimes V\otimes V\otimes V \tilde{\rightarrow} (\cdots\otimes V\otimes V)\otimes V. \tag{61}$$

The relation (49) gives the inverse map

$$\Phi_1(-q^{-1})\otimes v_0^* + \Phi_0(-q^{-1})\otimes v_1^* : (\cdots\otimes V\otimes V)\otimes V \tilde{\rightarrow} \cdots\otimes V\otimes V\otimes V, \tag{62}$$

where v_0^*, v_1^* are the dual basis of v_0, v_1. Therefore, we have

$$\sigma_1^z \quad = \quad \Phi_1(-q^{-1})\Phi_0(1) - \Phi_0(-q^{-1})\Phi_1(1). \tag{63}$$

In general, the correlation functions belong to the family of functions of the form

$$\mathrm{trace}_{\mathcal{H}_i} q^{2D}\Phi_{\varepsilon_1}(\zeta_1)\cdots,\Phi_{\varepsilon_n}(\zeta_n). \tag{64}$$

In CFT, the correlation functions are the matrix elements of the product of vertex operators between the highest weight vectors. The q-analogues of such matrix elements do not contain the lattice correlation functions. Instead, the trace functions (64) give the lattice correlation functions. The trace functions also contain the form factors of the local operators in the form

$$\mathrm{trace}_{\mathcal{H}_i} q^{2D}\Phi_{\varepsilon_1}(\zeta_1)\cdots,\Phi_{\varepsilon_n}(\zeta_n)\Psi^*_{\kappa_m}(\xi_m)\cdots,\Psi^*_{\kappa_1}(\xi_1). \tag{65}$$

This is because the excited states are given by (60).

I finish this talk with several remarks on the formula (65).

A direct computation of the trace is not practical because the trace is taken on the infinite dimensional space $\mathcal{H}_i$. However, it is possible to realize $\mathcal{H}_i$ as the Fock space of free bosons. In this realization, the operators $\Phi(\zeta)$ and $\Psi^*(\xi)$ are explicitly expressed in terms of bosonic currents. The integral formula for the trace functions follows from this.

The exchange relations (47-49) for the half transfer matrices induce a set of equations similar to (35) and (36) for the trace functions (64). Solving these equations under a certain analyticity condition which follows from the integral formula, we have

$$\frac{\mathrm{trace}_{\mathcal{H}_0} q^{2D}\left(\Phi_0(\zeta_1)\Phi_1(\zeta_2)+\Phi_1(\zeta_1)\Phi_0(\zeta_2)\right)}{\mathrm{trace}_{\mathcal{H}_0} q^{2D}\left(\Phi_0(\zeta_1)\Phi_1(\zeta_2)-\Phi_1(\zeta_1)\Phi_0(\zeta_2)\right)} = \frac{(-q^3\zeta^{-1};q^2)_\infty(-q\zeta;q^2)_\infty}{(q^3\zeta^{-1};q^2)_\infty(q\zeta;q^2)_\infty} \tag{66}$$

where $\zeta = \zeta_2/\zeta_1$. Baxter's result (11) follows from this.

Suppose that an operator $\mathcal{O}$ commutes with $\Psi^*(\xi)$ (in fact, the local operators, e.g., (63), do commute), then the trace functions $\mathrm{trace}_{\mathcal{H}_i}\mathcal{O}\Psi^*_{\kappa_m}(\xi_m)\cdots,\Psi^*_{\kappa_1}(\xi_1)$

satisfy exactly the same equations as (35) and (36) with pure imaginary ξ. Note, in particular, that the shift $\beta \to \beta + 2\pi i$ corresponds to the shift $\zeta \to q^2\zeta$. The relation (36) follows from $q^{2D}\Psi^*(\xi) = \Psi^*(q^2\xi)q^{2D}$ and the cyclicity of the trace. The relation (58) tells that the operators $\Phi(\zeta)$ and $\Psi^*(\xi)$ commute up to a simple factor $\tau(\zeta/\xi)$. With a suitable modification to cancel the factors $\tau(\zeta_i/\xi_j)$, the trace function (65), in general, gives a solution of the qKZ equation with level 0.

The connection between the XXZ model and the representaiton theory of $U_q(\widehat{sl}_2)$ fails in the massless phase. The reason for this is that the latter is singular when $|q| = 1$. The product of the intertwiners exhibit singularities there. However, the bosonic construction of the vertex operators satisfying the relevant exchange relations is possible ([17, 13, 10]). The integral formulas for the correlation functions and the form factors are, thus, available (so far, without a firm basis of the representation theory).

References

[1] R. J. Baxter. *Exactly Solved Models in Statistical Mechanics*. Academic, London, 1982.

[2] R. J. Baxter. Spontaneous staggered polarization of the F model. J. Stat. Phys., 9:145–182, 1973.

[3] A. A. Belavin, A. N. Polyakov and A. N. Zamolodchikov . Infinite conformal symmetries in two dimensional quantum field theory. *Nucl. Phya.*, B 241:333-380, 1984.

[4] B. Davies, O. Foda, M. Jimbo, T. Miwa, and A. Nakayashiki. Diagonalization of the XXZ Hamiltonian by vertex operators. *Comm. Math. Phys.*, 151:89–153, 1993.

[5] V. G. Drinfeld. Quantum Groups. In *Proceedings of the International Congress of Mathematicians.* 798–820, Berkeley, 1987.

[6] L. D. Faddeev and L. A. Takhtajan. Spectrum and scattering of excitations in the one-dimensional isotropic Heisenberg model. *J. Soviet Math.* 24: 241–267, 1984.

[7] O. Foda and T. Miwa. Corner Transfer Matrices and Quantum Affine Algebras. *Int. J. Mod. Phys.*, A7 supplement 1A:279, 1992.

[8] I. B. Frenkel and N. Yu. Reshetikhin. Quantum affine algebras and holonomic difference equations. *Comm. Math. Phys.*, 146: 1-60, 1992.

[9] M. Jimbo. A q analogue of $U(gl(N+1))$, Hecke algebra, and the Yang-Baxter equation. *Lett. Math. Phys.*, 11: 247–252, 1986.

[10] M. Jimbo, H. Konno and T. Miwa Massless XXZ model and degeneration of the elliptic algebra $A_{q,p}(\widehat{sl}_2)$ In the proceedings for Ascona meetings, 1996.

[11] M. Jimbo, K. Miki, T. Miwa and A. Nakayashiki. Correlation functions of the XXZ model for $\Delta < -1$. *Phys. Lett.* A 168: 256–263, 1992.

[12] M. Jimbo and T. Miwa. *Algebraic Analysis of Solvable Lattice Models.* CBMS Regional Conference Series in Mathematics vol. 85, AMS, 1994.

[13] M. Jimbo and T. Miwa. Quantum KZ equation equation with $|q| = 1$ and correlation functions of the XXZ model in the gapless regime. *J. Phys.*, A 29: 2923-2958, 1996.

[14] M. Jimbo, T. Miwa, and A. Nakayashiki. Difference equations for the correlation functions of the eight-vertex model. *J. Phys.*, A26:2199–2209, 1993.

[15] J. D. Johnson, S. Krinsky and B. M. McCoy Vertical-arrow correlation length in the eight-vertex model and the low-lyingexcitations of the X-Y-Z Hamiltonian. *Phys. Rev.*, A8;2526–2547, 1973.

[16] V. G. Knizhnik and A. B. Zmolodchikov. Current algebraand Wess-Zumino models in two dimensions. *Nucl. Phys.*, B247: 83–103, 1984.

[17] S. Lukyanov. Free field representation for massive integrable models. *Comm. Math. Phys.*, 167: 183–226, 1995.

[18] M. Sato, T. Miwa and M. Jimbo . Studies on holonomic quantum fields I. *Proc. Japan Acad.*, 53 A: 6-10. 1977.

[19] F. A. Smirnov. *Form Factors in Completely Integrable Models in Quantum Field Theory.* World Scientific, Singapore, 1992.

[20] A. Tsuchiya and Y. Kanie. Vertex operators in conformal field theory on $\mathbf{P}^1$ and monodromy representation of braid group. *Advanced Studies in Pure Mathematics*, 16: 297–372, 1988.

[21] T. T. Wu, B. M. McCoy, C. Tracy and E. Barouch. Spin-spin correlation functions for the two-dimensional Ising model: Exact results in the scaling region. *Phys. Rev.*, B13: 316–374, 1976.

[22] A. B. Zamolodchikov and Al. B. Zamolodchikov. *Annal. Phys.*, 120: 253–291, 1979.

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