Geometric Physics

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Abstract. Over the past two decades there has been growing interaction between theoretical physics and pure mathematics. Many of these connections have led to profound improvement in our understanding of physics as well as of mathematics. The aim of my talk is to give a non-technical review of some of these developments connected with string theory. The central phenomenon in many of these links involves the notion of *duality*, which in some sense is a non-linear infinite dimensional generalization of the Fourier transform. It suggests that two physical systems with completely different looking properties are nevertheless isomorphic if one takes into account "quantum geometry" on both sides. For many questions one side is simple (quantum geometry is isomorphic to classical one) and the other is hard (quantum geometry deforms the classical one). The equivalence of the systems gives rise to a rich set of mathematical identities. One of the best known examples of duality is known as "mirror symmetry" which relates topologically distinct pairs of Calabi-Yau manifolds and has applications in enumerative geometry. Other examples involve highly non-trivial "S-dualities" which among other things have found application to the study of smooth four manifold invariants. There have also been applications to questions of quantum gravity. In particular certain properties (the area of the horizon) of black hole solutions to Einstein equations have been related to growth of the cohomology of the moduli space of certain minimal submanifolds in a Calabi-Yau threefold. A central theme in applications of dualities is a physical interpretation of singularities of manifolds. The most well known example is the $A-D-E$ singularities of the $K3$ manifold which lead to $A-D-E$ gauge symmetry in the physical setup. The geometry of contracting cycles is a key ingredient in the physical interpretation of singularities. More generally, singularities of manifolds encode universality classes of quantum field theories. This leads not only to a deeper understanding of the singularities of manifolds but can also be used to "geometrically engineer" new quantum field theories for physics.

1. Introduction

The history of physics and mathematics is greatly interconnected. Sometimes new mathematics gets developed in connection with understanding physical questions (for example the development of Calculus was not independent of the questions raised by classical mechanics). Sometimes new physics gets developed from known mathematics (for example general theory of relativity found its natural setting in the context of Riemannian geometry). I believe we are now witnessing perhaps an unprecedented depth in this interaction between the two disciplines. It is thus a great pleasure to explain some of the recent progress which has been made in our understanding of quantum field theories, string theory and quantum gravity to a mathematical audience. The works I will be explaining here is a result of the work of many physicists and mathematicians.[1]

Many of the key elements in these recent advances have a deep mathematical content. These involve new predictions for answers to some very difficult mathematical questions as well as new interpretations of some old mathematical results. It also sometimes hints at the existence of whole new branches of mathematics which does not exist yet.

In preparing this talk, I have had to make some choices. First of all I have had to decide which topics to cover and which ones to leave out. This has been very difficult because there are many interesting interaction points between theoretical physics and pure mathematics today, and unfortunately I only have a very limited time here. My choice was motivated by the degree of my familiarity with the subject as well as by attempts at trying to give a unified exposition of the seemingly unrelated topics. Secondly I have had to assume a certain level of familiarity of this mathematical audience with physics. This is also unavoidable, if we are to make any connection to interesting new developments. However, I have tried to make this assumption in the weakest possible sense. Thirdly I have chosen a list of questions which I find interesting for physics which I hope the mathematicians will help us solve.

The organization of my talk is as follows: In section 2 I will describe the basic notion of *duality* which is the key notion in recent advances. In sections 3-5 I give examples of dualities. Section 3 is devoted to a review of what mirror symmetry is. Section 4 explains the physical interpretation of singularities of certain manifolds. Section 5 is devoted to the notion of black hole entropy and what duality predicts about that. Section 6 is devoted to a list of questions which I raise in connection with the topics discussed.

2. What is meant by Duality?

I will try to define a very general notion of duality first, a priori nothing to do with physics, and then try to be a little more particular in what it means in the physical context.

[1] I will not make any attempts to present a complete list of references to all the relevant literature, though some illustrative references, in the spirit of the presentation here will be given.

Suppose we have two classes of objects. Moreover suppose these two objects satisfy identical properties. Then in a mathematical context they usually will be called *isomorphic*. Very often this is a trivial isomorphism. For example if a property of geometry on a 2 dimensional plane is true, it will also be true for the mirror reflection of the same geometry (Fig. 1).

Fig.1*:* Reflection on the plane is an example of a "trivial" duality.

However there are times where the fact that the objects and operations are isomorphic is less trivial, because the maps between these two classes of objects is not so trivial. As an example, suppose we wish to solve a linear differential equation of the form

$$F = \sum_k a_k \frac{d^k}{dx^k} \psi(x) = 0$$

with constant coefficients a_k. Consider instead the *polynomial* equation in one variable p:

$$G = \sum a_k (ip)^k = 0$$

Apriori the two problems seem unrelated. In fact the second problem on the face of it sounds much simpler. However, as is well known the two problems are related by Fourier transform, and the general solution to the first problem is given by

$$\psi(x) = \int dp \phi(p) \exp(ipx) \delta(G(p))$$

This isomorphism of functions in x and functions in p with the map between them being Fourier transform allows us to solve a 'hard' problem in the x space setup in terms of an easy problem in the p space setup. Isomorphisms of this type which are *non-trivial* we will call *dualities*. As it is clear from this example dualities will be very useful in solving problems. Dualities very often transform a difficult problem in one setup to an easy problem in the other. In some sense very often the very act of 'solving' a non-trivial problem is finding the right 'dual' viewpoint.

Now I come to specializing this idea in the context of a physical system. Consider a physical system Q (which I will not attempt to define). And suppose this system depends on a number of parameters $[\lambda_i]$. Collectively we denote the space of the parameters λ_i by $\mathcal{M}$ which is usually called the moduli space of the coupling constants of the theory. The parameters λ_i could for example define the geometry of the space the particles propagate in, the charges and masses of particles, etc. Among these parameters there is a parameter λ_0 which controls how close the system is to being a classical system (the analog of what we call $\hbar$ in quantum mechanics). For λ_0 near zero we have a classical system and for $\lambda_0 \geq 1$ quantum effects typically dominate the description of the physical system. Typically physical systems have many observables which we could measure. Let us denote the observables by $\mathcal{O}_\alpha$. Then we would be interested in their correlation functions which we denote by[2]

$$\langle \mathcal{O}_{\alpha_1} ... \mathcal{O}_{\alpha_n} \rangle = f_{\alpha_1 ... \alpha_n}(\lambda_i)$$

Note that the correlation functions will depend on the parameters defining Q. The totality of such observables and their correlation functions determine a physical system. Two physical systems $Q[\mathcal{M}, \mathcal{O}_\alpha]$,$\tilde{Q}[\tilde{\mathcal{M}}, \tilde{\mathcal{O}}_\alpha]$ are dual to one another if there is an isomorphism between $\mathcal{M}$ and $\tilde{\mathcal{M}}$ and $\mathcal{O} \leftrightarrow \tilde{\mathcal{O}}$ respecting all the correlation functions. Sometimes this isomorphism is trivial and in some cases it is not. We are interested in the cases where this isomorphism is non-trivial. In such cases typically what happens is that a parameter which controls quantum corrections λ_0 on one side gets transformed to a parameter $\tilde{\lambda}_k$ with $k \neq 0$ describing some classical aspects of the dual side. This in particular implies that quantum corrections on one side has the interpretation on the dual side as to how correlations vary with some classical concept such as geometry. This allows one to solve difficult questions involved in quantum corrections in one theory in terms of simple geometrical concepts on the dual theory. This is the power of duality in the physical setup. Mathematics parallels the physics in that it turns out that the mathematical questions involved in computing quantum corrections in certain cases is also very difficult and the questions involved on the dual side are mathematically simple. Thus non-trivial duality statements often lead to methods of solving certain difficult mathematical problems.

One should note, however, that very rarely can one actually *prove* (even in the physics sense of this word) that two given physical systems are dual to one another. Often the existence of dualities between two systems is guessed at based on some physical consistency arguments. Testing many non-trivial consequences of duality conjectures leads us to believe in their validity. In fact we have observed that duality occurs very generically, for reasons we do not fully understand. This lack of deep understanding of duality is not unrelated to the fact that it leads to solutions of otherwise very difficult problems. At the mathematical level, evidence for duality conjectures amounts to checking validity of proposed solutions to certain difficult mathematical problems.

[2] One could attempt to define a physical system by an infinite dimensional bundle over $\mathcal{M}$ where the fiber space is identified with the space of observables $\mathcal{O}_\alpha$, together with a rank n multi-linear map from the fiber to C, for each n, satisfying some compatibility conditions.

In the next three sections I will consider examples of duality and some of its mathematical consequences. In section 3 we will start with the best understood duality known as mirror symmetry, which relates string theory on one target manifold with another. In section 4 we discuss how singularities of the geometry get related to gauge bundles for the dual theory. In section 5 we discuss a dual description of black hole geometry which is intimately related to properties of minimal submanifolds in Calabi-Yau manifolds.

3. Mirror Symmetry

String theory, which is the only known consistent framework for a quantum theory of gravity, involves the study of quantum properties of one dimensional extended objects. The spacetime picture corresponds to a two dimensional Riemann surface Σ mapped to a target spacetime Riemannian manifold M. The sliced Riemann surfaces give the picture of strings propagating in time (Fig. 2).

FIG.2: Strings propagating in spacetime span a Riemann surface known as the worldsheet.

In string theory we are instructed to "sum" over all such maps

$$\phi : \qquad \Sigma \to M$$

weighted with exp(−S(ϕ)) where $S(\phi)$ denotes the integral

$$S(\phi) = \int_\Sigma |d\phi|^2$$

where we use the metric on M to define $|d\phi|^2$. (For superstrings which is the case of most interest, there are also some fermionic fields, which I suppress in this discussion.)

One of the most amazing properties of string theory is that strings moving on one manifold may behave identically with strings moving on a different manifold. Any pair of manifolds M_1 and M_2 which behave in this way are called mirror pairs. Of course this would be a trivial duality if M_1 and M_2 are isomorphic Riemannian manifolds. The interesting dualities arise when M_1 and M_2 are distinct

Riemannian manifolds. In some cases M_1 and M_2 are topologically the same, but in some cases they are distinct even topologically. In such cases the equivalence of the two manifolds for string theory will be only a statement about correlation functions after summing over *all* maps ϕ. The act of summing over all maps ϕ is what we mean by the quantum theory. So only in the quantum theory, i.e. after summing over all ϕ the two computations would be related (i.e. we should not try to compare individual maps). The parameter controlling the significance of quantum corrections, for a fixed genus surface Σ, is the volume of M, $V(M)$. In particular, the parameter we called λ_0 in the previous discussion in this case is $\lambda_0 = 1/V(M)$ (and thus in the large volume limit the quantum corrections are suppressed).

The simplest example of mirror symmetry corresponds to choosing M_1 to be a circle of circumference L and M_2 to be a circle of circumference $1/L$. This is a case of mirror symmetry which can be rigorously proven (see [1] for a review). However here we will just illustrate why such a statement is not unreasonable.

This statement would definitely be unreasonable for point particle theories: If we consider a particle in a circle of size L, the momentum states are quantized as the allowed wave functions

$$\psi_n(x) = \exp(2\pi i n x/L)$$

compatible with the invariance under $x \to x + L$ gives the spectrum of allowed momenta (which for massless particles is the same as energy) to be n/L, where $n \in Z$. If we consider the circle of circumference $1/L$ the allowed energies are now nL. Thus the energy spectrum of the two theories do not match. The story changes dramatically for strings: We will still have the same excitations as in the point particle case, after all the string mapped to a point looks like a point particle. However we have in addition other states corresponding to winding states of the string around the circle. Consider the first circle of circumference L and assume a string wraps around it m times, then its energy is mL (I am working in units where the string tension is one). Now the full spectrum of momentum and winding states does have $L \to 1/L$ symmetry where in the process momentum states get exchanged with winding states (Fig. 3).

Fig.3: Momentum modes, with energy n/L get exchanged with winding modes with energy mL under mirror symmetry $L \to 1/L$.

There is one context in which a similar duality is already well known mathematically: Consider a $U(1)$ bundle on a circle. Then the choice of the bundle (i.e. the choice of the holonomy of $U(1)$ around the circle) is equivalent to the choice of a point on the dual circle. This also turns out to have a very important physical analog [2]. If we consider open strings, in addition to closed strings, we would be considering Riemann surfaces with boundaries. In such a case in addition to specifying the target geometry M where the closed strings are mapped to, we have to specify where the boundaries are mapped to. In general they could map to some subspaces of M of various dimensions p. Such a p-dimensional subspace of M is called a $p - brane$ or $Dp - brane$ (D signifying the fact that the maps from the Riemann surface have *Dirichlet* conditions in codimension p, and "brane" generalizing the terminology of membranes which are 2-branes, to the higher dimensional objects). Moreover it turns out that a Dp-brane will carry a $U(1)$ gauge field and so can be viewed as a sheaf in M. Physically a Dp-brane corresponds to some charged matter localized in a p-dimensional subspace of M. From the string viewpoint D-branes are regions where an open string can end on (Fig. 4).

Fig.4: A Dp brane is a subspace of the target manifold M where a string can end on.

Returning to the case of a circle, if we consider a $D1$ brane which includes the entire circle of circumference L, we can ask what happens under mirror symmetry to the D-brane. The answer is that it gets transformed to a $D0$ brane on the mirror. This is in accord with the mathematical fact mentioned before (where the holonomy of a $U(1)$ bundle gets transformed to the choice of a point on the dual circle). This has also a natural generalization to the case where we consider N D1 branes wrapping the S^1 which in physics leads to a $U(N)$ bundle on S^1 and choosing a flat $U(N)$ connection on S^1 amounts to choosing N points on the dual circle, i.e. it is transformed to N $D0$ branes on the mirror.

It is natural to ask how mirror symmetry extends in cases where the target manifold is more complicated than S^1. One simple example consists of taking a d-dimensional torus $T^d = (S^1)^d$ and doing inversion on each of the S^1's. The action of this on the Dp branes, viewed as subspaces $T^p \subset T^d$ is also clear where they get transformed to a dual $T^{*d-p} \subset T^{*d}$. However for more interesting examples we need the following idea [3].

1. *The Adiabatic Principle*

Consider a family of flat d-dimensional tori T^d *varying slowly*, i.e. *adiabatically* over some base space B. Consider the total space M_1 over B with T^d as the fiber. Consider another space consisting of the same base space B, where over each point we replace the fiber T^d with the mirror torus where all lengths are inverted. Call the total space M_2. Then it is natural to believe that the spaces M_1 and M_2 are mirror to one another. However the interesting examples arise when the assumption of adiabaticity is violated over some subspaces of B. For example the T^d may degenerate over some loci. If the category of objects we are dealing with is sufficiently nice one may hope that the mirror property will continue to hold. One nice category[4] seems to be when the base B is also d-dimensional and the total space is a Calabi-Yau d-fold (a Kähler manifold of complex dimension d whose bundle of holomorphic d-forms is trivial) where the fibers T^d are viewed as Lagrangian submanifolds relative to the Kähler form. In fact the non-trivial data specifying the geometry of the Calabi-Yau is precisely how the degeneration of T^d over B takes place. This construction corresponds to describing a hypersurface in a toric variety, in a degenerate limit. In a singular limit the Calabi-Yau may be viewed as a T^d fiber space over the base being a boundary of some simplex (in the sense of toric geometry), where T^d degenerates to T^k over $d-k$ dimensional subspaces of B. The data defining the mirror, after suitably rescaling the metric on B looks like the dual geometry where the regions where the T^d shrinks to T^k is replaced by the dual k-dimensional subspaces where the $T^{d-k} \subset T^d$ shrinks and the dual survives, this being consistent with the small/large radius exchange (Fig. 5). This gives what is known as Batyrev's construction of mirror pairs using the toric description.

[3] The presentation here of the mirror symmetry for more complicated target spaces does not follow the historical order of its discovery. Mirror symmetry was first conjectured to exist for Calabi-Yau manifolds in [3][4], with the concrete examples being found in [5] followed by a concrete application to counting holomorphic curves in [6]. The construction of mirror pairs was systematized by [7]. The presentation here follows the approach in [8] developed further in [9] which explains the construction of [7] from this viewpoint.

[4] There may well be other categories, such as the category of manifolds of $Sp(n)$, $Spin(7)$ or G_2 holonomy.

Fig.5: An application of inversion duality of tori when tori are varying leads to an explanation of mirror symmetry in more complicated examples.

2. Kähler-Complex Deformation Exchange

It would be nice to examine some of the consequences of the existence of mirror geometries. To get a feeling for this it is useful to start at the level of S^1 fibered trivially over $B = S^1$. This is a simple case, as a constant fibration admits the flat metric. Let R_f, R_b denote the radii of the fiber and a section respectively. Note that the complex structure (shape) of the torus is determined by

$$C = R_b/R_f$$

and its Kähler class (size) is determined by

$$K = R_b R_f$$

Now if we do mirror transform on the fiber S^1 it again leads to a torus. However since $R_f \to 1/R_f$ but $R_b \to R_b$ the parameters controlling the complex and Kähler deformations get exchanged:

$$C \leftrightarrow K \qquad under \quad mirror \quad transform$$

This turns out to be the general feature of mirror symmetry for Calabi-Yau manifolds, and the Kähler and complex structures always get exchanged. In the case of Calabi-Yau manifold of complex dimension d the number of complex moduli is determined by $h^{1,d-1}$ (where $h^{p,q}$ denotes the dimension of the cohomology of p-holomorphic and q anti-holomorphic forms). Thus if M and W are mirror Calabi-Yau manifolds we learn in particular that

$$h^{1,1}(M) = h^{1,d-1}(W) \qquad h^{1,d-1}(M) = h^{1,1}(W).$$

This in particular implies that the topology of the manifold and the mirror will in general be very different. In fact it turns out that $h^{p,q}(M) = h^{p,d-q}(W)$ for all p, q.

Moreover, as mentioned before, the parameter controlling quantum corrections is the Kähler class of the Calabi-Yau, which gets transformed under mirror transform to complex deformation parameter of the mirror. Thus the question of quantum corrections for one manifold get transformed to the question involving the variation of complex structure on the other, which is classical. This leads to some very non-trivial implications of mirror symmetry.

The most concrete prediction this leads to is to the question of counting the "number" of holomorphic curves mapped from a Riemann surface of genus g to the threefold. For example the intersection numbers of cycles in the Calabi-Yau receives a quantum correction coming from holomorphic curves (recall this is natural from the string theory viewpoint, where the worldsheet is a Riemann surface) (Fig. 6). This "quantum intersection theory" for triple intersections allows, in addition to the classical intersection, the possibility that the three cycles meet a holomorphic curve weighted by the quantum deformation parameter $q = e^{-A}$ where A is the area of the holomorphic curve[5].

Fig.6: Quantum intersection of three cycles A, B, C in addition to the classical piece has corrections where A, B, C meet on a holomorphic rational curve.

This very difficult mathematical problem, i.e. counting holmorphic curves in Calabi-Yau manifolds, gets transformed on the mirror to a question involving the variation of Hodge structures (in this case it is the study of how the middle dimensional $H^{p,d-p}$ cohomology elements vary as we vary the complex structure on the mirror). This is a well studied mathematical subject [6]. The genus 0 version of the prediction has been confirmed recently [11][12]. The higher genus version

[5] The fact that classical cohomology ring is deformed by instantons and gives rise to a quantum cohomology ring was pointed out in [3]. The precise definition of this deformation was given in [10].

[6] To be precise, the counting of genus 0 curves gets transformed to this question. The higher genus version gets transformed to a quantum version of variation of Hodge structure known as *Kodaira-Spencer theory of gravity* which is only slightly more complicated.

[13] has not been proven yet (except in some special cases), but there is little doubt that it is generally valid.

3. Extension to Bundles

It is clear from the discussion of D-branes in the context of circles that we can extend mirror symmetry to Calabi-Yau manifolds with bundles. In particular let $c \in \oplus_p H^{p,p}(M)$ denote the chern class of a holomorphic vector bundle on Calabi-Yau manifold M. Represent this by a collection of Poincaré dual holomorphic cycles. Consider D-branes wrapped over them. This is a D-brane made up of various even dimensional branes. Each (p,p) cycle projects to a p real dimensional subspace of B with typical fiber a p dimensional subtorus. On the mirror, the p dimensional subspace of T^d gets transformed to the dual torus T^{d-p}. Thus on the mirror Calabi-Yau, the whole bundle representated by the collection of D-branes is mirror to a submanifold C of real dimension d.[7] The condition that the original bundle be holomorphic translates to the condition that C is Lagrangian relative to the Kähler form on the mirror. If we further impose that the original bundle be stable, this translates to the cycle C being of minimal area. This extension of mirror symmetry to include bundles conjectured in [18] (see also related works [19][20][21][22]) has only recently been made and checks on its prediction are underway. It makes certain predictions for the enumerative geometry of holomorphic maps from Riemann surfaces with boundaries being mapped holomorphically to Calabi-Yau, with boundaries being mapped to Lagrangian cycles on it.[8]. For example the Ray-Singer Torsion associated to the bundle V is transformed to counting holomorphic maps from the annulus to the Calabi-Yau whose boundary is on the mirror minimal cycle.

4. Physical Interpretation of Geometric Singularities

One of the remarkable aspects of string theory is the existence of a few different types of consistent theories (5 in 10 dimensions and one in 11 dimensions) which are dual to one another. This is known as S-duality. For example, Type IIA strings in a 10 dimensional space having a $K3$ fibration ($K3$ being a Calabi-Yau manifold of complex dimension 2) is dual to heterotic strings in a space admitting a T^4 fibration. This is very surprising because in particular the two string theories and the two target spaces look very different. Moreover on the heterotic side one has to choose flat bundles of rank 16. Moreover as we change the size of the T^4 and the choice of the flat bundle (and some choice of a constant field belonging to

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[7] This leads to a new application of mirror symmetry: For example consider a rational elliptic surface inside a 3-fold. Then the study of rank N stable bundles on it gets transformed to the study of spectral curves on the dual rational elliptic surface (by viewing the bundle as D4 brane wrapped the rational elliptic surface and doing mirror symmetry along T^2 fiber)[14][15]. The Euler class of the moduli space can be computed using mirror symmetry techniques [16] (this prediction has been recently confirmed for the rank 2 case [17]).

[8] For this to make sense beyond Disc one should restrict to the category of stable bundles on one side and minimal Lagrangian submanifolds on the mirror.

$H^2(T^4)$) one can get various different gauge groups. For example one can obtain $SU(N), SO(2N)$ (for small enough N) and $E_{6,7,8}$. The question is how all this is reflected on the $K3$ geometry? It is well known that $K3$ can have singularities corresponding to contracting 2 spheres. Moreover the intersection matrix of the contracting 2 spheres is given by the Cartan matrix of the A-D-E groups. The appearance of the Dynkin structure for the $K3$ singularities appears mathematically as purely "accidental". However this accident gets explained in this duality context: One identifies the singular $K3$ geometries with A-D-E singularities with the points on the heterotic side with enhanced A-D-E gauge symmetry. The physical explanation of enhanced symmetries on the $K3$ side has to do with the existence of D2 branes, which can wrap around the contracting 2-cycles, and give rise to massless particles. The wrapped D2 branes encode in a beautiful way the connection of the bundle anticipated from the heterotic dual (Fig. 7). Thus the non-abelian enhancement of gauge symmetry on heterotic side is transformed to appearance of geometric singularities on the type IIA side.

FIG.7: A wrapped D2-brane over a sphere of blown up A-D-E- singularity is the origin of gauge symmetry enhancement when the spheres shrink.

Similar considerations suggest interesting physical interpretations whenever one has geometric singularities. For example if one considers a Calabi-Yau 3-fold, one has sometimes contracting S^3's. In this context there are two ways to get rid of the singularity. One either deforms the polynomial equations defining the manifold (which effectively gives a finite size to the contracting S^3's) or replaces the singular point by a higher dimensional geometry (in this case S^2's) which is known as blowing up the singularities, changing the geometry of the 3-fold in the process. The singular manifold can thus be viewed as belonging to two distinct families of Calabi-Yau manifolds. The physical interpretation of this is that there are two ways to get rid of the extra massless fields, one is by preserving a $U(1)^k$ gauge symmetry which is called the "Coulomb branch" (corresponding in type IIA string to blowing up S^2's) the other is going to the "Higgs branch" (which corresponds to making S^3's have finite volume)[23][24].

One can use these ideas to construct the geometric versions of quantum field theories with desired properties. This is called *geometric engineering of quantum* *field theories*. For example, if we have a shrinking CP^1 in $K3$ we already mentioned that this gives rise to $SU(2)$ gauge symmetry. If we fiber this over a complex curve, depending on what curve we choose we get different theories in the 4 left-over dimensions. For example if we consider the simple product with T^2, then we obtain a theory in four dimensions with $N = 4$ supersymmetric $SU(2)$ Yang-Mills theory. Moreover the coupling constant of the gauge theory $1/g^2$ (which appears in the action in 4 dimensions in the form $\frac{1}{g^2} TrF \wedge *F$) gets identified with the volume of T^2. As discussed before string theory has volume inversion symmetry for T^2. This implies, therefore, that $N = 4$ Yang-Mills should have $g \to 1/g$ inversion symmetry as well. This in fact was anticipated long ago [25]. This duality has interesting consequences for four-manifolds: Consider taking as the four left-over dimensions a smooth four manifold K. Then the (topological) partition function of $N = 4$ Yang-Mills is given by

$$F_{G,K}(q) = \sum_{instantons} q^n \chi(\mathcal{M}_n)$$

where $q = \exp(-1/g^2)$ and $\chi(\mathcal{M}_n)$ denotes the Euler characteristic of the moduli space of instantons of gauge group G (in the case at hand $G = SU(2)$) with instanton number n on K. The duality just discussed implies that this is a modular form (after shifting by an overall coefficient q^a for some constant a). This has been tested in some cases (see [26] and references therein). This modular form is a smooth invariant of K, for each group G.[9]

If we fiber the A_1 singularity instead of T^2 over a CP^1 we obtain an $N = 2$ supersymmetric gauge theory in 4 dimensions with $SU(2)$ gauge symmetry. If different singularities exist over different curves which intersect (what is sometimes called *colliding singularities*) we typically get “matter” in the physical language transforming according to a representation of the product of the two groups (Fig. 8) [27].

FIG.8*:* Matter arises where two loci of singularities intersect. The matter is localized at the intersection.

[9] The subgroup of $SL(2, Z)$ for which this is a modular form depends on G.

This geometric construction of quantum field theories allows us to have a new viewpoint in solving aspects of them. For example consider the $N=2$ supersymmetric $SU(2)$ gauge theory in 4 dimensions. As just mentioned this can be viewed as fibering a contracting CP^1 over a base CP^1. The instantons of this theory in four dimensions, which are relevant to questions involving Donaldson invariants of four manifolds, correspond to holomorphic curves mapped to a Calabi-Yau 3-fold whose local geometry is a line bundle over a CP^1 fibered over CP^1. In particular the instanton class in four dimension gets identified with the number of times the curve gets wrapped around the base CP^1. These can be counted thanks to mirror symmetry discussed before. Thus Donaldson invariants [28] through this geometric construction and by an applications of mirror symmetry can be reduced to Seiberg-Witten invariants [29][30].

Sometimes the physics of the singularities are unconventional. For example when a 4-cycle (say a CP^2) shrinks in a Calabi-Yau threefold, it gives rise to very interesting unconventional new physical theories which were not anticipated! This is thus a great source of insight into new physics. In particular what types of singularities occur as well as what are the ways to resolve them will be of extreme importance for unravelling aspects of this new physics. It is tempting to speculate that these singularities may also lead to new invariants for four manifolds.

5. Black Holes and Minimal Cycles

Black holes are solutions to the Einstein equations which represent matter with sufficient concentration in some region.[10] Consider a d dimensional spacetime. The idealized version of a black hole would correspond to a spherically symmetric distribution of possibly charged matter. This would correspond to solving Euler-Lagrange equations for the action of the form (suppressing all constants)

$$S = \int (R + \sum_i F_i \wedge *F_i)$$

where R denotes the scalar curvature of the metric and F_i denote the curvature of some $U(1)^k$ gauge fields. One solves these equation with the assumption of spherical symmetry with some asymptotic condition imposed on the metric which corresponds to a total mass M black hole and on the gauge fields with charge $Q_i = \int_{S^{d-2}} *F_i$.[11]

Black holes have a causal structure which separates it into two parts by a "horizon" $H = S^{d-2}$, for which the future light cone of points inside the sphere does not include exterior points (Fig. 9).

[10] The following discussion is somewhat oversimplified to make the essential point more clear.

[11] If d=4 we can also consider having magnetic charges $M_i = \int_{S^2} F_i$.

Fig.9: From the regions interior to the horizon no light can come out.

By some semiclassical arguments one expects that black hole carry entropy S, which is the logarithm of the number of its states, is given by

$$S = \frac{A(H)}{4}$$

where $A(H)$ denotes the $d-2$ dimensional "area" of the horizon H. For the black hole solution to make physical sense one finds a lower bound on mass for a fixed set of charges Q_i, namely $M^2 \geq \sum_i Q_i^2$. Physically what will happen is that if the mass is above this bound the black hole radiates and loses mass until it reaches this bound, at which point it becomes a stable stationary state. These are known as *extremal black holes.* The entropy, which is defined as a quarter of the horizon area now becomes

$$S = c_d M^{\frac{d-2}{d-3}}$$

where c_d is some universal constant, depending on d. It has been a challenge of quantum gravity to explain the microscopic origin of this entropy, i.e. what counting do we do to get this entropy.

In string theory, for large enough charges Q_i, the charged black holes are realized as branes wrapped around cycles of the Calabi-Yau, and the condition for extremality of the black hole is that the corresponding cycle be minimal in the given class. Thus the charge lattice corresponds to $H_*(M)$ where the target space is $R^d \times M$.[12]. Thus the question of black hole entropy gets transformed to counting of the "number" of minimal submanifolds for a fixed class $Q \in H_*(M)$. In case there are moduli for such cycles, what is meant by the "number" is the number of cohomology elements of the moduli space. The non-minimal surfaces correspond to non-extremal black holes which "decay" to the extremal ones.

I will now discuss one concrete example to illustrate how the counting works. Consider the 11 dimensional supergravity theory ("M-theory") on target space $R^5 \times T^6$ (which is closely related to type IIA on $R^4 \times T^6$), which I will use to

[12] The homology dimensions which are allowed charges correspond to the allowed dimensions of the branes in the corresponding theory.

count the number of black holes in 5 dimensions, with charges given by an element in $H_2(T^6, Z)$ (this is related to black hole count in [31]). Let us think of $T^6 = (T^2)^3$ and consider the 2-class of each T^2 being represented by e_i where $i = 1, 2, 3$. Let us consider an extremal black hole made of 2-branes whose class is $Ne_1 + Me_2 + Pe_3$. We will consider the regime of parameters where $N >> M, P >> 1$. Let Σ denote a holomorphic curve in the class $[\Sigma] = Me_2 + Pe_3$ (being holomorphic guarantees being minimal in that class). To construct a 2-surface in the class $Ne_1 + Me_2 + Pe_3$ we choose N points on Σ and attach a copy of the first T^2 on each of those points (Fig. 10).

FIG.10: A 2-brane constructed out of Σ and the attachment of N copies of T^2 at N points.

This gives rise to a degenerate minimal 2-cycle. The moduli of this $D2$ brane will in addition correspond to choosing a flat connection on it, which for each T^2 corresponds to choosing a point on the dual T^2. Thus this surface together with the choice of a flat connection is specified by N points in $\hat{T}^2 \times \Sigma$ where $\hat{T}^2$ denotes the dual torus. Of course the choice of N points has no ordering so that the moduli space of this minimal cycle, for a *fixed* Σ is given by

$$\mathcal{M}_N = \mathrm{Sym}^N(T^2 \times \Sigma)$$

Since we are interested in the regime where N is much larger than the other two parameters, we can treat Σ as fixed (i.e. the moduli degrees of freedom coming from it are negligible in comparison). We are thus interested in the growth for the cohomology of $\mathcal{M}_N$ for large N. This space is singular and this cohomology should be understood in the sense of the Hilbert Scheme. The answer is well known [32][33] and is given by the coefficient d_N of q^N in

$$F = \frac{\prod_n (1+q^n)^{b_{odd}}}{\prod_n (1-q^n)^{b_{even}}}$$

where $b_{odd} = b_{even} = 4(MP+2)$ denote the odd and even betti numbers of $T^2 \times \Sigma$. F has modular properties which allows one to estimate the growth of the coefficient of q^N, following Hardy-Ramanujan, to be

$$d_N \sim \exp(2\pi\sqrt{N(MP+2)})$$

Thus we obtain a prediction for the entropy to be

$$S = 2\pi\sqrt{N(MP+2)}$$

The computation of the area of this 5 dimensional black hole by solving the Einstein's equations in this case gives

$$S_{BH} = \frac{A(H)}{4} = 2\pi\sqrt{NMP}$$

which agrees with what we have found in the range of validity of the parameters $N >> M, P >> 1$.

6. A List of Questions

I will list a number of questions which I believe would be interesting to understand further.

1- I have discussed some aspects of mirror symmetry. The physical and mathematical properties of mirror symmetry without including the D-branes is more or less understood. The case involving the D-branes, which is mirror symmetry for (stable) sheaves on Calabi-Yau and is transformed to (minimal) Lagrangian mid-dimensional cycles on the mirror is stated in this note. However the prediction this entails has not been checked yet. In particular both sides of the mirror transform in this case, regardless of the relationship between the two, deserve further study. Even though some aspects of stable bundles on Calabi-Yau are known, it is rather far from a complete understanding. The properties of minimal Lagrangian cycles and enumerative questions in that context are even less understood. Thus the existence of mirror symmetry in this case may lead to many valuable mathematical insights into both questions.

2-We have mentioned that $A-D-E$ singularities of $K3$ lead to the appearance of the corresponding gauge group in physics. We have also noted that some other singularities, such as a contraction of CP^2 in a Calabi-Yau threefold leads to novel physics, not described by a conventional gauge theory. It is thus a pretty exciting link to develop further. To what extent can one classify singularity types of Calabi-Yau (and other Kähler) manifolds, for three and fourfolds? How about transitions among manifolds mediated through singularity types? What is a general way to think about all manifolds at once, having in mind their connectivity by passing through singular ones? Among all singularities is the appearance of $A-D-E$ singularity a rare phenomenon? If so, what explains the fact that we seem to live in a world with gauge symmetries?

3-Another issue we discussed was the counting of minimal submanifolds. This has some applications in the context of counting black hole states. There are many puzzles still to resolve in this context. In the context of minimal 2 dimensional submanifolds mirror symmetry gives us a way to count them in many cases of interest. However even here there are some puzzles: We consider a fixed class $Q \in H_2(M, Z)$ in a Calabi-Yau threefold M and ask how many black holes exist in that class. The predicted answer from solving the Einstein equations is given as follows. Consider an arbitrary Kähler metric k with volume 1 on Calabi-Yau M. Find the Kähler metric which minimizes the area of Q

$$V = k[Q]$$

Call the minimum value V_{min}, and assume this is achieved for a non-degenerate Kähler metric. Then the prediction for the entropy of the black hole [34], and thus the growth of moduli of holomorphic curves in the class Q is that it goes as

$$S = \exp(cV_{min}^{3/2})$$

where c is a universal constant independent of Calabi-Yau. Note that the exponent picks up a factor of $\lambda^{3/2}$ once we rescale $Q \to \lambda Q$. Mirror symmetry allows us to compute the Euler class (of an appropriate bundle) on the moduli space of curves and that has typical growth which upon the same rescaling of Q picks up only a λ in the exponent. The discrepancy of this growth with that obtained in mirror computation is presumably because the number that mirror symmetry computes is an Euler class, whereas the number the black hole degeneracy predicts is the growth of cohomologies of the moduli space. It also suggests there must be an enormous cancellation among even and odd cohomology states for such a dramatic change in the growth of states. It would be interesting to verify this.

For other types of black holes other counting problems arise. For example, for type IIB strings with target space being a Calabi-Yau threefold times R^4 we need to count the growth in the cohomology of the moduli space of minimal Lagrangian 3-submanifolds in a given class $Q \in H_3(M)$. The prediction from the black hole side is that if we denote by Ω the holomorphic 3-form on the Calabi-Yau and minimize

$$V = \frac{|\Omega(Q)|}{\sqrt{\int_M \Omega \wedge \overline{\Omega}}} \tag{1}$$

over the moduli space of complex structure of the Calabi-Yau, assuming that the minimum exists and does not correspond to a degenerate Calabi-Yau, then the growth in the cohomology of moduli space of the minimal submanifold in that class (together with a flat connection) is given by

$$S = \exp(c' V_{min}^2)$$

where c' is a universal constant. In order to verify such predictions we need to be able to count minimal Lagrangian submanifolds. The basic question is how to enumerate them and check this prediction? What is the analog of "mirror symmetry" which allows counting p branes with $p > 2$? In fact I would conjecture, based on a few examples (not predicted from physics) that for a Calabi-Yau of complex dimension d, if we consider real minimal Lagrangian submanifolds of dimension d and minimize V again as given by (1) then the growth of the cohomology of their moduli space (together with a flat connection) is given by

$$S = \exp(c(d)V_{min}^{d-1})$$

where $c(d)$ is a universal constant depending only on d. This formula is true for $d = 2, 1$ (in the $d = 1$ case it is vacuous and in the $d = 2$ case it can be verified) and is predicted to be true as discussed above for $d = 3$, and I am conjecturing it to be true for all d. Is this true? (Note that by mirror symmetry, this conjecture gets transformed to counting the growth of the cohomology of moduli of stable bundles on the mirror Calabi-Yau.)

4-We have seen many instances of dualities in physical systems and we have explained here some of its mathematical implications. We do not have a deep understanding of *why* these dualities even exist. Does studying the mathematical consequences of it shed any light on this question? In other words, why should seemingly difficult mathematical questions find answers in terms of very simple dual mathematical problems? What is the mathematical meaning of duality?

Given all this relation between physics and mathematics one recalls Wigner's thoughts on this relationship and in particular the "unreasonable effectiveness of mathematics" in solving physical problems. With recent developments in physics and its mathematical implications one may also reverse the arrow and wonder about the unreasonable effectiveness of physics in solving mathematical problems.

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References

[1] C. Vafa, in *Geometry, Topology, and Physics for Raoul Bott*, ed. S.-T. Yau, International Press 1995.

[2] J. Polchinski, Phys. Rev. Lett. 75 (1995) 4724.

[3] W. Lerche, C. Vafa and N. P. Warner, Nucl.Phys. B324 (1989) 427.

[4] L. Dixon in Proc. of the 1987 ICTP Summer Workshop in High Energy Physics and Cosmology, Trieste.

[5] B. R. Greene and M. R. Plesser, Nucl. Phys. B338 (1990) 15.

[6] P. Candelas, Xenia C. de la Ossa, Paul S. Green and L. Parkes, Nucl. Phys. B359 (1991) 21

[7] V. Batyrev, J. Alg. Geom. 3 (1994) 493.

[8] A. Strominger, S.-T. Yau and E. Zaslow, Nucl. Phys. B479 (1996) 243.

[9] N.C. Leung and C. Vafa, Adv. Theor. Math. Phys. 2 (1998) 91.

[10] E. Witten, Nucl. Phys. B340 (1990) 281.

[11] A. Givental, Internat. Math. Res. Notices (1996), 613 .

[12] B.H. Lian, K. Liu and S.-T. Yau, Asian J. Math. 1 (1997), 729.

[13] M. Bershadsky, S. Cecotti, H. Ooguri and C. Vafa, Comm. Math. Phys. 165 (1994) 311.

[14] R. Friedman, J.W. Morgan and E. Witten, *Vector Bundles over Elliptic Fibrations* alg-geom/9709029.

[15] M. Bershadsky, A. Johansen, T. Pantev and V. Sadov, Nucl. Phys. B505 (1997) 165.

[16] J. Minahan, D. Nemeschansky, C. Vafa and N.P. Warner, *E-strings and N=4 Topological Yang-Mills Theories*, hep-th/9802168.

[17] K. Yoshioka, *Euler characteristics of SU(2) instanton moduli spaces on rational elliptic surfaces*, math.AG/9805003.

[18] C. Vafa, *Extending Mirror Conjecture to Calabi-Yau with Bundles*, hep-th/9804131.

[19] M. Kontsevich, Proceedings of the 1994 ICM I, Birkäuser, Zürich, 1995, p.120; alg-geom/9411018.

[20] K. Fukaya, The Proceedings of the 1993 GARC Workshop on Geometry and Topology, H.J. Kim, ed., Seoul National University.

[21] A. Polishchuk and E. Zaslow, *Categorical Mirror Symmetry: The Elliptic Case*, math.AG/9801119.

[22] A.N. Tyurin, *Non-Abelian analogues of Abel's Theorem*, I.C.T.P. preprint (1997).

[23] A. Strominger, Nucl. Phys. B451 (1995) 96.

[24] B. Greene, D.R. Morrison and A. Strominger, Nucl. Phys. B451 (1995) 109.

[25] C. Montonen and D. Olive, Phys. Lett. B72 (1977) 117.

[26] C. Vafa and E. Witten, Nucl. Phys. B431 (1994) 3.

[27] S. Katz and C. Vafa, Nucl. Phys. B497 (1997) 146 ;
S. Katz, A. Klemm and C. Vafa, Nucl. Phys. B497 (1997) 173;
S. Katz, P. Mayr and C. Vafa, Adv. Theor. Math. Phys. 1 (1998) 53.

[28] S.K. Donaldson, Proc. London. Math. Soc. 50 (1985) 1.

[29] N. Seiberg and E. Witten, Nucl. Phys. B431 (1994) 484.

[30] G. Moore and E. Witten, *Integration over the u-plane in Donaldson theory*, hep-th/9709193.

[31] A. Strominger and C. Vafa, Phys. Lett. B379 (1996) 99.

[32] L. Göttsche, Math. Ann. 286 (1990) 193.

[33] F. Hirzebruch and T. Höfer, Math. Ann. 286 (1990) 255.

[34] S. Ferrara and R. Kallosh, Phys. Rev. D54 (1996) 1514; Phys. Rev. D54 (1996) 1525.

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