Dynamics:
A Probabilistic and Geometric Perspective

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Abstract. An overview of recent developments and open questions aiming at a global theory of general (non-conservative) dynamical systems.

1991 Mathematics Subject Classification: 58, 34, 60
Keywords and Phrases: uniformly hyperbolic system, attractor, sensitive dynamics, non-uniform hyperbolicity, physical measure, structural stability, stochastic stability, homoclinic orbit, heteroclinic cycle, partial hyperbolicity, dominated splitting, robust transitiveness, singular hyperbolicity, singular attractor.

1 Introduction

In general terms, Dynamics is concerned with describing for the majority of systems how the majority of orbits behave, specially as time goes to infinity. And with understanding when and in which sense this behaviour is robust under small modifications of the system. For instance, most gradient flows on a compact manifold have finitely many singularities, with almost every orbit converging to some of the attracting ones (stable equilibria). And the same is true about any nearby flow, with the same number of attractors. General systems can behave in much more complicated ways, though. Here I consider both discrete time systems – smooth transformations $f : M \to M$, possibly invertible – and continuous time systems – smooth flows or semi-flows $X^t : M \to M$, $t \in \mathbb{R}$ – on manifolds M.

In the early sixties, Smale was proposing the notion of uniformly hyperbolic system, a broad class that includes the diffeomorphisms and flows named after Anosov [4], most gradient-like systems, and the "horseshoe" map. See [101]. A *hyperbolic set*, or generalized horseshoe, is an invariant subset $\Lambda \subset M$ such that the tangent space over it splits into two invariant subbundles $T_\Lambda M = E^s \oplus E^u$ so that E^s is uniformly contracted by future iterates, and similarly for E^u in past iterates. The *system* is *uniformly hyperbolic*, or *Axiom A*, if its *limit set* – the closure of all future and past accumulation sets of orbits – is hyperbolic. A prototype is the diffeomorphism induced on the 2-torus by $(x, y) \mapsto (2x+y, x+y)$, with E^s and E^u corresponding to the eigenspaces of this linear map. This, just as many other uniformly hyperbolic systems, is also an example of "chaotic" (or *sensitive*) behaviour: orbits of typical nearby points move away from each other exponentially fast, under forward and backward iterations.

[1]Partially supported by PRONEX - Dynamical Systems, Brazil

Nevertheless, uniformly hyperbolic systems admit a very precise description of their behaviour: there are compact invariant subsets $\Lambda_1, \ldots, \Lambda_N$ that are transitive (dense orbits) and such that almost every forward orbit of the system accumulates on one of them [101]. And, though the dynamics near these *attractors* Λ_j may be quite "chaotic", it is strikingly well behaved from a statistical point of view: there exists a *physical probability measure* μ_j supported on Λ_j, such that the time average (δ_p stands for Dirac measure at p)

$$\lim_{n\to+\infty} \frac{1}{n} \sum_{j=0}^{n-1} \delta_{f^j(z)}, \qquad \text{or} \quad \lim_{T\to+\infty} \frac{1}{T} \int_0^T \delta_{X^t(z)}\, dt,$$

exists and coincides with μ_j for Lebesgue almost every point z whose orbit accumulates on Λ_j. Cf. Sinai, Ruelle, Bowen [100], [95], [20], [19].

Another major breakthrough was the proof that uniformly hyperbolic systems are, essentially, the structurally stable ones. This was completed by Mañé [63], and Hayashi [43] for flows, in the C^1 setting, after crucial contributions from several mathematicians, specially Anosov, Palis, Smale, Robbin, de Melo, Robinson. See [84] for an extended list of references. The notion of *structural stability*, introduced by Andronov-Pontryagin in the thirties, means that all nearby systems are equivalent up to continuous global change of coordinates.

On the other hand, striking examples like Newhouse's maps with infinitely many periodic attractors [74], or the "strange" attractors of Lorenz [56] and Hénon [44], showed that uniform hyperbolicity is too strong a condition for a general description of dynamics: systems can be persistently non-hyperbolic (persistently unstable). As the hope to describe generic dynamical systems in a uniformly hyperbolic scope was gradually abandoned, still other important developments were taking place concerning enlarged settings of dynamics.

Starting from Oseledets [78], Pesin [87] developed a theory of *non-uniform hyperbolicity*, dealing with general systems endowed with an invariant probability measure with respect to which almost every point exhibits asymptotic contraction and expansion along complementary directions (non-zero Lyapunov exponents). Then almost every point has a stable and an unstable manifold, whose points are exponentially asymptotic to it, respectively, in the future and in the past. See Katok-Hasselblatt [48] for an account of the theory and references.

There was also considerable progress in studying the modifications (bifurcations) through which a system may cease to be stable. Global bifurcations like *homoclinic tangencies* and *heteroclinic cycles*, that affect the system's behaviour on large regions of the ambient M, are accompanied by such a wealth of dynamical changes that one must aim at describing the main phenomena occurring for most nearby systems, specially in terms of probability in parameter space. See Palis-Takens [84] and Section 5 below.

And one could attain substantial understanding of some "chaotic" systems, such as Lorenz-like flows, quadratic maps of the interval, period-doubling cascades, and Hénon-like attractors. Since orbits are sensitive to initial conditions, and so essentially unpredictable over long periods of time, one focus on statistical properties of large sets of trajectories, a point of view pioneerly advocated by

Sinai and by Ruelle back in the seventies. See [106] and Sections 3, 4, 6 below.

Building on this, we are again trying to develop a global picture of Dynamics recovering, in a new and more probabilistic formulation, much of the paradigm of finitude and stability for most systems that inspired Smale's proposal about four decades ago. Palis conjectured that every dynamical system can be approximated by another having only finitely many attractors, supporting physical measures that describe the time averages of Lebesgue almost all points. This is at the core of a program [81] that also predicts that statistical properties of such systems are stable, namely under small random perturbations.

In this note I survey some of the recent, rather exciting progress in the general direction of such a program, as well as related open problems and conjectures, mostly in the context of general dissipative systems.

2 Setting the scenario

In what follows I refer mostly to transformations, since the definitions and results for flows are often similar. Except where otherwise stated, manifolds are smooth, compact, without boundary, and measures are probabilities on the Borel σ-algebra. Lebesgue measure means any measure generated by a smooth volume form.

Time averages of continuous functions $\varphi : M \to \mathbb{R}$

$$\lim_{n\to+\infty} \frac{1}{n}\sum_{j=0}^{n-1} \varphi(f^j(z))$$

are the most basic statistical data on the system's asymptotic behaviour. An invariant measure μ is a *physical measure* if the time average of every φ coincides with the spatial μ-average $\int \varphi\, d\mu$, for a positive Lebesgue measure subset of points $z \in M$. And the *basin* of μ is the set $B(\mu)$ of points z for which this happens.

Physical measures are often called *SRB measures*, after Sinai, Ruelle, Bowen, who first constructed them for Anosov systems [100] and then for general uniformly hyperbolic diffeomorphisms [95] and flows [20]. For these systems there are finitely many SRB measures $\mu_1, \dots, \mu_N$, and their basins cover Lebesgue almost all of the phase space M. Each support $\Lambda_i = \operatorname{supp}\mu_i$ is an *attractor*, meaning that it is an invariant transitive set whose basin of attraction has positive Lebesgue measure. An invariant set Λ is *transitive* if there exists $z \in \Lambda$ whose forward orbit $\{f^n(z) : n \geq 0\}$ is dense in Λ. The *basin of attraction* (or stable set) $B(\Lambda)$ is the set of points whose forward orbits accumulate in Λ. In this hyperbolic setting, as well as in all known cases that are relevant here, the basin contains a full neighbourhood of the attractor.

For systems preserving a smooth measure, Birkhoff's ergodic theorem ensures that time averages are defined Lebesgue almost everywhere. It is widely believed that the same should be true for *most* non-conservative systems, but this is not known, and there are examples showing that it is not the case for *all* systems. For instance, Bowen exhibited a simple flow on the plane where time averages fail to exist on a whole open region bounded by two saddle connections; see e.g. [106].

On the other hand, existence results for SRB measures are now available for some large classes of systems, as we shall see.

SRB measures are sometimes defined differently, by a property of absolute continuity of their conditional measures on unstable manifolds; see e.g. Eckmann-Ruelle [37]. The definition adopted above is a bit more general, but all the SRB measures we meet in the present paper also have this absolute continuity property.

Palis proposed a few years ago that, for a dense subset of all systems statistical properties should be essentially as nice as in the Axiom A case. In more precise terms, he conjectured that *every system can be approximated by another having only finitely many attractors* (approximation in the C^k topology, any $k \geq 1$) *supporting SRB measures whose basins cover a full Lebesgue measure subset of the manifold*; see [81]. He also conjectured that those properties should be very stable under small perturbations of the system. Here one thinks of modifications of the system along generic parametrized families, i.e. finite-dimensional submanifolds in the space of systems. For Lebesgue almost all parameters there should be finitely many attractors, supporting SRB measures whose basins cover nearly all of M, also in terms of Lebesgue measure. Moreover, time averages should not be much affected if small random errors in parameter space are introduced at each iteration: stochastic stability.

This last notion is most relevant when dealing with concrete situations modeled by mathematical systems (which are always only approximately correct): in many cases, features of the actual system that are unaccounted for by the model are well represented by random fluctuations around it. For a definition, let us consider first the situation where the initial map f has some attractor Λ supporting a unique SRB measure μ, and whose basin contains a trapping open region U: the closure of $f(U)$ is contained in U. One considers sequences x_j, $j \geq 0$, with $x_0 \in U$ and $x_{j+1} = g_j(x_j)$ for $j \geq 0$, where the maps g_j are chosen at random (independently) in the ϵ-neighbourhood of f, according to some probability $\mathcal{P}_\epsilon$. Here ϵ should not be too large, to ensure that these sequences x_j do not escape U. Then f is *stochastically stable* on the basin of Λ if for each continuous function φ

$$\lim_{n\to+\infty} \frac{1}{n} \sum_{j=0}^{n-1} \varphi(x_j) \quad \text{is close to} \quad \int \varphi \, d\mu,$$

for almost every random orbit $(x_j)_j$ (Lebesgue almost every $x_0 \in U$ and $\mathcal{P}_\epsilon$ almost every g_j, $j \geq 0$) if ϵ is small. More concretely, I propose to take these small random perturbations along generic parametrized families through f: $\mathcal{P}_\epsilon$ is given by Lebesgue measure in the corresponding parameter space.

There are other perturbation schemes, for instance, random orbits may be formed by choosing each x_{j+1} at random close to $f(x_j)$, following some probability measure $P_\epsilon(x_j, \cdot)$. The random noise $P_\epsilon(x, \cdot)$ is usually taken absolutely continuous with respect to Lebesgue measure, and supported on the ϵ-neighbourhood of $f(x)$ or, more generally, converging to Dirac measure at $f(x)$ as $\epsilon \to 0$. See Kifer [52], [53]. Stochastic stability with respect to this perturbation scheme is defined as before. Although it is not logically related to the notion in the previous paragraph, which corresponds to

$$P_\epsilon(x, A) = \mathcal{P}_\epsilon(\{g : g(x) \in A\}),$$

the two definitions agree for the systems known to be stable, such as uniformly hyperbolic attractors Kifer [53], Young [108], and other cases mentioned below.

So far, I restricted to attractors with a unique SRB measure and whose basin contains some trapping region: this is true for essentially all known cases, although it is not yet clear in which generality it holds. If the basin of attraction is just a positive Lebesgue measure set (or if one considers random noise which is not supported on small neighbourhoods), then random orbits may escape from it. In such cases, as well as for transitive attractors supporting several SRB measures, a more global notion of stochastic stability can be applied: denoting μ_i the SRB measures of f, time averages of each continuous φ along almost all random orbits should be close to the convex hull of the $\int \varphi \, d\mu_i$ when ϵ is small.

The main random perturbation scheme for flows X^t is by diffusion. That is, letting X be the corresponding vector field, one considers the flow ξ_t associated to the stochastic equation (for simplicity, pretend $M = \mathbb{R}^d$)

$$d\xi_t = X(\xi_t)\, dt + \epsilon A(\xi_t)\, dw_t \tag{1}$$

where $A(\cdot)$ is matrix-valued and dw_t is the standard Brownian motion. See e. g. Friedman [38]. Then stochastic stability is defined essentially as before, if X^t has a unique SRB measure μ: the time averages of each continuous φ over almost all stochastic orbits ξ_t should be close to $\int \varphi \, d\mu$ if ϵ is small. More generally, since solutions of (1) usually spread over the whole ambient manifold M, one should use a global notion of stability as in the previous paragraph.

Before proceeding, let me recall another probabilistic notion, expressing sensitivity of the dynamics, that plays an important role in the characterization of complex systems: decay of correlations. The definition applies to general maps f (or flows) endowed with some invariant measure μ, though the most interesting case is when μ is a physical measure. Informally, this notion can be motivated as follows. Sensitiveness means that orbits, in some sense, forget their initial state as time increases to infinity. So, given real functions φ and ψ on M, knowledge of $\varphi(z)$ should provide little information about $\psi(f^n(z))$ for large $n \geq 1$. This may be expressed in terms of their *correlations*

$$C_n(\varphi, \psi) = \int \varphi \, (\psi \circ f^n) \, d\mu - \int \varphi \, d\mu \int \psi \, d\mu,$$

that should converge rapidly to zero as time increases to infinity. In general, one must restrict to some subspace $\mathcal{F}$ of functions φ, ψ with a minimum amount of regularity. This is because the systems we deal with are actually deterministic (and, in many cases, reversible): loss of memory resulting from sensitiveness appears only at a coarse level of observation of the system, through quantities φ, ψ that do not distinguish nearby points well. One speaks of *(exponential) decay of correlations* in the space $\mathcal{F}$ if $C_n(\varphi, \psi)$ goes to zero (exponentially fast) as n goes to infinity, for all $\varphi, \psi \in \mathcal{F}$.

3 ONE-DIMENSIONAL MAPS

Let f_a be the real quadratic map given by $f_a(x) = x^2 + a$. If $a \notin [-1/4, 2]$ then the orbit $f^n(0)$ of the critical point goes to infinity as $n \to +\infty$, and so does the orbit of Lebesgue almost every point x. Let us look at the more interesting case $a \in [-1/4, 2]$. Then there exists a maximal compact interval I_a containing $x = 0$ and invariant under f_a, in the sense that $f_a(I_a) \subset I_a$. Two main types of behaviour are known, depending on the value of the parameter a.

A first type (periodic, uniformly hyperbolic, regular) corresponds to f_a having a periodic attractor, i.e. a point p such that $f_a^k(p) = p$ and $|(f_a^k)'(p)| < 1$ for some $k \geq 1$. Then, the orbit of Lebesgue almost every $x \in I_a$ converges to the orbit of p. It is easy to see that this behaviour corresponds to an open set of parameters, and it was conjectured for a long time that this set is also dense in $[-1/4, 2]$. This statement, known as the hyperbolicity conjecture, was eventually settled affirmatively by Swiatek with the aid of Graczyk [40], and by Lyubich [60].

A second kind of behaviour (chaotic, non-uniformly hyperbolic, stochastic) is displayed by maps f_a that admit an invariant measure μ_a absolutely continuous with respect to Lebesgue measure. It is a theorem of Jakobson [46] that this occurs for a set of parameters with positive Lebesgue measure. When it exists, such a measure μ_a is unique and ergodic, and it gives the time average of Lebesgue almost every $x \in I_a$, Blokh-Lyubich [13].

Do these cases exhaust all the possibilities for a full Lebesgue measure set of parameters? Remarkably, the answer is affirmative, as shown by Lyubich:

THEOREM 1 ([59]). *For Lebesgue almost every $a \in [-1/4, 2]$, the quadratic map f_a has either a periodic attractor or an absolutely continuous invariant measure.*

In particular, Palis' finitude conjecture in Section 2 holds in this context: Lebesgue almost every quadratic map admits a unique SRB measure (either a Dirac measure on a periodic orbit or an absolutely continuous measure), whose basin contains Lebesgue almost every bounded orbit. It is interesting to point out that quadratic maps without SRB measures do exist, cf. Hofbauer-Keller [45].

Most of this holds for general unimodal or multimodal maps of the interval or the circle, though the extension may be far from trivial. A proof of the hyperbolicity conjecture in a general setting of unimodal maps was announced by Kozlovski [54]. An analog of Theorem 1 is also conjectured for general families of one-dimensional maps, but this has not yet been proved.

Jakobson's theorem does extend beyond quadratic maps, and many general criteria for the existence of absolutely continuous invariant measures were obtained since then. This is the most interesting case from an ergodic point of view, and there are several works concerning statistical properties of non-uniformly hyperbolic maps in dimension one, such as the results of Keller-Nowicki [51] and Young [109] on exponential decay of correlations, and those of Collet [28], Katok-Kifer [49], Benedicks-Young [10], and Baladi-Viana [6] on stochastic stability.

Infinite-modal maps – one-dimensional maps with infinitely many maxima and minima – come up in many natural contexts of Dynamics, but they are mostly unexplored. Recently, Pacifico-Rovella-Viana [80] proved that non-uniform hyperbolicity is persistent – positive Lebesgue measure set of parameters – in a large class of parametrized families of infinite-modal maps, thus setting a way to a more complete study of such maps and their statistical properties. It is an interesting problem to carry out such a study.

4 Hénon-like attractors

This class of systems is modeled by the Hénon map [44]

$$(x, y) \mapsto f(x, y) = (1 - ax^2 + y, bx),$$

where a, b are real parameters. A main feature is the coexistence of hyperbolic and *folding* behaviour: at points away from the line $x = 0$ one may find complementary directions that are geometrically contracted and expanded by the derivative of the map; but these directions do not extend across the *critical region* $\{x \approx 0\}$, where the phase space is "folded" by the map.

For a large domain in parameter space, e.g. $1 < a < 2$ and b not too large, one may find some rectangle R which is positively invariant – f maps R to its interior – and this is the most interesting case. Computer pictures of the "strange attractor", where orbits of points inside R seem to accumulate, were produced by Hénon [44] for parameters $a \approx 1.4, b \approx 0.3$. But it was only some ten years ago that Benedicks-Carleson could prove that there is indeed a non-trivial (non-periodic) attractor, with positive probability in parameter space:

Theorem 2 ([7]). *For every sufficiently small $b > 0$ there exists a positive Lebesgue measure subset $E \subset \mathbb{R}$ so that for all $a \in E$ there exists a compact invariant subset $\Lambda \subset R$ such that $B(\Lambda)$ has non-empty interior, and $\|Df^n(z)\| \to +\infty$ exponentially fast when $n \to +\infty$, for some z with forward orbit dense in Λ.*

This was a major achievement, opening the way to a theory of *Hénon-like maps*, which are the first class of genuinely non-uniformly hyperbolic systems in dimension larger than 1 to be understood specially from an ergodic point of view (Lorenz-like flows can be reduced to hyperbolic maps, cf. Section 6).

On the one hand, it was shown that attractors combining hyperbolic and critical behaviour are a very general phenomenon occurring, with positive probability in parameter space, in many bifurcations of diffeomorphisms or flows: homoclinic tangencies Mora-Viana [64], saddle-node cycles Díaz-Rocha-Viana [35], Costa [31], saddle-focus connections Pumariño-Rodriguez [92]. Colli [30] proved that infinitely many of these attractors may coexist, for many parameter values, in the unfolding of homoclinic tangencies. Henceforth, I refer to all these attractors as *Hénon-like*.

On the other hand, Benedicks-Young proved that these non-hyperbolic attractors have, nevertheless, well defined statistical properties:

Theorem 3 ([11], [12]). *Let Λ be a Hénon-like attractor of a surface diffeomorphism f, as above. Then there exists a unique SRB measure μ supported on Λ, and (f, μ) is equivalent to a Bernoulli shift. Moreover, (f, μ) has exponential decay of correlations in the space of Hölder continuous functions.*

Their strategy in [11] was to find an ergodic invariant measure μ supported on Λ, with absolutely continuous conditional measures along Pesin's unstable manifolds. Then, the basin of μ must contain a positive Lebesgue measure set, cf. Pugh-Shub [90]. This construction of the SRB measure could not decide whether Lebesgue almost every point that is attracted to Λ is in $B(\mu)$ or, on the contrary, there are sizable sets ("holes") of points in $B(\Lambda)$ whose time average is not given by μ. This *basin problem* was raised by Sinai and by Ruelle back in the seventies, and is also related to the following question: is Lebesgue almost every orbit in the basin of attraction asymptotic to some orbit inside the attractor? For uniformly hyperbolic attractors the answers are well-known and affirmative, see Bowen [19].

Then, Benedicks and I solved both questions for Hénon-like attractors: there are no "holes" in their basins. More recently, we also proved that these attractors are stochastically stable, thus bringing the ergodic theory of these systems close to completion.

Theorem 4 ([9], [8]). *Let Λ be a Hénon-like attractor of a surface diffeomorphism f, as before, and μ be the SRB measure. Then*

$$B(\Lambda) = \bigcup_{\xi \in \Lambda} W^s(\xi) = B(\mu), \quad \text{up to zero Lebesgue measure sets.}$$

Moreover, (f, μ) is stochastically stable under small random perturbations.

The proofs of these results depend on an assumption of strong area dissipativeness, e.g. in Theorem 2 the Jacobian of f must be very small (much smaller than Hénon's $b \approx 0.3$). In particular, we are still far from understanding non-uniformly hyperbolic behaviour in area-preserving systems such as the conservative Hénon family $(x, y) \mapsto (1 - ax^2 + y, \pm x)$, or the *standard family* of maps on the 2-torus

$$f_k(x, y) = (-y + 2x + k \sin(2\pi x), x)$$

For the latter, Duarte [36] proved abundance of KAM islands for generic (Baire second category) large parameters k. But the standing conjecture is that, from a measure-theoretical point of view, non-uniform hyperbolicity – non-zero Lyapunov exponents on a positive Lebesgue measure subset, possibly even non-existence of elliptic islands – should prevail in parameter space. To settle this is a major challenge in Dynamics nowadays.

5 Homoclinic tangencies - Fractal dimensions

A *homoclinic tangency* is a non-transverse intersection between the stable manifold and the unstable manifold of some periodic point p. In this section I want to explain why this phenomenon is a main ingredient for non-hyperbolic dynamics: homoclinic tangencies are always an obstruction to hyperbolicity and, for low dimensional systems such as surface diffeomorphisms, this is likely to be *the* essential obstruction.

Palis conjectured that every surface diffeomorphism can be C^k approximated by another which either is uniformly hyperbolic or has a homoclinic tangency. This was recently established by Pujals-Sambarino, for $k = 1$:

Theorem 5 ([91]). *The set of diffeomorphisms on a surface M which are either uniformly hyperbolic or have a homoclinic tangency is dense in* $\mathrm{Diff}^1(M)$.

Their arguments, inspired by Mañé's proof of the C^1 stability conjecture [61], [62], [63], have other important consequences, including the following corollary of Theorem 5 that gives a partial converse to Newhouse's theorem [75]: C^1 open sets where coexistence of infinitely many periodic attractors occurs densely must contain diffeomorphisms with homoclinic tangencies.

There are other results showing that specific phenomena of complicated dynamics, such as saddle-node cycles, cascades of bifurcations, or Hénon-like attractors, can be approximated by maps with homoclinic tangencies; see Newhouse-Palis-Takens [77], Catsigeras [25], Ures [103]. Conversely, surface diffeomorphisms with homoclinic tangencies are approximated by others exhibiting any of these phenomena; see Newhouse [75], Yorke-Alligood [107], Mora-Viana [64], Colli [30]. In these situations one gets approximation in the C^k sense, any $k \geq 1$, and so these results indicate that the space of non-hyperbolic C^k surface diffeomorphisms should be rather homogeneous, even if there is little hope to settle the general case $k \geq 2$ of the conjecture above in a near future.

Let f_μ, $\mu \in \mathbb{R}$, be a generic parametrized families of diffeomorphisms on a surface M, such that $f = f_0$ has a homoclinic tangency. What can one say about the dynamics of f_μ, for the majority of parameters μ close to zero? In some cases f_μ turns out to be uniformly hyperbolic for a set of parameters H with full Lebesgue density at $\mu = 0$:

$$\lim_{\varepsilon \to 0} \frac{\mathrm{Leb}(H \cap [-\varepsilon, \varepsilon])}{2\varepsilon} = 1.$$

This is due to Palis-Takens [82], [83], extending Newhouse-Palis [76], where a main assumption is that the periodic point p is in a hyperbolic set Λ whose *Hausdorff dimension* $HD(\Lambda)$ is less than 1. On the other hand, Palis-Yoccoz [86] showed that this is generically not true if the Hausdorff dimension is larger than 1.

These works, as well as Newhouse [74], displayed a crucial role played by fractal dimensions and related geometric invariants in the theory of bifurcations, and inspired some general problems about Cantor subsets of the real line with important consequences in the dynamical setting. Another conjecture of Palis claimed that for generic regular Cantor sets $K_1, K_2 \subset \mathbb{R}$, the arithmetic difference $K_2 - K_1 = \{a_2 - a_1 : a_1 \in K_1, a_2 \in K_2\}$ either has zero Lebesgue measure or contains an interval. A *regular Cantor set* is one which is generated by a smooth expanding map defined on a finite union of intervals. The space of regular Cantor sets inherits a topology from the space of such expanding maps, and the word generic refers to a residual (Baire second category) subset in this topology. The arithmetic difference always has measure zero if $HD(K_1) + HD(K_2) < 1$, so the interesting case of the conjecture corresponds to the sum being larger than 1. This was achieved a couple of years ago by Moreira-Yoccoz who, in fact, proved a stronger statement:

Theorem 6 ([73]). *There exists an open and dense subset of the space of pairs of regular Cantor sets (K_1, K_2) with $HD(K_1)+HD(K_2) > 1$, such that K_1 intersects stably some translate $K_2 + t$.*

Partial results had been obtained by Moreira [71], who introduced the notion of *stable intersection*: given any $\tilde{K}_1$ close to K_1 and $\tilde{K}_2$ close to K_2+t, then $\tilde{K}_1 \cap \tilde{K}_2$ is non-empty. In particular, $K_2 - K_1$ contains an interval around t. Theorem 6 has the following important translation in the dynamical setting [72]: for a generic family of diffeomorphisms f_μ unfolding a homoclinic tangency the set of parameters for which f_μ is either uniformly hyperbolic or has *persistent homoclinic tangencies* has full density at $\mu = 0$. This second possibility corresponds to intervals in parameter space where densely one observes homoclinic tangencies, cf. [75].

Of course, one also wants to describe the structure of the limit set $L(f_\mu)$, for most small values of μ, specially when it is not uniformly hyperbolic. Palis-Yoccoz announced recently that $L(f_\mu)$ does have a property of *weak hyperbolicity* for a set of parameters with full density at zero, if the Hausdorff dimension of the horseshoe Λ involved in the tangency is not too large, e.g. $HD(\Lambda) < 3/2$. Roughly, the part of the limit set that is related to the unfolding of the tangency looks like a saddle-type version of the Hénon attractor: in particular, its stable and unstable sets have zero Lebesgue measure.

Several of these results hold in any dimension, or have been subsequently extended to that generality, see Viana [104], Palis-Viana [85], Romero [93], Gonchenko-Shil'nikov-Turaev [39], and references therein. As a rule, results involving fractal dimensions are much harder in higher dimensions, and this is a subject of current research. On the other hand, for high dimensional diffeomorphisms and flows, new key phenomena enter the scene, besides homoclinic tangencies, and problems and conjectures must be restated accordingly. This I discuss in the next sections.

6 Singular flows

In the early sixties, Lorenz [56] observed that the solutions of a simple differential equation in dimension 3,

$$\dot{x} = -10x + 10y, \quad \dot{y} = 28x - y - xz, \quad \dot{z} = -\frac{8}{3}z + xy \tag{2}$$

related to a model of atmospheric convection, seemed to depend sensitively on the initial point. Thus, in practice, their behaviour over long periods of time can not be effectively predicted (and so neither can the weather, according to Lorenz): one would need to know the initial point with infinite precision.

Geometric models were proposed by Afraimovich-Bykov-Shil'nikov [1] and Guckenheimer-Williams [42], to interpret the behaviour observed by Lorenz in the equation (2). These are smooth flows X^t in three dimensions, admitting a trapping region U – the closure of $X^t(U)$ is contained in U for every $t > 0$ – such that the maximal invariant set $\Lambda = \cap_{t>0} X^t(U)$ contains both a singularity (equilibrium point) and regular orbits dense in Λ. The flow leaves invariant a foliation of U, a key property that permits to reduce the dynamics to that of an expanding map of the interval. Moreover, these attractors are robust: the maximal invariant set in U of any nearby flow Y^t also has all these properties.

These Lorenz models attracted a lot of attention, and their geometric, dynamical, and ergodic properties are now well understood: in particular, they support a unique SRB measure and they are stochastically stable. See e.g. Bunimovich [22], Collet-Tresser [29], Kifer [52], Pesin [88], Sataev [97], and references therein. On the other hand, Lorenz' original conjecture that a sensitive attractor Λ exists in the specific system (2) remained an open problem for more than three decades. Remarkably, a positive solution has just been announced by Tucker [102].

With these examples in mind, let us call a compact invariant set Λ of a flow X^t a *singular transitive set* if it is the maximal invariant set $\Lambda = \bigcap_{t\in\mathbb{R}} X^t(U)$ in some open neighbourhood U, and contains both singularities and dense regular orbits. We also call Λ a *singular (or Lorenz-like) attractor* if U can be taken positively invariant (trapping), and a *singular repeller* if it is a singular attractor for the flow X^{-t} obtained from X^t by reversing the direction of time. In general, we say that the singular transitive set Λ is C^1 *robust* if $\Lambda_Y = \cap_{t\in\mathbb{R}} Y^t(U)$ is also a singular transitive set for any flow Y^t in a C^1 neighbourhood of X^t.

Robust singular transitive sets are a main novelty in the dynamics of flows, relative to discrete time systems. In the last few years, Morales-Pacifico-Pujals have been developing a general theory of such sets, specially in the 3-dimensional case. A related goal is to characterize the flows whose singularities and periodic orbits are robustly hyperbolic, meaning that they remain so for every C^1 nearby flow, see [66]. Morales-Pacifico-Pujals construct new types of flows with singular attractors, some of which can be obtained from hyperbolic flows through a single bifurcation [65], [70], [67]. Most specially, they prove that a C^1 robust singular transitive set Λ must have the following hyperbolicity property [68]. A compact invariant set Λ is *singular hyperbolic* for the flow X^t if there exists a decomposition of the tangent space

$$T_\Lambda M = E^1 \oplus E^2$$

invariant under every DX^t, where E^1 is 1-dimensional and (norm) contracting, and E^2 is 2-dimensional and volume expanding. The latter may contain directions that are contracted, but the decomposition must be *dominated*: possible contraction along E^2 is weaker than the contraction along E^1. We also say that Λ is singular hyperbolic for X^t if it is singular hyperbolic for the dual flow X^{-t}.

Theorem 7 ([68]). *Let Λ be a C^1 robust singular transitive set of a flow on a 3-dimensional manifold M. Then all the singularities in Λ have the same stable dimension, either 1 or 2. In the first case Λ is a singular repeller, in the second one it is a singular attractor. In either case, Λ is a singular hyperbolic set.*

A key tool in Theorem 7, and in other important results in this area, is Hayashi's connecting lemma [43]: a system exhibiting some unstable manifold accumulating on a stable manifold may be C^1 perturbed to have the two invariant manifolds intersect.

A next step is to understand the structure of singular hyperbolic sets. In this direction, Morales-Pacifico-Pujals can give a pseudo Markov description reminiscent of [20], and they also have made progress towards a converse to Theorem 7, characterizing when a singular hyperbolic set is C^1 robustly transitive. In the proof of the theorem they also get that in the attractor case the eigenvalues at the singularities $\lambda_1 < \lambda_2 < 0 < \lambda_3$ must satisfy $\lambda_2 + \lambda_3 > 0$, just as in the classical Lorenz models. A dual fact holds in the repeller case.

Rovella [94] had given the first examples of singular transitive attractors that, although not robust, are *persistent* in a probabilistic sense: positive probability in parameter space, in generic parametrized families of flows through the initial one. For this he considered a modification of the geometric Lorenz flows where the eigenvalues at the singularity satisfy $\lambda_2 + \lambda_3 < 0$ instead. New examples are provided by the extended model for the behaviour of the Lorenz equations over a large parameter range proposed by Luzzatto-Viana [58], [57]: a main novelty with respect to the usual geometric models and Rovella's flows is that these systems admit no invariant foliation. Moreover, Pacifico-Rovella-Viana [80], [79] proved that global spiral attractors exist, as conjectured by Sinai, in fact they occur persistently in many families of flows. These are attractors containing a saddle-focus singularity (two contracting complex and one real expanding eigenvalue), which forces an extremely complicated spiraling geometry.

The theory of singular flows in dimension larger than 3 is mostly open. Until very recently it was not even known whether robust transitive attractors can contain singularities with unstable dimension larger than 1, an old problem posed by the introduction of the geometric Lorenz models in the seventies. This was solved by Bonatti-Pumariño-Viana [17] who proved that such *multidimensional Lorenz-like attractors* do exist, with arbitrary unstable dimension $k \geq 1$. Moreover, they support a unique SRB measure. Examples persisting in codimension 2 subsets of flows were found by Morales-Pujals [69].

Let me also briefly comment on piecewise smooth maps, an important class of systems including e.g. Poincaré maps of flows with singularities, some Markov or non-Markov extensions of smooth maps, and billiards. See [50]. Liverani [55] proved exponential decay of correlations for area-preserving uniformly hyperbolic piecewise smooth maps. Young [110] extended this to the dissipative case, and also proved exponential decay of correlations for planar Sinai billiards [99]. Chernov [26], [27] extended these results to arbitrary dimension. Alves [2] constructed absolutely continuous invariant measures for piecewise expanding maps with countably many domains of smoothness, in any dimension.

7 Cycles - partial hyperbolicity

For high dimensional maps and flows, more generally than homoclinic tangencies one must take into account *heteroclinic cycles*: periodic points with variable stable dimensions cyclic related through intersections between their invariant manifolds. A general version of the conjecture at the beginning of Section 5 was also proposed by Palis: every diffeomorphism can be C^k approximated by another which either is uniformly hyperbolic or has a homoclinic tangency or a heteroclinic cycle.

A key fact about uniformly hyperbolic diffeomorphisms (or flows), is that the limit set $L(f)$ can be partitioned into finitely many *basic pieces* $\Lambda_1, \ldots, \Lambda_K$ (among which are the attractors of f) that are invariant, transitive, and *isolated*: each Λ_i is the maximal invariant set in a neighbourhood U_i. In fact, Λ_i is C^1 *robustly* *transitive*: the continuation $\Lambda_i(g) = \cap_{n\in\mathbb{Z}} g^n(U_i)$ of Λ_i is also transitive, for any diffeomorphism g C^1 close to f. See [101]. Can one find something on the way of such a decomposition for general diffeomorphisms? Recently, there has been some remarkable progress towards understanding how the building blocks could look like. Let Λ be an isolated C^1 robustly transitive set of a diffeomorphism f. What can be said about Λ?

For surface diffeomorphisms, Mañé [61] proved that Λ must be a hyperbolic set. He also observed that this can not be true in higher dimensions: there exist open sets of C^1 diffeomorphisms of the 3-torus which are transitive in the whole ambient, and yet have periodic saddles with different stable dimensions (so they can not be Anosov diffeomorphisms). Notice that C^1 robustly transitive diffeomorphisms that are not uniformly hyperbolic had been exhibited before by Shub [98], in dimension 4 or higher. In both constructions, the diffeomorphisms admit a continuous invariant splitting $TM = E^s \oplus E^c \oplus E^u$ such that E^s is contracting, E^u is expanding, and they dominate E^c. Bonatti-Díaz [14], building on Díaz[32], gave the first examples of robustly transitive diffeomorphism with central bundle E^c having dimension larger than 1.

Next, Díaz-Pujals-Ures [33] proved that C^1 robustly transitive sets of diffeomorphisms in dimension 3 must be partially hyperbolic. A compact set Λ invariant under a diffeomorphism f is *partially hyperbolic* if there are $C > 0, \lambda < 1$, and an invariant splitting of the tangent space $T_\Lambda M = E^1 \oplus E^2$ which is *dominated*

$$\|Df^n|E^1_z\|\,\|(Df^n|E^2_z)^{-1}\| \le C\lambda^n \text{ for all } z \in M \text{ and } n \ge 1$$

and such that either E^1 is contracting or E^2 is expanding: either

$$\|Df^n|E^1\| \le C\lambda^n \text{ for all } n \ge 1, \quad \text{or} \quad \|(Df^n|E^2)^{-1}\| \le C\lambda^n \text{ for all } n \ge 1.$$

It is common to write the splitting $E^1 \oplus E^2$ as $E^s \oplus E^c$ in the first case, and as $E^c \oplus E^u$ in the second one, and I shall keep this convention in what follows. Still in dimension 3, Bonatti observed that C^1 robustly transitive sets need not be strongly partially hyperbolic (three invariant subbundles), see [18] for other examples. Also related to this, Díaz-Rocha [34] prove that near a diffeomorphism with a heteroclinic cycle there are others with either homoclinic tangencies or robustly transitive sets that are strongly partially hyperbolic.

In [18], Bonatti and I also constructed the first examples of robustly transitive diffeomorphisms having neither contracting nor expanding subbundles. Our examples, e.g. in the 4-torus, do admit a dominated splitting, though, with E^1 volume contracting and E^2 volume expanding. Then, Bonatti-Díaz-Pujals [16] rounded off this series of results, by proving that a dominated splitting is indeed a necessary condition for robust transitivity, in any dimension. Summarizing:

Theorem 8. *Let Λ be a C^1 robustly transitive set of $f : M \to M$.*

1. *([61]) If $\dim M = 2$ then Λ is a hyperbolic set.*
2. *([33]) If $\dim M = 3$ then Λ is a partially hyperbolic set.*
3. *([16]) If $\dim M \ge 4$ then Λ admits a dominated splitting.*

Actually, Mañé [61] had proved a stronger fact than 1 above, implying that a transitive isolated set of a surface diffeomorphism either is hyperbolic or its continuation for some C^1 near map contains infinitely many periodic attractors or repellers. This is also extended to any dimension in [16], with hyperbolicity replaced by existence of a dominated splitting.

Diffeomorphisms with infinitely many periodic attractors or repellers are still a mystery: little is known apart from the fact that they are generic in some open sets of $\mathrm{Diff}^2(M)$, cf. [74], [75], [85], and of $\mathrm{Diff}^1(M)$ if $\dim M \geq 3$, cf. [15]. Pujals-Sambarino report some progress in the direction of proving that such diffeomorphisms can be approximated by others having (codimension 1) homoclinic tangencies, in the C^1 topology. This would be an important step towards incorporating this phenomenon into the theory. Another point of view is to try to show that it is negligible from a probabilistic point of view. It is not yet known if coexistence of infinitely many attractors or repellers corresponds to zero Lebesgue probability sets in parameter space, for generic families of maps. But Araújo [5] proves that some general maps with random noise have only finitely many attractors, including one-parameter families of diffeomorphisms through homoclinic tangencies (as originally considered in [74]) with small random errors in parameter space.

8 Ergodic properties of partially hyperbolic systems

Then, a central problem is to understand the structure and properties of partially hyperbolic transitive sets or, more generally, invariant transitive sets supporting a dominated splitting. Here is a couple of my favourite questions: Do these sets have some shadowing property (approximation of pseudo-orbits by actual orbits)? Can one give some description of the dynamics in symbolic terms (semi-conjugacy to a shift map)?

In general, these questions are wide open, but for C^2 diffeomorphisms on a surface Pujals-Sambarino [91] provide a rather precise description of sets Λ with a dominated splitting: if all the periodic points in Λ are hyperbolic saddles, then it is the union of a hyperbolic set and finitely many invariant closed curves which are normally hyperbolic and support an irrational rotation.

On the other hand, there is substantial progress in the ergodic theory of partially hyperbolic systems. Much of the foundations concerning invariant foliations were set by Brin-Pesin [21], and they investigated the relations between topological properties of these foliations and ergodic properties of the system, specially when it preserves volume. This was pursued more recently by Grayson-Pugh-Shub [41], leading to several other results providing conditions for a diffeomorphism to be *stably ergodic*: every volume preserving diffeomorphism in a C^1 neighbourhood is ergodic with respect to Lebesgue measure.

For general partially hyperbolic attractors Λ, Pesin-Sinai [89] constructed *Gibbs u-states*: invariant measures with absolutely continuous conditional measures along strong-unstable leaves (leaves of the unique integral foliation of E^u). Then Carvalho [23] proved that in some cases, e.g. diffeomorphisms derived from Anosov ones, these Gibbs u-states are SRB measures. Kan [47] gave examples of transitive partially hyperbolic diffeomorphisms having more than one SRB measure, with the basin of each of these measures dense in the ambient.

In [105], I introduced a class of maps exhibiting non-hyperbolic attractors with a multidimensional character: there are several expanding directions (positive Lyapunov exponents) at Lebesgue almost every point in the basin of attraction. The simplest case corresponds to cylinder maps like

$$\varphi : S^1 \times \mathbb{R} \to S^1 \times \mathbb{R}, \quad \varphi(\theta, x) = (g(\theta), a(\theta) - x^2)$$

where g is strongly expanding, and $a(\cdot)$ is a convenient Morse function (diffeomorphisms in compact manifolds and/or higher dimensions may be constructed along similar lines). They present some notable differences with respect to low dimensional non-hyperbolic systems such as unimodal or Hénon maps, in particular they are robust (not just metrically persistent): chaotic behaviour – several positive Lyapunov exponents – occurs for a full open set of perturbations. In this context, Alves obtained the first examples of SRB measures with non-uniform multidimensional expansion:

Theorem 9 ([2]). *Every map in a neighbourhood of φ in the space of C^3 self maps of $S^1 \times \mathbb{R}$ admits an absolutely continuous invariant measure μ. Moreover, this measure is unique and ergodic.*

These last results inspired two general statements of existence and finitude of SRB measures for partially hyperbolic attractors that I condense in the following theorem. They concern partially hyperbolic diffeomorphisms whose central direction is either mostly contracting – negative Lyapunov exponents along E^c – or mostly expanding – positive Lyapunov exponents along E^c. Without going into technicalities (nor maximum generality) let me say that, given a diffeomorphism f partially hyperbolic over the whole M with invariant splitting $TM = E^c \oplus E^u$, then E^c is *mostly contracting* if $\|Df^n(z)v\| \to 0$ exponentially fast as $n \to +\infty$, for every $v \in E^c_z$ and Lebesgue almost every $z \in M$. And, given f with invariant splitting $TM = E^s \oplus E^c$, we say that E^c is *mostly expanding* if $\|Df^n(z)v\| \to \infty$ exponentially fast as $n \to +\infty$, for every $v \in E^c_z$ and Lebesgue almost every $z \in M$, like in (3) below.

Theorem 10. *Let f be a partially hyperbolic C^2 diffeomorphism on a manifold M. We have*

1. *([18]) If the central direction is mostly contracting, then the Gibbs u-states of f are SRB measures, there are finitely many of them, and their basins cover a full Lebesgue measure subset of M.*

2. *([3]) If the central direction is mostly expanding, then Lebesgue almost every point is in the basin of some SRB measure. If the central Lyapunov exponents are bounded away from zero then there are finitely many SRB measures.*

Pushing part 1 of the theorem further on, Castro [24] has just proved exponential decay of correlations for a large class of partially hyperbolic attractors. Related to the examples of Kan [47] I mentioned before, which also fit in this setting, it is interesting to mention that if all the leaves of the strong-unstable foliation are dense in M then there is a unique SRB measure [18]. Is this generic among the transitive diffeomorphisms satisfying the assumptions of part 1?

The proof of part 2 includes a generalization of Ruelle's theorem [96] on the existence of absolutely continuous invariant measures for uniformly expanding maps. Let $f : M \to M$ be any C^2 covering map which is *non-uniformly expanding* in the sense that ($m(L) = 1/\|L^{-1}\|$ is the minimum expansion of a linear map L)

$$\liminf_{n\to+\infty} \frac{1}{n} \log \prod_{j=0}^{n-1} m(Df(f^j(z))) > 0 \tag{3}$$

Lebesgue almost everywhere. Then f has some ergodic invariant measure absolutely continuous with respect to Lebesgue measure and, indeed, the basins of such measures cover almost all of M. There is a version of this last result for piecewise smooth maps, assuming that most points do not visit the singular set (where the map fails to be smooth, or the derivative fails to be surjective) too close too often; see [3].

Such results suggest that non-uniform hyperbolicity may suffice for a system to have good statistical properties. In this spirit, I state the following

Conjecture: *If a smooth map has only non-zero Lyapunov exponents at Lebesgue almost every point, then it admits some SRB measure.*

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