Dynamical Systems – Past and Present

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Introduction

It is a great honor for me to be invited to present a lecture at this International Congress of Mathematicians here in Berlin. This town (and its Academy) brings to mind a distinguished mathematical tradition in the last century, and I want to mention the names of Jacobi, Dirichlet and Weierstraß; they all contributed to the beginning of the topic of this lecture.

It was 94 years ago that the last ICM took place in Germany. This was in 1904 in Heidelberg (where, incidentally, the anniversary of Jacobi's 100th birthday was celebrated).

This long hiatus is, of course, not an accident, if one remembers that Germany was the scene of World War I , World War II and the Nazi terror. It was the time when Germany spread devastation and fear over the world. It was the time when – in the words of my friend Stefan Hildebrandt – Germany stepped out of the community of civilized countries. Even though these events lie more than half a century back I feel compelled to recall these terrible times since I myself lived through this dark period, having been born in this country.

During these times also science was trampled, and many eminent scientists were kicked out of their positions which caused irreparable damage. More than one third of the faculty of German universities was dismissed between 1933 and 1938! This reminds me of the Hilbert story, which I learned from my teacher Franz Rellich in Göttingen: When Hilbert – who was old and retired – was asked at a party by the newly appointed Nazi-minister of education: "Herr Geheimrat, how is mathematics in Göttingen, now that it has been freed of the Jewish influences" he replied: "Mathematics in Göttingen? That does not *exist* anymore!"

We must never forget this low point of German history – yet we also must put it behind us and look ahead. It is gratifying to see so many mathematicians who have come to Berlin to partake in this Congress. Let us celebrate this occasion as a new beginning at the end of this century.

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In this lecture I will present what I consider significant advances in the field of dynamical systems during the last 50 years. This field had a tremendous expansion in this time and my task would be impossible without severe restrictions. I will restrict myself to Hamiltonian systems – just as Birkhoff understood the concept in his book “Dynamical Systems” in 1927! Even there I will not attempt a survey, but rather select some topics, which in my view illustrate the dramatic changes that occurred during the past half century in this field. Clearly this lecture is not meant for experts, but for a wide audience.

As guide line I will use the stability problem for Hamiltonian systems, which still holds many fascinating problems. After some historical remarks I will discuss some applications of Kolmogorov's theorem on invariant tori (1954), then in Section 3 Xia's solution of the Painlevé problem, in Section 4 completely integrable systems, and, if time permits, in Section 5 the role of minimizers in the Aubry–Mather theory. Because of the limited time, I will omit many related topics, even some of great interest. The activity in symplectic geometry, which grew partially out of the Poincaré–Birkhoff fixed point theorem and led to most remarkable results will be discussed in other lectures at this meeting. Also ergodic theory and hyperbolic systems are active fields which I will not touch at all.

I Historical Remarks

a) The stability problem for Hamiltonian systems is an old unsolved problem which fascinated many mathematicians in the past. It was motivated by celestial mechanics and the stability problem for the planetary system. This is modeled by the N-body problem where N masspoints (of positive masses m_j) move in Euclidean space $\mathbf{R}^2$ or $\mathbf{R}^3$. One asks for bounded orbits avoiding collisions. More precisely, if r_{ij} is the distance between the i^{th} and j^{th} masspoints we require that along the orbits the expression

$$\Delta = max_{1 \leq i < j \leq N} \quad \{r_{ij}, \; \frac{1}{r_{ij}}\}$$

is bounded *for all times!*

The simplest solutions of this kind are the periodic solutions, represented by closed curves in the phase space. Therefore there was a great interest in establishing the existence of periodic solutions, and Poincaré devised perturbation methods as well as topological arguments for this purpose. However, the periodic solutions forms an exceptional set in phase space and therefore are of limited interest for the understanding of the dynamical behavior – unless one can prove their stability.

The question of stability requires not only finding single orbits with bounded Δ but an open set in phase space of such solutions, accounting for the imprecise knowledge of the initial values. In other words, one is interested in an open set in phase space in which Δ is bounded and to which the orbits are confined for all times!

In spite of the modern advances in this field this is still an *open problem*! It is conceivable that (for $N \geq 3$) the complement of all orbits which exist for all time and with Δ bounded forms a dense set in phase space. This would mean that by arbitrary small changes of the initial states one would find orbits which ultimately escape or end up in collisions!

In this connection it is interesting to read a statement of Charlier from the year 1907 about the question of the stability of the planetary system: "It still has to be considered as an open problem, although one would hardly be considered as a phantastic prophet if one expresses the conjecture that one does not have to wait for many decades for its solution." So much for prediction about open problems!

b) To proceed more constructively one replaced the quest for periodic solutions by that for quasi-periodic ones. These are given by generalized Fourier series of the form

$$(*) \qquad x(t) = Re(\sum_{j \in Z^d} c_j e^{i(j,\omega)t}), \quad \omega = (\omega_1, \omega_2, ..., \omega_d)$$

where the frequencies $\omega_1, \omega_2, ..., \omega_d$ are rationally independent real numbers. Perturbation theory of classical mechanics led to such series expansions for the solutions already in the last century. However, the convergence of these series became a notorious problem. The difficulty is due to the so-called small divisors – powers of terms of the form $(j, \omega), j \in \mathbf{Z}^d \backslash (0)$ – entering the coefficients. Since the frequencies are rationally independent these expressions are not zero, but they become arbitrarily small. This convergence problem – which would lead to the existence of quasi-periodic solutions – has been of central interest at the end of the last century, particularly to Dirichlet, Weierstraß (here in Berlin), Poincaré and others.

c) This problem *has* been solved half a century later! We turn to the fundamental theorem of Kolmogorov, which assures precisely the existence of such solutions, for Hamiltonian systems:

$$\dot{q}_k = H_{p_k}, \dot{p}_k = -H_{q_k}, \ (k = 1, 2, ...n)$$

or , combining q, p to a vector $x \in \Omega \subseteq \mathbf{R}^{2n}$ we write this in the form

$$\dot{x} = JH_x, \qquad J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}, \qquad x \in \Omega.$$

The corresponding flow will be denoted by φ^t.

A more geometrical formulation for quasi-periodic solutions is given by an embedding of a torus $T^d = \mathbf{R}^d / \mathbf{Z}^d$,

$$u : T^d \to \Omega$$

such that the "Kronecker flow" $\kappa^t : \theta \to \theta + \omega t$ on T^d is mapped into the flow φ^t restricted to the torus $u(T^d)$, i. e.

$$u \circ \kappa^t = \varphi^t \circ u.$$

Torus embedding

Then $u(T^d)$ is an invariant torus of the flow, and the orbits on it are indeed quasi-periodic. Moreover, by Kronecker's theorem, each of these orbits is dense on this torus; that means that this torus is a minimal set for the flow φ^t.

At the International Congress ICM 1954 in Amsterdam Kolmogorov announced the remarkable theorem: For a Hamiltonian system with Hamiltonian H of n degrees of freedom, close to an "integrable" one with Hamiltonian H_0 and compact energy surfaces, there exists a set of such invariant tori of dimension $d = n$. Moreover, they form a set of positive measure in phase space.

We will come to the concept of integrable systems in Section 4; here it is sufficient to know that these are Hamiltonian systems with sufficiently many integrals whose level sets are (if they are compact) invariant tori carrying quasi-periodic orbits. The theorem asserts that under small perturbations many of these quasi-periodic orbits persist.

Here is not the place to give the precise formulation of this basic result. But we want to point out some important consequences:

1) The union of these tori, generally, does not form an open set. Since it forms a set of positive measure, these tori are not exceptional!

2) The union of these tori is generally nowhere dense, so that a nearby orbit may not be bounded and may escape if $n \geq 3$, while for $n = 2$ the 2-dimensional tori can be used as boundaries of a domain on the three-dimensional energy surface, providing genuine stability results (some of which we will mention below).

Since the set of the constructed invariant tori have a relatively large measure one is led to a modified concept of stability: Instead of requiring that *all* orbits of a certain neighborhood are bounded for all times, one asks that most (in measure) orbits are bounded. This could be called "stability in measure", a concept which in applications is often sufficient, and which can be assured also for systems of three or more degrees of freedom.

3) This theorem provides a proof of the convergence of the series ($*$) provided the frequencies ω satisfy some Diophantine condition, thus answering the question of the last century.

The proof of Kolmogorov's theorem was published in 1963 by V. I. Arnold. The proof of a related theorem in a simpler situation, namely about the existence of invariant curves of area-preserving mappings in the plane had been published in 1962 by the speaker. It has become customary to refer to this technique as KAM theory.

For the plane three body problem the existence of a set of positive measure of quasi-periodic orbits has been established (Arnold 1963) but even for this problem in $\mathbb{R}^3$ one encounters difficulties which have not yet been overcome.

d) To return to the exciting history of this problem, we want to mention that Weierstraß had a keen interest in this topic. In the Wintersemester 1880/81 he taught a course "Über die Störungen in der Astronomie" hier in Berlin. In his correspondence with S. Kovalevskaya (1878) (Acta Math. 35, 30) he asserts that he found a series expansion for the solutions of the 3-body-problem, and tried, though in vain, to prove its convergence. He was aware of a remark made by Dirichlet to Kronecker in 1858 that he had found a method to approximate solutions of the N-body problem successively. Dirichlet died soon afterwards, and no written records were found. Later Weierstraß suggested this problem to Mittag–Leffler as a prize question. This prize, sponsored by the Swedish king, was awarded to Poincaré, although actually he did not solve this problem. But his famous prize-paper contained so many new ideas that there was unanimity in awarding the prize to him. This story can be read in many places now; here I wanted to point out the little known connection of this problem with the mathematicians in Berlin of the last century!

II Applications, Mappings

a) There are many applications of KAM theory to old problems of celestial mechanics. Most interesting are the stability results for systems of two degrees of freedom. We want to single out the stability proof of the periodic solutions in Hill's lunar theory.

These problems have more historical interest, and nowadays are at most of academic interest to astronomers. However, since many physical phenomena can be described by Hamiltonian systems it is not surprising that the stability theory has a multitude of other applications. I want to mention just two.

b) The early 1950's was the time of the construction of high energy accelerators in the USA, Europe at CERN and other places. In these machines charged particles are accelerated in a huge circular tube to tremendous velocities. This tube is brought to near vacuum state, so as to avoid any slow-down of the particles by the gas. For the successful working of the acceleration process one has to keep the (majority of the) particles from hitting the wall of the vacuum chamber for a long time. This is to be achieved by an appropriate magnetic field which allows to the particles to be trapped in the interior of the vacuum chamber. Since the motion of charged particles in a magnetic field is governed by Hamiltonian systems we are dealing with the stability in question.

At that time a new principle was introduced to improve this stability behavior, which led to the "Alternating Gradient Synchrotron" (AGS) which was built in Brookhaven, NY. This was a "true" application, since the stability behavior was one essential factor for the decision whether such a machine could be built.

Since the theory was not yet so well developed, one resorted to numerical experiments. If I may include some personal experiences: When I first visited the Courant Institute in 1953, there was a lot of activity in calculating the iterates of section maps to decide about the stability of the fixed points. This was done in connection with the AGS machine. These computations were carried out on a UNIVAC still using punch cards! Nowadays everybody can do the same thing on a PC using MATLAB in a few minutes. Let me illustrate to you what such computer pictures yielded: At least in the two-dimensional case the calculations showed much more optimistic results than could be true!

c) By a standard procedure one can reduce the study of a flow to that of a mapping, the so-called "Poincaré mapping". In particular one is interested in studying the stability of a fixed point of an area-preserving mapping, say φ, in the plane.

Poincaré section map

A necessary condition for stability under iteration of φ is that the linearized mapping is similar to a rotation. One speaks of an elliptic fixed point. In the following I show you some pictures of some 1000 iterates of points under a nonlinear area-preserving map. Near the fixed point the iterates of a point seem to organize themselves on a smooth curve, if one is close enough to the fixed point, indicating stability. The mapping chosen is a simple polynomial mapping, but the output is typical for such mappings.

The computations show that the iterates of a point fall on curves, surrounding the fixed point, making its stability evident. At some distance this curve patterns breaks up, leaving a certain stability region. The problem at the time was: Find a method to construct these curves and the stability region!

What one should have known even then was that there could not be a family of closed curves, and that the calculations were oversimplifying. If one uses more accurate computations and applies a microscope to them one will discover that between such curves there are regions with complicated dynamics (regions of instability in the terminology of G. D. Birkhoff).

Nevertheless, the set of invariant curves form a set of relatively large measure, as follows from KAM theory, so that stability of the fixed point is guaranteed. The orbit structure is amazingly complex for such simple mappings, as here for a polynomial map! Incidentally the region of instability contains "Mather sets" and complicated motions which nowadays would be called "chaotic". That these phenomena really occur for the typical area-preserving mapping, even in the case of real analytic mappings, has been established by Zehnder (1973) and in a sharper form by Genecand (1990).

Thus in this case the early calculations gave a misleading simplification of the situation. Still they were of great importance for stimulating this activity.

d) The Störmer problem.

Another large scale confinement region is known in the magnetic field of the earth. With the advent in 1957 of satellites it was soon discovered that the earth was surrounded by (two) belts of charged particles caused by its magnetic field. Since the beginning of the century it was known that such charged particles were present above the atmosphere and were responsible for the aurora borealis (and australis). It was Störmer (incidentally president of the ICM 1936 in Oslo) who made calculations of the orbits of these charged particles moving in the magnetic field of the earth, which he modelled as a magnetic dipole field. This is an interesting nonlinear Hamiltonian system.

The satellite measurements led to the discovery of two regions surrounding the earth, the so-called van Allan belts, in which the charged particles were trapped. It turns out that it is an example of a magnetic bottle to which the stability theory is applicable (M. Braun 1970).

It is interesting to realize the dimensions involved: For electrons the "cyclotron radius" is of the order of a few kilometers and the corresponding period of oscillation about one millionth of a second! The "bounce period" of travel from the north pole to the south pole and back is a fraction of a second.

In addition to the natural van Allan belts several artificial radiation belts have been made by the explosion of high-altitude nuclear bombs since 1958. Some of these so created belts had a life time up to several years – which shows the long stability of these experiments as well as the irresponsibility for carrying them out! Some 30 years ago these tests have been stopped.

Störmer problem

Van Allan belt

e) Hill's lunar problem.

In 1878 Hill developed a theory for the motion of the moon, which attracted great attention and impressed also Poincaré deeply. Later G. D. Birkhoff wrote: "A highly important chapter in theoretical dynamics began to unfold with the appearance in 1878 of G. H. Hill's researches on the lunar theory". He established the existence of 2 periodic solutions on the energy surface

$$\frac{1}{2}(\dot{u}^2+\dot{v}^2)-\frac{1}{r}-\frac{3}{2}u^2 = const < 0$$

of the model equation of the equations of the moon:

$$\begin{cases} \ddot{u}-2\dot{v} = -\dfrac{u}{r^3}+3u \\ \ddot{v}+2\dot{u} = -\dfrac{v}{r^3} \end{cases}$$

where $r=\sqrt{u^2+v^2}$. Nowadays this result has, of course, been derived in much simpler ways. But it took nearly a century till it was possible to prove the stability of Hill's orbits. This is an application of KAM theory in a rather singular situation. (see Conley, Kummer).

III Painlevé Problem

a) Besides the stable behavior we find, of course, unstable motions in Hamiltonian systems, in particular, in the N-body problem. Here we want to discuss a recently discovered, most extreme form of instability, namly a motion of the N-body problem in which the greatest mutual distance became unbounded in finite time! This is rather unexpected and hard to visualize, and seems to contradict (naive) energy considerations!

b) Actually this is related to an old problem raised by Painlevé in his lectures on celestial mechanics in 1895. (Incidentally, later in 1904, Painlevé was one of the four plenary speakers at the ICM in Heidelberg). What led to the quest for such strange solutions? Originally Painlevé was interested in the study of *all* possible singularities of the solutions of the N-body. It is obvious that collisions of two or more masspoints give rise to singularities, the so-called "collision singularities". They can be characterized by the property that the positions of the masspoint approach a definite position in configuration space. Such singularities, especially double and triple collisions have been studied extensively (Levi–Civita, C. L. Siegel et al).

Painlevé asked whether also other noncollision singularities could possibly exist, and the title of this Section refers to this question. Obviously they do not exist for the Kepler problem, and it was known to Painlevé that also for the three-body problem such singularities can not occur. So the problem referred to the N-body problem for $N \geq 4$ only.

To describe the situation briefly we denote by $q_j \in \mathbf{R}^3, j = 1, 2, .., N$ the position of the masspoints of mass m_j, and by $r_{ij} = |q_i - q_j| > 0$ their distances. The Newton potential is given by

$$-U = \sum_{i<j} \frac{m_i m_j}{r_{ij}}.$$

If at $t = T$ a singularity occurs then one has $U \to -\infty$, hence $\min r_{ij} \to 0$ as $t \to T - 0$. In 1908 von Zeipel discovered that a noncollision singularity can occur only if in addition also

$$\max_{i<j} r_{ij} \to \infty \quad \text{for} \quad t \to T;$$

as a matter of fact this property characterizes a noncollision singularity! Thus the quest for noncollision singularities is the same as that for the extreme form of instability we started with!

c) This makes the situation clearly very unlikely! Nevertheless, J. Xia was able to construct such a weird solution for the 5-body problem in $\mathbf{R}^3$. Here is a schematic view of the solution discovered by Jeff Xia in 1992: We consider two doublestars (P_1, P_2) and (Q_1, Q_2), both of equal masses, moving symmetrically on two planes perpendicularly to the z-axis. These approximately elliptical orbits are chosen so that the angular momentum is zero. Now we add a fifth masspoint, a "shuttle", traveling back and forth on the z-axis between these double stars.

Xia's model

Choosing the parameters appropriately one can achieve that the shuttle experiences a huge acceleration at each near-encounter (near triple collision!) so that the return times decrease so fast that they add up to a finite number.

d) Now the history of this solution is not so straight-forward; it came from quite independent investigations, based on work of Conley, McGehee and Mather some 25 years ago. It originated in the investigation of the neighborhood of triple collisions by Conley and McGehee around 1974, which revealed hyperbolic behavior near such a triple collision, which for an individual solution had already been observed by Siegel. Using this hyperbolic behavior Mather and McGehee succeeded (1974) in constructing a noncollision singularity even for the colinear four body problem! However, their solution had a shortcoming: It involved infintely many double collisions, which were unavoidable in the one-dimensional situation. Nevertheless, it was the first breakthrough for this problem. To find a solution free from this blemish took 18 more years! In 1992 Jeff Xia succeeded in constructing a noncollision solution for the five-body problem, thus solving the almost 100 year old problem! The proof is very intricate and subtle, but the underlying principle is to pass close to a sequence of triple collisions, and to use their instability at each step to reverse the shuttle with tremendous acceleration. It is an extra difficulty to verify that one can avoid collisions on the way.

An earlier attempt is due to Gerver (1984), who constructed another configuration for the five-body problem leading to non-collision singularity, but the details for a complete proof have not yet published.

Clearly this solution is not of any astronomical significance. Why do I present it: It shows, in one example, the progress gained from the study of hyperbolical dynamical systems which provided the understanding and the tools for the solution of this problem. It also reminds us of the efforts that go in the studies of singularities in partial differential equations, e. g. of the Navier-Stokes equation, provided they exist! One usually thinks of singularities as a local phenomenon, but even this (simple!) classical example of ordinary differential equations exhibits such complicated singularities of nonlocal type, whose existence was doubted for a long time.

IV Integrable Systems

a) All stability results for Hamiltonian systems – aside from trivial exceptions – depend on how well a given system can be approximated by an integrable one! Since these integrable systems are very rare this seems a hopeless proposition.

In the last 30 years, this topic has received immense attention from mathematicians and physicists alike. Its rapid development has affected many branches of mathematics, such as PDE, scattering theory, differential geometry, even algebraic geometry and others. Moreover, it has led to technical applications, as for example in transmission of optical pulses in fibers.

It is one of the fields which attained a certain popularity. Most scientists have heard the catch words “solitons”, “Korteweg–de Vries equations”.

This is all the more surprising as this subject is a very old one having its origin in the last century! At the time of Euler and Jacobi integrable system were of great interest since they could be solved "by quadrature", i. e. more or less explicitly which was of great importance, since existence theorems were not available then. Roughly speaking a Hamiltonian system of n degrees of freedom is called "completely integrable" if it possesses n integrals of the motion, whose mutual Poisson brackets vanish. In view of E. Noether's theorem this means that the systems admit an n-dimensional commutative group action (via symplectic transformations). In the compact case this would be a torus action. In short, these are particularly simple systems, and the structure of the flow can be described fairly easily. For 2 degrees of freedom rotational symmetric systems are completely integrable since they admit the angular momentum and the energy as integrals. In this case the "integrability" is obvious.

Now there are a number of integrable systems whose integrals and whose symmetries are not at all obvious and one speaks loosely of "hidden symmetries". Who would expect the geodesic flow on an ellipsoid with different axes to be integrable! This was discovered by Jacobi in 1838. He wrote to Bessel: "Yesterday I solved the equations for the geodesic lines on an ellipsoid with three different axes by quadrature. These are the simplest formulae of the world, Abelian integrals, which turn into elliptic integrals if two of the axes become equal". Today we would say that the solutions lie on a 2-dimensional torus, which is the real part of the Jacobian variety of a hyperelliptic curve of genus 2. They are with the exceptions of the geodesics passing through the focal points quasi-periodic. This statement can be generalized to ellipsoids of any dimension, which was done already in Jacobi's lectures. There are many other such examples, such as Euler's two fixed center problem, where one studies the motion of a masspoint under the Newton attraction of two fixed mass points, or the Kovalevskaya top.

Geodesics on an ellipsoid   Lift to the unit tangent bundle

The symmetry in these example certainly is "hidden". It was revealed only by analytical methods, namly by solving the Hamilton–Jacobi equation by separation of variables. Later this became a favorite topic for tricky exercises in mechanics.

No wonder that the topic became dormant.

The interest in integrable systems waned when Poincaré showed that, generically, Hamiltonian systems do not possess integrals besides the Hamiltonian itself. The field became obsolete.

b) The revival, or rediscovery, of this dormant field is most surprising. It is no exaggeration to say that this subject was initiated by a computer experiment! In 1965 Kruskal and Zabusky investigated a partial differential equation obtained by replacing the viscosity term in the Burgers equation by a third order derivative(dispersion)-term to see what it does to the shock solutions:

$$u_t + uu_x + u_{xxx} = 0$$

The equation was known in the literature as the Korteweg–de Vries equation, and it played a role in the theory of water waves, but the discoveries of Kruskal et al was absolutely new and totally unexpected. They found a strange phenomenon about the interaction of wave solutions:

The equation admits a family of wave solutions of different velocities, and the interaction between them appeared to be absolutely clean, so that after the interaction the waves reappeared in the same form and shape as before.[1]

In general, for other evolution equations, one would expect a scattering and a loss of the waves after the interaction. Kruskal coined the term "soliton" for these waves because they seemed to retain their identity.

After this observation, based on the numerical calculations, the search for an explanation of this extraordinary phenomenon began. I can not describe here the dramatic development that ensued. Here just some stages: The first guess was that the equation must possess more conserved quantities than the standard 3 (energy, mass and momentum), and after some efforts of a group some 10 integrals were found by laborious hand calculation. Ultimately one could extend these to an infinite number, and a method for solving these equations by inverse scattering methods was devised by Kruskal and his coworkers (1968).

It did not take long until C. Gardner discovered a Poisson bracket in function space, with respect to which the Korteweg–de Vries equation is Hamiltonian. Moreover the Poisson bracket of the integrals vanished, in short the KdV turned out to be the first example of an integrable Hamiltonian system of infinite degrees of freedom! This was the start of an intense activity. One was the discovery that the integrals, in fluid dynamics called conservation laws, could be viewed

[1] A video demonstrating this was presented at the lecture. It can be seen in the abstract of this manuscript in the electronic version of these Proceedings.

as eigenvalues of a simple operator, the one-dimensional Schrödinger operator $L = -D^2 + q(x)$, with $q = -u/6$, in which the solution of the KdV figures as the potential. In other words the flow defined by the KdV defines a deformation of this operator which leaves the spectrum unchanged. The observation of the iso-spectral deformations by P. D. Lax fruitfully led to many other discoveries, in particular, of several other integrable partial differential equations, as well as to further new insights.

c) By analogy with the finite-dimensional case one would expect that this partial differential equation can be solved explicitly! What are the (hidden) symmetries. Here are some highpoints which I want to single out:

i) If one subjects the KdV to the periodic boundary condition $u(x+1,t) = u(x,t)$, i. e. if one considers the solutions on the circle, then all the solutions are almost periodic in t. (McKean and Trubowitz 1976) This is a most unexpected property for a nonlinear partial differential equation. It is the reflection of the integrability of the equation. For the geodesics on an ellipsoid, for example, all solutions are quasi-periodic, with the exception of the orbits through the focal points. In the case of the KdV such exceptions do not exist! The proof is based on the fact that the isospectral manifolds are infinite-dimensional tori, which can be interpreted as the real part of the Jacobian variety of a Riemann surface (complex curve) of infinite genus, on which the flow is linear. This curve is obtained as follows. It has been known for a long time that the spectrum of the one-dimensional Schrödinger operator with periodic potential has a "band" spectrum, that is, it consists, in general, of infinitely many intervals clustering at $+\infty$. Now consider the double covering of the complex plane and glue the 2 sheets along these intervals, in the customary fashion. This gives the desired complex hyperelliptic curve whose genus is equal to the number of intervals — if it is finite.

ii) Inverse spectral theory: In spectral theory it is an old question to construct the potential of an operator from the spectrum, which is the inverse of the usual question of spectral theory. The answer is usually too hard, or the solution not unique. But the question for all the potentials having a "finite gap" potential has been answered by S. Novikov and his coworkers in 1976:

Given a set of finitely many disjoint intervals, one of which is half-infinite stretching to $+\infty$, find all potentials having these intervals as spectrum. The answer is given in terms of the hyperelliptic functions on the above mentioned hyperelliptic curve. In case of a single (half-infinite) intervall the potential is a constant, for 2 intervals (genus 1) the potentials is an elliptic function (Lamé equation) etc.

d) It is another startling fact that the soliton theory has down-to-earth applications to communication theory. Here the underlying equation is not the KdV but the nonlinear Schrödinger equation

$$iu_t + u_{xx} + |u|^2\, u = 0$$

which also was recognized as an integrable system (Zakharov, Shabat 1971) using ideas of P. Lax. This equation also possesses "solitons" with extraordinary stability properties. This fact was used by physicists (Hazegewa (1973), Mollenauer (1980)) for signal transmission in optical fibres. Here the solitions describe the envelope of a wave train.

This approach has been used with success to transmit ultrashort pulses over large distance (∼10'000 km) with less loss than one encounters with standard methods.

e) It is impossible to even touch on the many ramifications that have evolved from the study of integrable system. The question why iso-spectral deformation gives rise to systems respecting a symplectic form has led to interesting applications of Kac–Moody algebras. The old Schottky problem asks for the characterization of those Abelian tori which are Jacobian varieties of an algebraic curve. In 1986 Shiota found an answer in terms of the solutions of the "KP-equation", a partial differential equation, generalizing the Korteweg-de Vries equation, thus connecting this problem of algebraic geometry with integrable partial differential equations.

On the side of analysis the question has been raised and answered whether the KAM technique can be applied to partial differential equations, e. g. can one establish the existence of quasi-periodic solutions for the perturbed KdV: $u_t + uu_x + u_{xxx} = \epsilon(g(x,u))_x$ where g is a real analytic function, periodic in x. For small values one finds indeed quasi-periodic solutions of this equation. The necessary theory is highly technical. It has been developed by Kuksin, and subsequently by Pöschel, Craig and Wayne and Bourgain. However, one has to point that in this case the solutions so obtained form a "small" subset in the phase space.

V Breakdown of Stability

a) What happens when the perturbation from the integrable system gets larger and larger? It turns out that the structure of the invariant tori and the stability of the system breaks down! However, the invariant tori degenerate into some invariant sets, generally Cantor sets, the so-called Aubry–Mather sets. This is the object of a theory discovered independently by the physicist Aubry and by John Mather. They were motivated by entirely different questions: Aubry by stable states in a simple model for one-dimensional crystals in solid state physics, while Mather studied invariant sets for area-preserving mappings. Both theories were ultimately recognized to be the same. This (Aubry–Mather) theory (1982) brought a significant advance to dynamical systems, but is also related to an interesting development in differential geometry.

The underlying idea of this theory can be illustrated with the simple model problem of the geodesic flow on a two-dimensional torus $T^2 = \mathbf{R}^2/\mathbf{Z}^2$. We give a $\mathbf{Z}^2$-periodic metric, say g on $\mathbf{R}^2$. The corresponding geodesic flow gives rise to a Hamiltonian system on the cotangent bundle $T^*(T^2)$ and the unit-cotangent bundle $\mathcal{E} = T^*(T_1^2)$ as three-dimensional energy surface. For the flat metric, denoted by g_0, all geodesics are straight lines, and a family of parallel lines lift to an invariant torus on $\mathcal{E}$. According to the KAM theory many of these tori persist under perturbation, namly those for which the slope is an irrational number which is badly approximable by rationals. In particular, the orbits between two such tori are trapped, and the flow on $\mathcal{E}$ is certainly not ergodic.

On the other hand, about 10 years ago V. Donnay found smooth metrics, say g_1 on the 2-torus for which the geodesic flow is ergodic. Consequently the structure of invariant tori must break down if we deform g_0 to g_1.

b) To understand the situation we project the flow on such an invariant torus into the configuration space, i. e. $\mathbf{R}^2$. One finds that the orbits on such an invariant torus project into a $\mathbf{Z}^2$-invariant foliation made up of geodesics.

In the terminology of the Calculus of Variations this is a "field of extremals". It is classical result, going back to Weierstraß, that the geodesics belonging to an extremal field are "minimizers", i. e. any segment of such a geodesic minimizes the length between its endpoints. In other words, all orbits belonging to an invariant torus project into minimizers. This is — or can be taken as — the clue to the Aubry-Mather theory. The goal then is to study the minimizers among all geodesics. This is generally a strict subset of the set of all geodesics. As a matter of fact, by a classical theorem of E. Hopf, a metric for which all geodesics are minimizers, is necessarily flat ($K = 0$). The minimizers on a torus had already been studied by Hedlund, after earlier work by his teacher M. Morse (1924), who called them "geodesics of class A".

Projection of an invariant torus into a minimal foliation

c) These minimizers (or geodesics of class A) intersect each other at most once, as do straight lines. Moreover, they have the crucial property that they are trapped in a strip bounded by two straight lines whose distance D depends only on the metric, not on the individual minimizer.

"Trapping" of minimizers

In particular, one can associate with any minimizer a direction, say $\theta(\mathrm{mod} 2\pi)$. Moreover, for each value of $\alpha = \theta/2\pi(\mathrm{mod} 2\pi)$ the set of these minimizers, $\mathcal{M}_\alpha$ can be shown to be non-empty. Now at least if α is irrational one can put together the corresponding extremal field from these not intersecting minimizers of $\mathcal{M}_\alpha$ to obtain a minimial foliation, provided these minimizers are dense on the torus, and the lift of this foliation recovers the invariant torus. However, it is possible, as simple examples of "bumpy" metrics show, that these minimizers may not be dense, if projects on the torus. In that case these minimizers provide only a "lamination", covering only a part of the torus.

Bumpy metric (after Bangart)

In this case these recurrent elements of $\mathcal{M}_\alpha$ lift to an invariant set, in fact, a unique invariant set, which turns out to be a minimal set associated with any value of α. This is the Mather set in question, to which the invariant torus deteriorate under deformation of the metric.

I want to show with this indication, that Mather sets are obtained in a very natural way by selecting out of all orbits only those which are minimizers (and not just stationary).

d) This selection principle of minimizers out of the class of all solutions of the Euler equations is a very useful priniciple, also for elliptic partial differential equation derived from a variational problem, satisfying a Legendre condition. It has proved useful in differential geometry. It is possible to extend the Mather theory to minimal foliations on a higher dimensional torus, where the orbits are replaced by minimal surfaces of codimension 1. It is even more interesting to study such minimizers of codimension 1 on manifolds where the reference metric has negative curvature. The crucial trapping property mentioned above holds also in this situation and leads to most interesting new results. This development is due to Gromov, who introduced the term "trapping". I can not and need not enter into this field since it has been presented in an ICM 1994 lecture by V. Bangert. Since then he and Urs Lang have obtained very general results about the so-called asymptotic Plateau problem.

VI Concluding Remarks

I hope to have shown to you that the subject of dynamical systems holds a vast number of connections to other fields — even with the restrictions I imposed on myself.

Most striking to me is the development of integrable systems (some 30 years ago) which did not grow out of any given problem, but out of a phenomenon which was discovered by numerical experiments in a problem of fluid dynamics. Intelligent studies and deep insight opened up to a novel field impinging on differential geometry, algebraic geometry and mathematical physics, including applications in communication of fiber optics. This illustrates that one is ill-advised to try to direct or predict the development of mathematics. In a time of dangerous specialization we should feel free to use all tools available to us, and use them with proper taste. To me, it seems idle to argue whether to prefer solving of challenging problems, building abstract structures, or working on applications. Rather we should keep an open mind when we approach new problems, and not forget the unity of mathematics. In the words of Birkhoff: "It is fortunate that the world of mathematics is as large as it is".

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